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Knots with only two strict essential surfaces

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Abstract We consider irreducible 3–manifolds M that arise as knot complements in closed 3–manifolds and that contain at most two connected strict essential surfaces. The results in the paper relate the boundary slopes of the two surfaces to their genera and numbers of boundary components. Explicit quantitative relationships, with interesting asymptotic properties, are obtained in the case that M is a knot complement in a closed manifold with cyclic fundamental group.

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We dedicate this paper to Andrew Casson, in honor of his 60th birthday.

Introduction

It is well known that if K is a torus knot of type (p, q) in S^3, then the exterior of $M(K)$ contains exactly two connected essential surfaces up to isotopy. (See Section 1 for precise definitions of "essential surface" and other terms used in this introduction.) One of the connected essential surfaces in $M(K)$ is an essential annulus whose numerical boundary slope with respect to the standard framing is pq. The other connected essential surface in $M(K)$ has boundary slope 0 (ie, is a spanning surface) and has genus $(p-1)(q-1)/2$. Note that the genus of the surface with boundary slope 0 is less than the boundary slope of the other essential surface.

This relationship between two quantities that are computed in entirely different ways may appear coincidental. However, one of the results of this paper, Corollary 9.6, asserts that a similar, although weaker, inequality holds for any knot K in a homotopy 3–sphere Σ such that $M(K)$ is irreducible and has only two essential surfaces up to isotopy. In this case one of the essential surfaces

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has boundary slope 0 with respect to the standard framing and the other has boundary slope $r \neq 0$. If the surface with boundary slope 0 has genus $g \geq 2$ and if $r \neq \infty$ then

$$\frac{g-1}{4\log_2(2g-2)} \leq r^2.$$

Thus g is bounded above by a function of r which grows only slightly faster than a quadratic function.

This result illustrates the general theme of this paper. We consider knots in a closed, orientable 3–manifold Σ, whose complements are irreducible and contain at most two connected essential surfaces that are *strict* in the sense of 1.15. For such knots our results relate the boundary slopes of the connected strict essential surfaces in the knot exteriors to their intrinsic topological invariants (genera and numbers of boundary curves). Our deepest results concern the case in which $\pi_1(\Sigma)$ is cyclic. (This includes the case $\Sigma = S^3$.) However, we also obtain some non-trivial results of this type for an arbitrary Σ; studying a general knot exterior in an arbitrary Σ is equivalent to studying an arbitrary irreducible *knot manifold*, ie, a compact, irreducible, orientable 3–manifold whose boundary is a torus.

In Section 6 we give a general qualitative description of irreducible knot manifolds that contain at most two connected strict essential surfaces up to isotopy. Theorem 6.7 includes a complete classification of the irreducible knot manifolds that contain at most one isotopy class of connected strict essential surfaces; in particular they are all Seifert fibered spaces. This completes a partial result proved in [6]. Theorem 6.7, with Proposition 6.4, also provides a dichotomy among the irreducible knot manifolds that contain exactly two connected strict essential surfaces up to isotopy. One subclass of such knot manifolds, called exceptional graph manifolds, are defined by an explicit classification (6.3). The complementary subclass, called non-exceptional two-surface knot manifolds, are homeomorphic to compact cores of one-cusped finite-volume hyperbolic 3–manifolds. Furthermore, the two connected strict essential surfaces in a non-exceptional two-surface knot manifold are bounded, have distinct boundary slopes, and have strictly negative Euler characteristic.

The subsequent sections are devoted to studying the relationships among the boundary slopes and intrinsic topological invariants of connected strict essential surfaces in non-exceptional two-surface knot manifolds. Theorem 7.4 applies to an arbitrary non-exceptional two-surface knot manifold M and asserts, roughly speaking, that when the two connected strict essential surfaces F_1 and F_2 in M are isotoped into standard position with respect to each other, they cut each other up into disks and annuli. This easily implies Corollary 7.6, which asserts that for $i=1,2$ we have

$$|\chi(F_i)| \leq \frac{m_1 m_2 \Delta}{2},$$

where m_i denotes the number of boundary components of F_i, and Δ denotes geometric intersection number of the boundary slopes of F_1 and F_2. (Thus if M is a knot exterior in a closed manifold, and if with respect to some framing of the knot the numerical boundary slope of F_i is $s_i = p_i/q_i$, then $\Delta = |p_1q_2 - p_2q_1|$.)

For the case where the non-exceptional two-surface knot manifold M arises as the complement of a knot K in a closed 3–manifold Σ with cyclic fundamental group, our main results are Theorem 9.5 and Theorem 11.16. If F_i, m_i and Δ are defined as above, if g_i denotes the genus of F_i and q_i the denominator of its numerical boundary slope with respect to any framing of K, and if $g_2 \geq 2$, Theorem 9.5 asserts that

$$\left(\frac{q_1}{\Delta}\right)^2 \leq \frac{4m_2^2 \log_2(2g_2-2)}{g_2-1}.$$

For the case of a knot in a homotopy 3–sphere whose exterior is irreducible and has only two essential surfaces up to isotopy, Theorem 9.5 specializes to Corollary 9.6. However, Theorem 9.5 applies more generally to knots in S^3 and other lens spaces whose exteriors contain three non-isotopic essential surfaces of which only two are strict. The figure eight knot and its sister are well known examples of such knots. Theorem 11.16 gives a somewhat different inequality under the same hypotheses as Theorem 9.5. An examination of the inequality in the statement of 11.16 will reveal that it may be written in the form

$$\frac{q_1^2}{\Delta} \leq \frac{m_2|\chi_1|}{m_1|\chi_2|} f(|\chi_2|),$$

where $\chi_i = \chi(F_i) = 2 - 2g_i - m_i$ for $i = 1,2$, and $f(x)$ is a function of a positive variable x which grows more slowly than any positive power of x. Theorem 11.16 is more difficult to prove than Theorem 9.5, and in a sense that we shall explain in 11.18 it is qualitatively stronger than Theorem 9.5, although it does not imply the latter theorem.

This paper, like our earlier papers [6] and [7], is based on the idea of using character variety techniques to study the essential surfaces in a knot exterior. Sections 2–5 are foundational in nature. Much of the work in these sections consists of refining and systematizing material that has its origins in such papers as [5], [4], and [3], concerning character varieties, actions on trees, essential surfaces, and the norm on the homology of the boundary of a knot manifold that was first used in the proof of the Cyclic Surgery Theorem.

The material in Section 7 is crucial to the proofs of all the main results of the paper. This section centers around the study of the norm on the plane $H_1(M;\mathbb{R})$ in the case where M is a non-exceptional two-surface knot manifold M. In this case the ball of any radius with respect to the norm is a parallelogram for which the slopes of the diagonals are the boundary slopes of the strict essential surfaces F_1 and F_2 in M. The extra information needed to determine the shape of such a parallelogram is the ratio of the norms of the boundary slopes. Unlike the slopes themselves, this ratio has no obvious topological interpretation. Theorem 7.2 asserts that this ratio is bounded above by a topologically defined quantity associated to the surfaces F_1 and F_2. This quantity, denoted $\kappa(F_1,F_2)$, is defined in 7.1.

A key ingredient in the proof of Theorem 7.2 is the study of degrees of trace functions of non-peripheral elements of $\pi_1(M)$. This appears to be the first application of information of this type to the topology of 3–manifolds.

Theorem 7.4, the result mentioned above which asserts that when F_1 and F_2 are isotoped into standard position they cut each other up into disks and annuli, is relatively easy to derive from Theorem 7.2. The results about knot exteriors in a manifold with cyclic fundamental group depend on combining Theorem 7.2 with a fundamental result from [4] relating the norm on $H_1(M;\mathbb{R})$ to cyclic Dehn fillings of M. Combining these directly gives a purely topological result, Theorem 7.7, which asserts that

$$\frac{q_1^2}{\Delta} \leq 2\kappa(F_1,F_2).$$

The deepest theorems in the paper, Theorems 9.5 and 11.16, which relate the boundary slopes and intrinsic topological invariants of F_1 and F_2, are proved by combining Theorem 7.7 with combinatorial results, Proposition 9.4 and Proposition 11.15, which relate $\kappa(F_1,F_2)$ to more familiar topologically defined quantities. These results depend on combining graph-theoretical arguments, given in Sections 8 and 10 respectively, with material in 2– and 3–manifold topology that is presented in Sections 9 and 11 respectively.

The theorems proved in this paper remain true if the condition that M is a non-exceptional two-surface manifold is replaced by a condition that is weaker, but more technical. This condition is described in 7.3. Computational evidence suggests that there are many examples of knot exteriors in lens spaces that satisfy this condition.

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1 Conventions

1.1 In this paper the results about manifolds may be interpreted in the smooth or PL category, or in the category in which objects are topological manifolds, embeddings are locally flat, and polyhedra contained in manifolds are tame. All of these categories are equivalent in dimensions ≤ 3, and we will often implicitly choose one for the proof of a particular result.

We shall generally denote the unit interval $[0,1] \subset \mathbb{R}$ by I. The Euler characteristic of a compact polyhedron P will be denoted by $\chi(P)$, and the cardinality of a finite set X by $\#(X)$.

1.2 Base points will often be suppressed when the choice of a base point does not affect the truth value of a statement; for example, if f is a map between path-connected spaces X and Y, to say that $f_\sharp\colon \pi_1(X) \to \pi_1(Y)$ is injective means that for some, and hence for every, choice of base point $x \in X$, the homomorphism $f_\sharp\colon \pi_1(X,x) \to \pi_1(Y, f(x))$ is injective.

1.3 Suppose that M is a manifold or a polyhedron, that $x \in M$ is a base point and that $(\widetilde{M}, p)$ is a regular covering space of M. Each choice of basepoint $\widetilde{x} \in p^{-1}(x) \subset \widetilde{M}$ determines an action of $\pi_1(M,x)$ on $\widetilde{M}$. An action of $\pi_1(M,x)$ on $\widetilde{M}$ will be termed *standard* if it arises in this way from some choice of basepoint $\widetilde{x} \in p^{-1}(x)$. Note that if x and y are any base points in X and if $J\colon \pi_1(M,x) \to \pi_1(M,y)$ is the isomorphism determined by some path from x to y, then pulling back a standard action of $\pi_1(M,y)$ via J gives a standard action of $\pi_1(M,x)$.

1.4 A path connected subspace A of a path connected space X will be termed "π_1–injective" if the inclusion homomorphism from $\pi_1(A)$ to $\pi_1(X)$ is injective. More generally, if A is a subspace of a space X, to say that A is π_1–injective in X will mean that each path component of A is π_1–injective in the path component of X containing it.

1.5 Suppose that F is a properly embedded codimension–1 submanifold of a manifold M. By a *collaring* of F in M we mean an embedding neighborhood of F, and $h\colon F \times [-1,1] \to M$ such that $h(x,0) = x$ for every $x \in F$ and $h(F \times [-1,1]) \cap \partial M = (\partial F) \times [-1,1]$. If h is a collaring of F, we shall set $V_h = h(F \times [-1,1])$, $V_h^{+1} = h(F \times [0,1])$ and $V_h^{-1} = h(F \times [-1,0])$. We define a *collar neighborhood* of F to be a set that has the form V_h for some collaring h of F.

By definition, the submanifold F is *two-sided* if it has a collaring. A *transverse orientation* of a two-sided submanifold F is an equivalence class of collarings, where two collarings h and h' are defined to be equivalent if $V_h^{+1} \cap V_{h'}^{-1} = F$.

1.6 As usual, we define a *homotopy* to be a continuous map $H\colon X \times I \to Y$ where X and Y are spaces, and for each $t \in I$ we denote the map $x \mapsto H(x,t)$ by H_t. The *inverse* $\bar{H}$ of the homotopy $H\colon X \times I \to Y$ is defined by $\bar{H}(x,t) = H(x, 1-t)$. By a *reparametrization* of H we mean a map $H'\colon X \times [a,b] \to Y$, where $[a,b] \subset \mathbb{R}$ is a non-degenerate interval, defined by $H'(x,t) = H(x,\alpha(t))$, where $\alpha\colon [a,b] \to I$ is a homeomorphism with $\alpha(a) = 0$.

A homotopy $H\colon X \times I \to Y$ is a *composition* of homotopies $H^1, \dots, H^k\colon X \times I \to Y$ if there are real numbers $t_0, \dots, t_k$ with $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $H|_{[t_{i-1},t_i]}$ is a reparametrization of H^i for $i = 1, \dots, k$.

A *path* in a space Y, ie, a map from I to Y, may be regarded as a homotopy $\{\star\} \times I \to Y$, where $\{\star\}$ is a one-point space. By specializing the definitions given above we obtain definitions of the inverse of a path, a reparametrization of a path and a composition of paths.

1.7 Suppose that M is a compact manifold and that $F \subset M$ is a properly embedded submanifold of codimension 1. By a *homotopy in* (M,F) we shall mean a homotopy $H\colon K \times I \to M$, where K is some polyhedron, such that $H(K \times \partial I) \subset F$; we may regard H as a map of pairs $H\colon (K \times I, K \times \partial I) \to (M,F)$.

Now suppose that we are given a transverse orientation of $F \subset M$, and an element ω of $\{-1,+1\}$. A homotopy H in (M,F) will be said to *start on the* ω *side* (or, respectively, to *end on the* ω *side* if for some $\delta > 0$ we have $H(K \times [0,\delta]) \subset V_h^{\omega}$ (or, respectively, $H(K \times [1-\delta, 1]) \subset V_h^{\omega}$), where h is a collaring of F realizing its transverse orientation; the condition is independent of the choice of a collaring realizing the given transverse orientation.

A homotopy H in (M,F) is a *basic homotopy* if $H^{-1}(F) = K \times \partial I$. Note that every basic homotopy starts on the ω side and ends on the ω' side for some $\omega, \omega' \in \{-1,+1\}$.

Specializing these definitions to the case in which K is a point, we obtain the definitions of a path in (M,F), of a basic path in (M,F), and of a path in (M,F) which starts or ends on the ω side.

A basic path α in (M,F) will be termed *essential* if it is not fixed-endpoint homotopic to a path in F. A basic homotopy $H\colon (K \times I, K \times \partial I) \to (M,F)$

will be termed *essential* if for every $x \in K$ the basic path $\alpha_x\colon t \mapsto H(x,t)$ in (M,F) is essential. Note that the condition that α_x be essential depends only on the connected component of x in K.

Now suppose that F is a properly embedded, codimension–1 submanifold F of a compact manifold M, and that k is a positive integer. A homotopy $H\colon (K\times I, K\times\partial I)\to(M,F)$ will be called a *reduced homotopy of length k in (M,F)* if we may write H as a composition of k essential basic homotopies $H^1,\dots,H^k$ in such a way that, given a transverse orientation of F, for each $i\in\{1,\dots,k-1\}$ there is an element ω_i of $\{-1,+1\}$ such that H^i ends on the ω_i side and H^{i+1} starts on the $-\omega_i$ side. Note that this condition is independent of the choice of transverse orientation. Note also that, for any choice of transverse orientation, H starts on the same side as H_1 and ends on the same side as H_k.

We define a *reduced homotopy of length* 0 in (M,F) to be a map H from K to F. In this case we set $H_0 = H_1 = H$. If H is a reduced homotopy of length 0 and H' is a reduced homotopy of length ≥ 0 for which H'_1 (or H'_0) is equal to H, we define the *composition* of H with H' (or of H' with H) to be H'.

1.8 By a *closed curve* in a topological space X we mean a map $c\colon S^1\to X$. If c is a closed curve in a manifold M, and F is a properly embedded submanifold of codimension 1 in M, we define the *geometric intersection number of c with F*, denoted $\Delta(c,F)$ (or $\Delta_M(c,F)$ when we need to be more explicit), to be the minimum cardinality of $g^{-1}(F)$, where g ranges over all closed curves homotopic to c.

1.9 A *simple closed curve* in a manifold M is a connected closed 1–manifold $C\subset M$. With a simple closed curve C we can associate a closed curve c in M, well-defined modulo composition with self-homeomorphisms of S^1, such that $c(S^1)=C$. If $F\subset M$ is a properly embedded submanifold of codimension 1, the *geometric intersection number* $\Delta(C,F)=\Delta(c,F)$ is well-defined, since composing c with a self-homeomorphism of S^1 clearly does not change its geometric intersection number with F.

In particular, for any two simple closed curves C and C' in a closed 2–manifold, $\Delta(C,C')$ is the geometric intersection number of c and C' in the familiar sense.

1.10 If T is a 2–dimensional torus, we define a *slope* on T to be an isotopy class of homotopically non-trivial simple closed curves in T. If s_1 and s_2 are slopes, we shall write $\Delta(s_1,s_2)=\Delta(C_1,C_2)$ for any simple closed curves C_i realizing the slopes s_i.

The isotopy classes of homotopically non-trivial *oriented* simple closed curves in T are in natural bijective correspondence with elements of $H_1(T;\mathbb{Z})$ which are *primitive* in the sense of not being divisible by any integer > 1. Thus there is a natural two-to-one map from the set of primitive elements of $H_1(T;\mathbb{Z})$ onto the set of slopes on T. We shall denote this map by $\alpha \mapsto \langle\alpha\rangle$. We have $\langle\alpha\rangle = \langle\alpha'\rangle$ if and only if $\alpha' = \pm\alpha$.

If T is a 2–torus and α and β are primitive elements of $H_1(T;\mathbb{Z})$, then $\Delta(\langle\alpha\rangle, \langle\beta\rangle)$ is the absolute value of the homological intersection number of α and β.

1.11 If C is a non-empty closed 1–manifold in a 2–torus T, and C has no homotopically trivial components, then all components of C have the same slope s. We call s the *slope* of C.

Let C_1, C_2 be closed 1–manifolds, with no homotopically trivial components, in a torus T. Let s_i and m_i denote respectively the slope and the number of components of C_i. Then C_1 and C_2 are isotopic to 1–manifolds C_1^0 and C_2^0 such that $\#(C_1^0 \cap C_2^0) = m_1 m_2 \Delta(s_1, s_2)$. If $\#(C_1 \cap C_2) = m_1 m_2 \Delta(s_1, s_2)$ we shall say that C_1 and C_2 *intersect minimally*. This implies that no arc in C_1 is fixed-endpoint homotopic to any arc in C_2.

1.12 An *essential surface* in an irreducible, orientable 3–manifold M is a two-sided properly embedded surface in M which is non-empty and π_1–injective, and has no 2–sphere components and no boundary-parallel components.

1.13 We define a *knot manifold* to be a connected, compact, orientable 3–manifold M such that ∂M is a torus.

We will say that a knot manifold is *hyperbolic* if it is homeomorphic to the compact core of a complete hyperbolic manifold with finite volume.

If M is a knot manifold we will say that an element $\gamma \in \pi_1(M)$ is *peripheral* if it is conjugate to an element of the subgroup $\mathrm{im}(\pi_1(\partial M) \to \pi_1(M))$.

If K is a (tame) knot in a closed, orientable 3–manifold Σ, the *exterior* of K, defined to be the complement of an open tubular neighborhood of K, will be denoted by $M(K)$. Note that $M(K)$ is well-defined up to ambient isotopy in Σ, and in particular up to homeomorphism, and that it is a knot manifold. A *meridian* of K is a non-trivial simple closed curve in the torus $\partial M(K)$ which bounds a disk in the tubular neighborhood $\Sigma - \mathrm{int}\, M(K)$. Such a curve exists and is unique up to isotopy. Thus there is a well-defined *meridian slope* in the torus $\partial M(K)$. A *meridian class* for K is a primitive element μ of

$H_1(\partial M(K);\mathbb{Z})$ such that $\langle\mu\rangle$ is the meridian slope. According to 1.10, K has exactly two meridian classes, and they differ by a sign.

We define a *framing* for K to be an ordered basis (μ,λ) for $H_1(\partial M(K);\mathbb{Z})$ such that μ is a meridian class. In the special case where Σ is an integer homology 3–sphere we define a framing (μ,λ) to be *standard* if λ generates the kernel of the inclusion homomorphism $H_1(\partial M(K);\mathbb{Z}) \to H_1(M(K);\mathbb{Z})$.

If (μ,λ) is an arbitrary framing for K, there is a bijective correspondence between slopes in $\partial M(K)$ and elements of $\mathbb{Q}\cup\{\infty\}$ defined by

$$\langle\alpha\rangle \mapsto \omega(\alpha,\lambda)/\omega(\alpha,\mu),$$

where ω denotes homological intersection number. If C is a non-empty closed 1–manifold in $\partial M(K)$ whose components are homotopically non-trivial, we define the *numerical slope* of C, with respect to a given framing, to be the element of $\mathbb{Q}\cup\{\infty\}$ corresponding to the slope of C (in the sense of 1.10).

Note that if C is a non-empty closed 1–manifold in $\partial M(K)$, and if the numerical slope of C in a given framing is written in the form p/q, where p and q are relatively prime integers and $q>0$, then $q=\Delta(s,\mathfrak{m})$, where s denotes the slope of C and $\mathfrak{m}$ denotes the meridian slope of K in particular q is independent of the choice of framing. For this reason, if s is a slope on $\partial M(K)$, it is natural to refer to $\Delta(s,\mathfrak{m})$ as the *denominator* of s.

1.14 Let M be a compact orientable 3–manifold such that every component of ∂M is a torus. Let T be a component of ∂M. If F is an essential surface in M that meets T then $\partial F\cap T$ is a 1–manifold in T having no homotopically trivial components. Thus by 1.11, $\partial F\cap T$ has a well-defined slope s, which we call the *boundary slope of F on T*.

If F is a bounded essential surface in a knot manifold M then we will refer to the boundary slope of F on ∂M simply as the *boundary slope of F*.

If K is a knot in a closed orientable 3–manifold Σ and if F is a bounded essential surface in $M(K)$ then we define the *numerical boundary slope* of F with respect to any given framing (μ,λ) to be the numerical slope of ∂F with respect to (μ,λ), in the sense of 1.13.

We define a *boundary class* of a bounded essential surface F in a knot manifold M to be a primitive element α of $H_1(\partial M;\mathbb{Z})$ such that $\langle\alpha\rangle$ is the boundary slope of F. According to 1.10, a bounded essential surface in a knot manifold has exactly two boundary classes, and they differ by a sign.

1.15 Suppose that M is a compact, orientable irreducible 3–manifold whose boundary components are tori. A connected essential surface in M is called a *semi-fiber* if either F is a fiber in a fibration of M over S^1, or F is the common frontier of two 3–dimensional submanifolds of M, each of which is a twisted I–bundle with associated ∂I–bundle F. An essential surface $F \subset M$ is termed *strict* if no component of F is a semi-fiber. A strict essential surface has no disk components, since an irreducible knot manifold which has an essential disk must be a solid torus, and the essential disk in a solid torus is a fiber.

1.16 Since a semi-fiber in a bounded 3–manifold M must meet every component of ∂M, any essential surface that is disjoint from at least one component of ∂M must be strict.

1.17 Suppose that the orientable 3–manifold M is either a Seifert fibered 3–manifold or an I–bundle over a surface. We will define a surface in M to be *vertical* if it is a union of fibers, and to be *horizontal* if it is everywhere transverse to the fibers. If M is an I–bundle over a surface B, then the *vertical boundary* of M is the inverse image of ∂B under the projection map.

It is known that if M is a Seifert fibered manifold then an essential surface in M is either isotopic to a vertical surface or to a horizontal surface. (A stronger version of this statement, for essential laminations, is proved in [2]. See also [8, Section II.7] and [12].) It is clear that the manifold obtained by splitting a Seifert fibered manifold along a horizontal surface has a natural I–bundle structure, and hence that horizontal essential surfaces in Seifert fibered manifolds are never strict.

An essential vertical annulus in a Seifert fibered manifold is the inverse image under the Seifert fibration map of an essential properly embedded arc in the base surface. An essential vertical annulus in an I–bundle is the inverse image under the fibration map of an essential simple closed curve in the base.

Suppose that M is a trivial I–bundle and that F is a properly embedded π_1–injective surface in M such that all components of ∂F are contained in the same component B of the ∂I–bundle associated to M. It follows from [13, Proposition 3.1] that F is isotopic to a horizontal surface by an ambient isotopy that preserves the vertical boundary of M, and that each component of F is parallel to a subsurface of B.

As a consequence of this fact we observe that if M is a trivial I–bundle, and F is a properly embedded π_1–injective surface in M such that ∂F is contained in the vertical boundary of M, then F is isotopic to a horizontal surface by an ambient isotopy that preserves the vertical boundary of M.

Suppose that M is an I–bundle and that A is a disjoint union of properly embedded annuli in M none of which is parallel to an annulus contained in the ∂I–bundle associated to M. It follows from [13, Lemma 3.4] in the case that M is a trivial I–bundle, and from [2, Lemma 2] in the twisted case that A is isotopic to a vertical surface.

Suppose that F is a properly embedded π_1–injective surface in an I–bundle M such that ∂F is contained in the vertical boundary of M. Then F is isotopic to a horizontal surface. This follows from [13, Proposition 3.1 and Proposition 4.1].

1.18 A closed curve c in a path-connected space X defines a conjugacy class in $\pi_1(X)$, which we shall denote by $[c]$.

1.19 By a *graph* we mean a CW–complex of dimension ≤ 1. Thus a graph Γ has an underlying space, which we shall denote by $|\Gamma|$, and which need not be connected; $|\Gamma|$ is a disjoint union of 0–cells, called *vertices*, and open 1–cells, called *edges*. Each edge has the structure of an affine interval.

The vertices in the frontier of an edge will be called its *endpoints*; each edge has either one or two endpoints. We sometimes will need to consider *oriented edges* in a graph. For every edge e there are two oriented edges whose underlying edge is e; these will be called *orientations* of e. If ω is an oriented edge we shall denote by $|\omega|$ the underlying edge of ω and by $-\omega$ the opposite orientation to ω. An oriented edge ω has an *initial vertex* denoted $\mathrm{init}(\omega)$ and a *terminal vertex* denoted $\mathrm{term}(\omega)$. The *valence* of a vertex v is the number of oriented edges whose initial vertex is v.

In the last four sections of this paper, the underlying space $|\Gamma|$ of a graph Γ will often arise as a subpolyhedron of a PL manifold.

A *subgraph* of a graph Γ is a graph Γ' such that $|\Gamma'| \subset |\Gamma|$ and every vertex or edge of Γ' is a vertex or edge of Γ. A graph Γ is said to be *connected* if $|\Gamma|$ is connected, and a *component* of a graph Γ is a subgraph C such that $|C|$ is a connected component of $|\Gamma|$.

An *edge path* of length $n > 0$ in a graph Γ is a sequence $(\omega_1, \ldots, \omega_n)$ of oriented edges of Γ such that $\mathrm{term}(\omega_i) = \mathrm{init}(\omega_{i+1})$ for $i = 1, \ldots, n-1$. If $\omega_i \neq -\omega_{i+1}$ for $i = 1, \ldots, n-1$ then we will say that the edge path is *reduced.* The *track* of an edge path $(\omega_1, \ldots, \omega_n)$ is the subgraph of Γ whose edges are $|\omega_1|, \ldots, |\omega_n|$ and whose vertices are their endpoints.

An *arc* in a graph Γ is a subgraph A of Γ such that $|A|$ is homeomorphic to a (possibly degenerate) closed interval in $\mathbb{R}$. A *circuit* in Γ is a subgraph C of Γ such that $|C|$ is homeomorphic to S^1.

The *length* of a finite graph Γ is the number of edges of Γ.

2 Essential surfaces

In this section we collect several general results about essential surfaces which will be used in Sections 5–7.

The next two results are proved in [9]. We restate them here for completeness.

Proposition 2.1 (Proposition 1.1 of [9]) *Suppose that F is a bounded essential surface in an irreducible knot manifold M, and suppose that α is a path in F which has its endpoints in ∂F and is fixed-endpoint homotopic in M to a path in ∂M. Then α is fixed-endpoint homotopic in F to a path in ∂F.*

Proposition 2.2 (Proposition 1.3 of [9]) *Let M be a compact orientable irreducible 3–manifold containing an essential torus T and let M' be the manifold obtained by splitting M along T and let $q\colon M' \to M$ denote the quotient map. Let F be a connected properly embedded surface in M which is not isotopic to T. Then F is a strict essential surface if and only if it is isotopic to a surface S transverse to T such that*

(1) *each component of $q^{-1}(S)$ is essential in the component of M' containing it; and*

(2) *some component of $q^{-1}(S)$ is a strict essential surface in the component of M' containing it.*

Proposition 2.3 *For any essential surface F in a hyperbolic knot manifold M we have $\chi(F) < 0$.*

Proof Since M is orientable and F is two-sided, F must be orientable. Since M is irreducible and is not a solid torus, F cannot be a 2–sphere or a disk, and the hyperbolicity of M implies that F cannot be a torus. Now suppose that F is an annulus. Consider the submanifold N of M which is the closure of a regular neighborhood of $F \cup \partial M$. The frontier of N has either one or two components, each of which is a torus containing a simple closed curve that is isotopic to a core curve of F. In particular, no frontier component of N can be contained in a ball. Furthermore since F is essential, no frontier component of N can be boundary-parallel. Since M has no essential tori, it follows that each frontier component of N bounds a solid torus, which must be a component of $\overline{M-N}$. Since F is a π_1–injective annulus and each component of $\overline{M-N}$ is a solid torus it follows that M is Seifert-fibered. This contradicts the fact that M is hyperbolic. □

Proposition 2.4 *Let S and F be essential surfaces in a compact, irreducible, orientable 3–manifold M. Suppose that F is connected, and that every component of S is isotopic to F. Then there exist a collaring h of F in M and a finite set $Y \subset [-1,1]$ such that S is isotopic to $h(F \times Y)$.*

Proof Let $\mathcal{N}$ denote the set of all connected submanifolds N of M such that either (a) N is a component of S, or (b) N is 3–dimensional, each component of $\mathrm{frontier}_M N$ is a component of S, and N can be given the structure of a trivial I–bundle in such a way that $N \cap \partial M$ is the vertical boundary of N. Then $\mathcal{N}$ is finite, and is non-empty since $S \neq \emptyset$. Hence we may choose $N_0 \in \mathcal{N}$ which is maximal with respect to inclusion.

We claim that $S \subset N_0$. Suppose this is false. Then some component S_0 of S is disjoint from N_0. Let us choose a component S_1 of $\mathrm{frontier}_M N_0$. Since S_0 and S_1 are isotopic to F and hence to each other, it follows from [13, Lemma 5.3] that $S_0 \cup S_1$ is the associated I–bundle of a trivial I–bundle $H \subset M$. In particular $H \in \mathcal{N}$. Hence if $H \supset N_0$, we have a contradiction to the maximality of N_0. The other possibility is that $H \cap N_0 = S_1$. However, in this case, since $H, N_0 \in \mathcal{N}$, it is clear that $H \cup N_0 \in \mathcal{N}$, and we again have a contradiction to maximality. This proves our claim.

Now let M_0 denote a regular neighborhood of N_0 in M. Then M_0 may be given the structure of a trivial I–bundle in such a way that $M_0 \cap \partial M$ is the vertical boundary of M_0. Since $S \subset N_0$, we may regard S as a π_1–injective surface in M_0 whose boundary is contained in the vertical boundary of M_0. It now follows from 1.17 that S is isotopic to a horizontal surface in M_0. This implies the conclusion of the proposition. □

Proposition 2.5 *Let F be a connected essential surface in a compact, irreducible, orientable 3–manifold M. Let h denote a collaring of F in M. Suppose that Y is a finite subset of $[-1,1]$, and let S denote the essential surface $h(F \times Y) \subset M$. Let $y_0 \in Y$ be given, let $K \subset F$ be a compact polyhedron and set $K_0 = h(K \times y_0) \subset S$. Then we have $t_S(K_0) \geq \#(Y) \cdot t_F(K)$.*

Proof Set $\nu = \#(Y)$. If $\nu = 1$ the assertion is trivial. If $\nu > 1$ we may assume without loss of generality that $\{-1, +1\} \subset Y$. According to Definition 3.12, what we need to prove is that if for a given positive integer θ there is a reduced homotopy $H\colon (K \times I, K \times \partial I) \to (M, F)$ of length $\theta - 1$ such that, for some $c \in I$, the map H_c is the inclusion map $K \hookrightarrow F \subset M$, then there is a reduced homotopy $H'\colon (K \times I, K \times \partial I) \to (M, S)$ of length $\nu\theta - 1$ such that, for some $c' \in I$, the map H'_c is the inclusion map $K_0 \hookrightarrow S \subset M$.

Let us give F the transverse orientation determined by the collaring h. We may write H as a composition of $\theta - 1$ essential basic homotopies $H^1, \ldots, H^{\theta-1}$ in such a way that for each $i \in \{1, \ldots, \theta - 2\}$ there is an element ω_i of $\{-1, +1\}$ such that H^i ends on the ω_i side and H^{i+1} starts on the $-\omega_i$ side. Let ω_0 and $\omega_{\theta-1}$ denote the elements of $\{-1, +1\}$ such that H^1 starts on the $-\omega_0$ side and $H^{\theta-1}$ ends on the $\omega_{\theta-1}$ side.

We set $M' = M - h(F \times (-1, 1)) = \overline{M - V_h}$, and we fix a map $q\colon M' \to M$ such that $q(h(x, j)) = x$ for every $(x, j) \in F \times \{-1, 1\}$, and such that q maps $M - V_h$ homeomorphically onto $M - F$. For $i = 1, \ldots \theta - 1$, since H^i is a basic homotopy, there is a homotopy $\widetilde{H}^i\colon K \times I \to M' \subset M$ such that $H^i = q \circ \widetilde{H}^i$. Now for $i = 0, \ldots, \theta - 1$, define a homotopy $J^i\colon K \times I \to M$ by $J^i(x, t) = h(x, -\omega_i(2t - 1))$. Since $\{-1, +1\} \subset Y$, the $\widetilde{H}^i$ are essential basic homotopies and the J^i are reduced homotopies of length $\nu - 1$. We may define the required homotopy H' to be a composition of $J^0, \widetilde{H}^1, J^1, \widetilde{H}^2, \ldots, \widetilde{H}^{\theta-1}, J^{\theta-1}$. (In particular for $\theta = 1$ we have $H' = J^0$.) □

Remark 2.6 The inequality in Proposition 2.5 can presumably be shown to be an equality, but we will not need this.

3 Dual surfaces

The material in this section overlaps with material that has been presented in [4] and [3], but we have found it convenient to provide a self-contained account of it.

3.1 By a *tree* we mean a graph T such that $|T|$ is 1–connected. Since an edge in a tree is determined by its endpoints and its endpoints are always distinct, a tree has the structure of a geometric simplicial complex arising from the affine structure on the edges. If T is a tree, $E_T \subset |T|$ will denote the set of all midpoints of edges of T. For any two vertices s, s' of T there is a unique arc having s and s' as endpoints. The length of this arc will be denoted by

$d_T(s,s')$, or simply by $d(s,s')$ when it is clear which tree is involved. If we regard d_T as a distance function, the set of vertices of T becomes an integer metric space.

3.2 Suppose that Γ is a group. By a Γ*–tree* T we will mean a tree T equipped with a simplicial action of Γ. More explicitly, this means an action on the underlying space $|T|$ under which vertices are always carried to vertices, and edges are carried to edges via affine homeomorphisms. In general we will leave the action itself unnamed and implicit in the notation for a Γ–tree; the effect of an element $\gamma \in \Gamma$ on a point x of $|T|$ will ordinarily be denoted $\gamma \cdot x$. We will say that a Γ–tree T is *trivial* if for some vertex s of T we have $\Gamma \cdot s = s$.

If $\rho\colon \Gamma \to G$ is a homomorphism of groups and if T is a G–tree then we may define a simplicial action of Γ on the tree T by $\gamma \cdot x = \rho(\gamma) \cdot x$ for any point x in $|T|$. The resulting Γ–tree will be called the it pull-back of the G–tree T via ρ.

Definition 3.3 Let Γ be a group. A Γ–tree T will be termed *bipartite* if for every vertex s of T and every $\gamma \in \Gamma$, the integer $d_T(s, \gamma \cdot s)$ is even.

A Γ–tree is said to be *without inversions* if for every $\gamma \in \Gamma$ and every edge e of T such that $\gamma \cdot e = e$, the element e fixes both endpoints of e (and hence fixes e pointwise). Note that a bipartite Γ–tree is in particular a Γ–tree without inversions.

The term "bipartite" is motivated by the following result.

Proposition 3.4 *Suppose that Γ is a group and that T is a bipartite Γ–tree. Then the set of vertices of T is a disjoint union of two Γ–invariant subsets X_0 and X_1 such that each edge of T has one endpoint in X_0 and one endpoint in X_1. In particular, the quotient graph T/Γ is bipartite.*

Proof Fix a vertex $s_0 \in T$. For $i = 0, 1$, define X_i to be the set of all vertices s of T such that $d(s_0, s) \equiv i \pmod 2$. If s and s' are any two vertices of T, we have $d(s,s') = d(s_0,s) + d(s_0,s') - 2l$, where l is the length of the intersection of the arcs joining s_0 to s and to s'. In particular, $d(s,s') \equiv d(s_0,s) + d(s_0,s')$ $\pmod 2$. It follows that the distance between two vertices of X_0 or between two vertices of X_1 is even, while the distance between a vertex of X_0 and a vertex of X_1 is odd. The definition of a bipartite Γ–tree therefore implies that X_0 and X_1 are Γ–invariant. Furthermore, if two vertices s and s' are joined by an edge of T then $d(s,s')$ is the odd number 1, and hence one of the vertices s, s' must be in X_0 and the other in X_1. ☐

Definition 3.5 Let Γ be a group and let T be a Γ–tree. We shall define the *length* of an element $\gamma \in \Gamma$ relative to the Γ–tree T, denoted $\lambda_T(\gamma)$, by

$$\lambda_T(\gamma) = \min_s d_T(s, \gamma \cdot s),$$

where s ranges over the vertices of T. It is clear that conjugate elements of Γ have the same length. If $\Gamma = \pi_1(X)$ for some path-connected space X, and if c is a closed curve in X, then we will set

$$\lambda_T(c) = \lambda_T(\gamma)$$

where γ is an arbitrary element of the conjugacy class $[c]$ (see 1.18).

Definition 3.6 Let M be a 3–manifold, let $(\widetilde{M}, p)$ denote its universal covering space, and let T be a $\pi_1(M)$–tree without inversions. We shall say that a map $f\colon \widetilde{M} \to |T|$ is *equivariant* if it is $\pi_1(M)$–equivariant with respect to the action of $\pi_1(M)$ on T and some standard action (see 1.3) of $\pi_1(M)$ on $\widetilde{M}$. We shall say that f is *transverse* if it is transverse to E_T. If $f\colon \widetilde{M} \to |T|$ is a transverse equivariant map, we have $f^{-1}(E_T) = p^{-1}(S)$ for a unique properly embedded surface $S \subset M$. The surface S will be denoted by S_f.

When we are given a 3–manifold M and a $\pi_1(M)$–tree T without inversions, we define a *T–surface* in M to be a surface that has the form S_f for some transverse equivariant map $f\colon \widetilde{M} \to |T|$, where $\widetilde{M}$ denotes the universal covering space of M.

Remark 3.7 In Definition 3.6, $\pi_1(M)$ is understood to be defined in terms of an unspecified base point. It follows from the remark on change of base point in 1.3 that if x and y are points of M, if T is a $\pi_1(M, x)$–tree, and if we give T the structure of a $\pi_1(M, y)$–tree by pulling back the action of $\pi_1(M, x)$ via the isomorphism $J\colon \pi_1(M, y) \to \pi_1(M, x)$ determined by some path from y to x, then a map $f\colon \widetilde{M} \to |T|$ is equivariant (in the sense of 3.6) when we regard T as a $\pi_1(M, x)$–tree if and only if it is equivariant when we regard T as a $\pi_1(M, y)$–tree. From this it follows that the statements made in this section are independent of the choice of a base point, and in accordance with the convention described in 1.2, base points will be suppressed.

Proposition 3.8 *If M is an orientable 3–manifold and T is a $\pi_1(M)$–tree without inversions which is non-trivial (see 3.2), then any T–surface in M is non-empty.*

Proof Let $\widetilde{M}$ denote the universal cover of M. Suppose that $f\colon \widetilde{M} \to |T|$ is a transverse equivariant map such that $S_f = \emptyset$. Then f maps $\widetilde{M}$ into a component C of $|T| - E_T$. Such a component contains only one vertex, say s. Since f is equivariant, Γ must leave C invariant, and since Γ acts simplicially on T it must fix s and therefore be a trivial action. □

Remark 3.9 A non-trivial homomorphism $f\colon \pi_1(M) \to \mathbb{Z}$ determines a non-trivial $\pi_1(M)$–tree T, where $|T|$ is the real line, and the vertices of T are the integers. If $\phi \in H^1(M;\mathbb{Z})$ corresponds to f under the natural isomorphism between $\mathrm{Hom}(\pi_1(M),\mathbb{Z})$ and $H^1(M;\mathbb{Z})$, and if S is a T–surface, then S is an essential surface which represents the class that is the Poincaré–Lefschetz dual of ϕ in $H_2(M,\partial M;\mathbb{Z})$.

Proposition 3.10 *If M is an orientable 3–manifold and T is a bipartite $\pi_1(M)$–tree, then for any T–surface $S \subset M$ there are closed subsets A_0 and A_1 of M which are 3–dimensional submanifolds, such that* $A_0 \cap A_1 = \mathrm{frontier}\, A_0 = \mathrm{frontier}\, A_1 = S$.

Proof According to Proposition 3.4, the set of vertices of T is a disjoint union of two Γ–invariant subsets X_0 and X_1, such that each edge of T has one endpoint in X_0 and one endpoint in X_1. For $i = 0, 1$, let Y_i denote the union of the closures of all components of $|T| - E_T$ which contain vertices in X_i. Then the Y_i are Γ–invariant, and $Y_0 \cap Y_1 = \mathrm{frontier}\, Y_0 = \mathrm{frontier}\, Y_1 = E_T$. Now suppose that $S \subset M$ is a T–surface, so that $S = S_f$ for some transverse equivariant map $f\colon \widetilde{M} \to |T|$, where $(\widetilde{M}, p)$ denotes the universal covering space of M. Since f is $\pi_1(M)$–equivariant and transverse to E_T, the closed set $\widetilde{A}_i = f^{-1}(Y_i) \subset \widetilde{M}$ is a $\pi_1(M)$–invariant 3–dimensional submanifold, and $\widetilde{A}_0 \cap \widetilde{A}_1 = \mathrm{frontier}\, \widetilde{A}_0 = \mathrm{frontier}\, \widetilde{A}_1 = f^{-1}(E_T) = p^{-1}(S)$. Hence $\widetilde{A}_i = p^{-1}(A_i)$ for some closed set $A_i \subset M$ which is a 3–dimensional submanifold, and $A_0 \cap A_1 = \mathrm{frontier}\, A_0 = \mathrm{frontier}\, A_1 = S$. □

Proposition 3.11 *Suppose that M is a compact, orientable 3–manifold, that T is a $\pi_1(M)$–tree without inversions, and that $S \subset M$ is a T–surface. Then for any closed curve c in M we have $\lambda_T(c) \le \Delta_M(c, S)$.*

Proof Set $\Delta = \Delta_M(c, S)$ and $E = E_T$. We may assume c to be chosen within its homotopy class so that $\#(c^{-1}(S)) = \Delta$. Let $(\widetilde{M}, p)$ denote the universal covering of M. According to the definition of a T–surface, we have $S = S_f$ for some transverse equivariant map $f\colon \widetilde{M} \to |T|$; in particular, $f^{-1}(E) = p^{-1}(S)$.

We first consider the degenerate case in which $f(p^{-1}(c(S^1)))$ contains no vertex of T. In this case, f maps each component of $p^{-1}(c(S^1))$ into a single edge e of T. It then follows from equivariance that some element of $[c]$ leaves e invariant, and hence fixes the endpoints of e since T is a $\pi_1(M)$–tree without inversions. Hence we have $\lambda_T(c) = 0$ in this case, and the conclusion follows.

We may therefore assume that f maps some point $\widetilde{x} \in p^{-1}(c(S^1))$ to a vertex s of T. After reparametrizing c if necessary we may assume that $p(\widetilde{x}) = c(1)$, where 1 is the standard base point of S^1. Let $q\colon I \to S^1$ be a path representing a generator of $\pi_1(S^1, 1)$, set $\alpha = c \circ q$, choose a lift $\widetilde{\alpha}\colon I \to \widetilde{M}$ of α, and set $\beta = f \circ \alpha\colon I \to |T|$. Then $\beta(0) = s$ and $\beta(1) = \gamma \cdot s$ for some element γ of $[c]$ in $\pi_1(M)$.

Since $p^{-1}(S) = f^{-1}(E)$, we have

$$\#(\beta^{-1}(E)) = \#(\alpha^{-1}(S)) = \Delta.$$

Hence if $\mathcal{E}$ denotes the set of edges of T whose midpoints lie in $\beta(I)$, we have $\#(\mathcal{E}) \leq \Delta$. If X denotes the subgraph of T consisting of all vertices of T and of those edges that belong to $\mathcal{E}$, then β can clearly be deformed to a path in X, and hence to an arc in X. This arc has length at most Δ since X has at most Δ edges. Hence

$$d_T(s, \gamma \cdot s) = d_T(\beta(0), \beta(1)) \leq \Delta,$$

and by the definition of translation length we have $\lambda_T(c) \leq \Delta$. □

Definition 3.12 Let S be an essential surface in a compact, orientable, irreducible 3–manifold M. Let $K \subset S$ be a compact polyhedron which is π_1–injective in S. We define the *thickness* of K (relative to S) to be the supremum of all integers $\theta > 0$ for which there is a reduced homotopy $H\colon (K \times I, K \times \partial I) \to (M, S)$ of length $\theta - 1$ such that, for some $t \in I$, the map H_t is the inclusion map $K \hookrightarrow S \subset M$. The thickness of K will be denoted by $t_S(K)$, or by $t(K)$ when there is no danger of confusion. Note that $t_S(K)$ is either a strictly positive integer or $+\infty$. Moreover, if S is a semi-fiber then $t_S(S) = +\infty$, and hence $t_S(K) = +\infty$ for any compact π_1–injective polyhedron $K \subset S$.

Theorem 3.13 *Suppose that M is an irreducible knot manifold and that T is a non-trivial (see 3.2) bipartite $\pi_1(M)$–tree. Then there is an essential T–surface $S \subset M$ which has the following properties.*

(1) *For any closed curve c in ∂M we have $\lambda_T(c) = \Delta_{\partial M}(c, \partial S)$.*

(2) *If $K \subset S$ is any π_1–injective, connected, compact polyhedron such that $\chi(K) < 0$, if $t \leq t_S(K)$ is a positive integer, and if $\Theta \leq \pi_1(M)$ is the*

subgroup defined up to conjugacy by $\Theta = \mathrm{im}(\pi_1(K) \to \pi_1(M))$*, then* Θ *fixes an arc of length* t *in* T*.*

The next seven lemmas are needed for the proof of Theorem 3.13.

Lemma 3.14 *Suppose that* M *is a compact orientable* 3*–manifold and that* T *is a* $\pi_1(M)$*–tree without inversions. Let* $(\widetilde{M}, p)$ *denote the universal covering space of* M *and fix a standard action of* $\pi_1(M)$ *on* M*. Suppose that* $L \subset K \subset M$ *are compact polyhedra and that* $\widetilde{K}$ *is a union of components of* $p^{-1}(K) \subset \widetilde{M}$*. Set* $\widetilde{L} = \widetilde{K} \cap p^{-1}(L)$*. Suppose that* H *is a subgroup of* $\pi_1(M)$ *which stabilizes* $\widetilde{K}$ *and that* V *is a connected* H*–invariant subset of* $|T|$ *such that* $\mathrm{frontier}_{|T|} V$ *contains no vertices of* T*. Suppose that* $g_L\colon \widetilde{L} \to \bar{V}$ *is a PL map such that* $g_L(h \cdot x) = h \cdot g_L(x)$ *for all* $x \in \widetilde{L}$ *and* $h \in H$*. Then* g_L *may be extended to a PL map* $g_K\colon \widetilde{K} \to \bar{V}$ *such that* $g_K(\widetilde{K} - \widetilde{L}) \subset V$*, and* $g_K(h \cdot x) = h \cdot g_K(x)$ *for all* $x \in \widetilde{K}$ *and* $h \in H$*.*

Proof Fix a triangulation of K in which L is a subcomplex, and give $\widetilde{K}$ the triangulation inherited from that of K. For $i = -1, 0, 1, 2, 3$, let $\widetilde{K}^{(i)}$ denote the i–skeleton of $\widetilde{K}$ (so that $\widetilde{K}^{(-1)} = \emptyset$), and set $\widetilde{L}^{(i)} = \widetilde{K}^{(i)} \cap \widetilde{L}$. We shall recursively construct, for $i = -1, 0, 1, 2, 3$, a piecewise-linear map $g^{(i)}\colon \widetilde{K}^{(i)} \cup \widetilde{L} \to \bar{V}$ which extends g_L, maps $\widetilde{K}^{(i)} - \widetilde{L}^{(i)}$ into V, and is H–equivariant in the sense that $g_i(h \cdot x) = h \cdot g_i(x)$ for all $x \in \widetilde{K}^{(i)} \cup \widetilde{L}$ and $h \in H$. We take $g^{(-1)} = g_L$. Suppose that $g^{(i)}$ has been constructed for a given $i \leq 2$. Let $\mathcal{D}$ be a complete set of orbit representatives for the action of H on the set of $(i+1)$–simplices of $\widetilde{K}$ that are not contained in $\widetilde{L}$. For each $\delta \in \mathcal{D}$ we extend $g^{(i)}|_{\partial\delta}$ to a PL map $h_\delta\colon \bar{\delta} \to \bar{V}$; the extension exists because V, being a connected subset of the underlying space of the tree T, is contractible. Furthermore, since $\mathrm{frontier}_{|T|} V$ contains no vertices of T, there is a neighborhood N of $\mathrm{frontier}_{|T|} V$ relative to $\bar{V}$ such that N is a 1–manifold with boundary and $\mathrm{frontier}_{|T|} V \subset \partial N$. Hence by general position we may choose the extension h_δ so that it maps the open simplex δ into V.

For each point $x \in \widetilde{K}^{(i+1)} - (\widetilde{K}^{(i)} \cup \widetilde{L}^{(i+1)})$ there exist a unique $\gamma \in \pi_1(M)$ and a unique $\delta \in \mathcal{D}$ such that $\gamma \cdot x \in \delta$. We set $g^{(i+1)}(x) = \gamma^{-1} \cdot h_\delta(\gamma \cdot x)$. For $x \in \widetilde{K}^{(i)} \cup \widetilde{L}$ we set $g^{(i+1)}(x) = g^{(i)}(x)$. The extension $g^{(i+1)}$ of $g^{(i)}$ defined in this way is clearly piecewise-linear and H–equivariant. Since V is H–invariant and since $g^{(i)}(\widetilde{K}^{(i)} - \widetilde{L}^{(i)}) \subset V$, we have $g^{(i+1)}(\widetilde{K}^{(i+1)} - \widetilde{L}^{(i+1)}) \subset V$. □

Lemma 3.15 *Suppose that* M *is a compact orientable* 3*–manifold and that* T *is a* $\pi_1(M)$*–tree without inversions. Then there exists a transverse equivariant map* f *from the universal cover* $\widetilde{M}$ *of* M *to* $|T|$*.*

Proof First we fix a standard action of $\pi_1(M)$ on $\widetilde{M}$ and apply Lemma 3.14, taking $K = M$, $L = \emptyset$, $\widetilde{K} = \widetilde{M}$, $H = \pi_1(M)$, $V = |T|$, and taking $g_L = g_\emptyset$ to be the empty map. This gives a $\pi_1(M)$–equivariant PL map $g = g_M\colon\ \widetilde{M} \to |T|$.

If we subdivide the triangulation of $\widetilde{M}$, and subdivide the simplicial complex T, so that g is simplicial, then g is transverse to every non-vertex point in the subdivision of T. In particular, every (open) edge e of T contains a point to which g is transverse. Let $\mathcal{E}$ denote a complete set of orbit representatives for the action of $\pi_1(M)$ on the set of edges of T. For each $e \in \mathcal{E}$ choose a point $z_e \in e$ such that g is transverse to z_e, and set $E_0 = \{\gamma \cdot z_e\colon\ \gamma \in \pi_1(M), e \in \mathcal{E}\}$. Since T is a $\pi_1(T)$–tree without inversions, E_0 contains exactly one point in each edge of T, and there is a $\pi_1(M)$–equivariant self-homeomorphism η of $|T|$ such that $\eta(E_0) = E_T$. Then $f = \eta \circ g$ is a transverse equivariant map. □

Lemma 3.16 *Suppose that M is a compact orientable 3–manifold and that T is a $\pi_1(M)$–tree without inversions. Let $(\widetilde{M}, p)$ denote the universal cover of M, and suppose that $f\colon\ \widetilde{M} \to |T|$ is a transverse equivariant map. Suppose that S_f is the frontier of a compact 3–dimensional submanifold A of M. Suppose that $X \subset A$ is a compact connected 3–manifold with the following properties:*

- (i) *every component of* $\mathrm{frontier}_A X$ *is a properly embedded 2–manifold $C \subset A$ with $\partial C \subset \mathrm{int}\, S_f$; and*
- (ii) *for some component $\widetilde{X}$ of $p^{-1}(X)$, $f(\widetilde{X} \cap p^{-1}(S_f))$ is a single point.*

Then $\mathrm{frontier}_M \overline{X - A}$ *is a T–surface.*

Remarks 3.17 (1) Of course condition (i) in the hypothesis of Lemma 3.16 holds vacuously in the special case where X is a component of A, since then $\mathrm{frontier}_A X = \emptyset$.

(2) If f is a transverse equivariant map then it follows from the definition of S_f that $f(p^{-1}(S_f)) \subset E_T$. Thus condition (ii) in the hypothesis of Lemma 3.16 may be paraphrased by saying that $f(\widetilde{X} \cap p^{-1}(S_f))$ is a single point of E_T.

(3) Condition (ii) in the hypothesis of Lemma 3.16, together with the equivariance of f, implies that for *every* component $\widetilde{X}$ of $p^{-1}(X)$, $f(\widetilde{X} \cap p^{-1}(S_f))$ is a single point of E_T.

Proof of Lemma 3.16 We fix a standard action of $\pi_1(M)$ on $\widetilde{M}$ that makes f a $\pi_1(M)$–equivariant map. According to the hypotheses, we may choose a component $\widetilde{X}_0$ of $p^{-1}(X)$ and a point $\mu \in E_T$ (cf Remark 3.17(2)) such that $f(\widetilde{X}_0 \cap p^{-1}(S_f)) = \{\mu\}$. We denote by e the edge of T whose midpoint is μ, and by H the stabilizer of $\widetilde{X}_0$ in $\pi_1(M)$. Then H stabilizes $\widetilde{X}_0 \cap p^{-1}(S_f)$, and by the $\pi_1(M)$–equivariance of f it follows that H fixes μ. Since T is a $\pi_1(M)$–tree without inversions it follows that H fixes e.

We denote by B the closure of $M - A$ in M, and by C the closure of $A - X$ in A. We set $F = \text{frontier}_A X \subset \partial C$ and $J = X \cap S_f \subset \partial B$. We denote by Z a submanifold of C such that $Z \cap \partial C = F$ and such that the pair (Z, F) is homeomorphic to $(F \times I, F \times \{0\})$; we set $F^\sharp = (\partial Z) - F$. (Note that if X is a component of A then $F^\sharp = F = \emptyset$, cf Remark 3.17(1).) Likewise, we denote by Y a submanifold of B such that $Y \cap \partial B = J$ and such that the pair (Y, J) is homeomorphic to $(J \times I, J \times \{0\})$; we set $J^\sharp = (\partial Y) - J$. Then the 3–manifold $X^\sharp = X \cup Y \cup Z \subset M$ deform-retracts to X. Hence the component $\widetilde{X}_0^\sharp$ of $p^{-1}(X^\sharp)$ containing $\widetilde{X}_0$ is precisely invariant under H, in the sense that $\gamma \cdot \widetilde{X}_0^\sharp = \widetilde{X}_0^\sharp$ for any $\gamma \in H$, while $(\gamma \cdot \widetilde{X}_0^\sharp) \cap \widetilde{X}_0^\sharp = \emptyset$ for any $\gamma \in \pi_1(M) - H$. Note that $\text{frontier}_M X^\sharp = J^\sharp \cup F^\sharp$, and hence that $\text{frontier}_{\widetilde{M}} \widetilde{X}_0^\sharp = \widetilde{J}_0^\sharp \cup \widetilde{F}_0^\sharp$, where $\widetilde{J}_0^\sharp = p^{-1}(J^\sharp) \cap \widetilde{X}_0^\sharp$ and $\widetilde{F}_0^\sharp = p^{-1}(F^\sharp) \cap \widetilde{X}_0^\sharp$.

Let $\widetilde{A}_0$ denote the component of $p^{-1}(A)$ containing $\widetilde{X}_0$, and let V denote the component of $|T| - E_T$ containing $f(\text{int}\,\widetilde{A}_0)$. Then

$$f(\widetilde{F}_0^\sharp) \subset f(\widetilde{X}_0^\sharp \cap p^{-1}(A)) \subset f(\widetilde{A}_0) \subset \bar{V}.$$

Note that V is one of the two components of $|T| - E_T$ whose closures contain μ; we shall denote the other one by W. Since H fixes e, it leaves V and W invariant. Since f is transverse to E_T and maps $\widetilde{X}_0 \cap p^{-1}(S_f)$ to μ, every component of $p^{-1}(B)$ which meets $\widetilde{A}_0$ must be mapped into $\bar{W}$ by f. In particular we have

$$f(\widetilde{J}_0^\sharp) \subset f(\widetilde{X}_0^\sharp \cap p^{-1}(B)) \subset \bar{W}.$$

We set $\widetilde{F}_0 = p^{-1}(F) \cap \widetilde{X}_0$, and we define a map $g_0\colon \widetilde{J}_0^\sharp \cup \widetilde{F}_0^\sharp \cup \widetilde{F}_0 \to |T|$ to agree with f on $\widetilde{J}_0^\sharp \cup \widetilde{F}_0^\sharp$ and to map $\widetilde{F}_0$ to μ. Then g_0 is well-defined since $(\widetilde{J}_0^\sharp \cup \widetilde{F}_0^\sharp) \cap \widetilde{F}_0 \subset p^{-1}(S_f) \cap \widetilde{X}_0 \subset f^{-1}(\{\mu\})$, and it is H–equivariant because H fixes μ. Now set $P = X \cup Y \subset X^\sharp$, and note that $\widetilde{X}_0^\sharp$ is the union of the two H–invariant sets $\widetilde{Z}_0 = \widetilde{X}_0^\sharp \cap p^{-1}(Z)$ and $\widetilde{P}_0 = \widetilde{X}_0^\sharp \cap p^{-1}(P)$, and that $\widetilde{Z}_0 \cap \widetilde{P}_0 = \widetilde{F}_0$. It follows from Lemma 3.14 that $g_0|_{\widetilde{F}_0^\sharp \cup \widetilde{F}_0}$ may be extended to a PL H–equivariant map $g_Z\colon \widetilde{Z}_0 \to \bar{V}$ such that $g_Z(\widetilde{Z}_0 - (\widetilde{F}_0^\sharp \cup \widetilde{F}_0)) \subset V$, and that $g_0|_{\widetilde{J}_0^\sharp \cup \widetilde{F}_0}$ may be extended to a PL H–equivariant map $g_P\colon \widetilde{P}_0 \to \bar{V}$ such that $g_P(\widetilde{P}_0 - (\widetilde{J}_0^\sharp \cup \widetilde{F}_0)) \subset W$. Now define a map $g_{X^\sharp}\colon \widetilde{X}_0^\sharp \to \bar{V}$ to agree with g_Z on $\widetilde{Z}_0$ and with g_P on $\widetilde{P}_0$. Since $g_{X^\sharp}$ is H–equivariant and agrees with f on $\text{frontier}_{\widetilde{M}} \widetilde{X}_0^\sharp$, and since $\widetilde{X}_0^\sharp$ is precisely invariant under H, there is a unique $\pi_1(M)$–equivariant map $f'\colon \widetilde{M} \to |T|$ which agrees with $g_{X^\sharp}$ on $\widetilde{X}_0^\sharp$ and with f on $\widetilde{M} - \pi_1(M) \cdot \widetilde{X}_0^\sharp$. (For any $x \in \widetilde{X}_0^\sharp$ and any $\gamma \in \pi_1(M)$ we set $f'(\gamma \cdot x) = \gamma \cdot g_{X^\sharp}(x)$; the precise invariance of $\widetilde{X}_0^\sharp$ and the H–equivariance of $g_{X^\sharp}$ guarantee that f' is well-defined.)

If we set $S' = \operatorname{frontier}_M \overline{X - A} = (S_f - (S_f \cap A)) \cup F$, it follows from the construction of f' that $(f')^{-1}(E_T) = p^{-1}(S')$. The construction also shows that the restriction of f' to a small neighborhood of $\widetilde{X}_0^\sharp$ is transverse to E_T. Since f' is $\pi_1(M)$–equivariant and agrees with f outside $\pi_1(M) \cdot \widetilde{X}_0^\sharp$, it is everywhere transverse to E_T. Hence f' is a transverse equivariant map and $S_{f'} = S'$. In particular, S' is a T–surface. □

The following slight variant of Lemma 3.16 will also be useful. The proof will show that it is essentially a special case of 3.16.

Lemma 3.18 *Suppose that M is a compact orientable 3–manifold and that T is a $\pi_1(M)$–tree without inversions. Let $(\widetilde{M}, p)$ denote the universal cover of M, and suppose that $f\colon \widetilde{M} \to |T|$ is a transverse equivariant map. Suppose that S_f is the frontier of a compact 3–dimensional submanifold A of M. Suppose that $X \subset A$ is a compact connected 3–manifold with the following properties:*

- (i) *every component of* $\operatorname{frontier}_A X$ *is a properly embedded 2–manifold $C \subset A$ with $\partial C \subset \operatorname{int} S_f$;*
- (ii) *$X \cap S_f$ is connected;*
- (iii) *$\pi_1(X \cap S_f) \to \pi_1(X)$ is surjective.*

Then $\operatorname{frontier}_M \overline{X - A}$ *is a T–surface.*

Proof We will prove this by showing that the hypotheses of Lemma 3.18 imply those of Lemma 3.16. The only point to check is that condition (ii) of 3.16 follows from the hypotheses of Lemma 3.18. If $\widetilde{X}$ is any component of the covering space $p^{-1}(X)$ of X, the surjectivity of $\pi_1(X \cap S_f) \to \pi_1(X)$ implies that the induced covering space $\widetilde{X} \cap p^{-1}(S_f)$ of $X \cap S_f$ is connected. Since f maps $p^{-1}(S_f)$ into the discrete set E_T, it must map the connected subset $\widetilde{X} \cap p^{-1}(S_f)$ to a single point. □

Definition 3.19 Let M be an orientable 3–manifold and let $(\widetilde{M}, p)$ denote its universal covering. Let T be a bipartite $\pi_1(M)$–tree, and let $f\colon \widetilde{M} \to |T|$ be a transverse equivariant map. We shall say that f has a *folded boundary-annulus* if there is an annulus $R \subset \partial M$ such that $\operatorname{int} R$ is a component of $\partial M - \partial S_f$, and for some component $\widetilde{R}$ of $p^{-1}(R)$, the components of $\partial\widetilde{R} = \widetilde{R} \cap p^{-1}(E_T)$ are mapped by f to the same point of E_T. We shall say that f has a *big folded I–bundle* if there is a submanifold X of M which is an I–bundle over a compact, connected surface of negative Euler characteristic, such that (i) $Y = X \cap S_f$ is the associated ∂I–bundle of X, (ii) Y is π_1–injective in S_f, and (iii) for some component $\widetilde{X}$ of $p^{-1}(X)$, the set $\widetilde{X} \cap p^{-1}(Y) = \widetilde{X} \cap f^{-1}(E_T)$ is mapped by f to a single point of E_T.

Remark 3.20 Suppose that M is an orientable 3–manifold, whose universal cover we denote by $(\widetilde{M}, p)$. Suppose that T is a bipartite $\pi_1(M)$–tree T, and that $f\colon \widetilde{M} \to |T|$ is a transverse equivariant map. Suppose that a submanifold X of M is an I–bundle over a compact, connected surface of negative Euler characteristic, that $Y = X \cap S_f$ is the associated ∂I–bundle of X, and that Y is π_1–injective in S_f. If $\widetilde{X}$ is any component of $p^{-1}(X)$, then $\widetilde{X}$ is a covering space of X, and is therefore a connected I–bundle whose associated ∂I–bundle is $\widetilde{Y} = \widetilde{X} \cap p^{-1}(Y)$. Thus $\widetilde{Y}$ has at most two components, and since $\widetilde{Y} \subset p^{-1}(S_f)$, each component of $\widetilde{Y}$ must be mapped by f to a point of E_T. Furthermore, $\widetilde{Y}$ has exactly two components if and only if $\widetilde{X}$ is a trivial I–bundle. If we assume that the transverse equivariant map f has no big folded I–bundles, then $f(\widetilde{Y})$ cannot be a single point; hence in this case the I–bundle $\widetilde{X}$ must be trivial, and f must map the two components of $\widetilde{Y}$ to distinct points of E_T.

Lemma 3.21 *Suppose that M is an irreducible knot manifold and that T is a non-trivial bipartite $\pi_1(M)$–tree. Then there is a transverse equivariant map $f\colon \widetilde{M} \to |T|$ such that (i) S_f is essential and (ii) f has no folded boundary-annuli or big folded I–bundles.*

Proof We denote by $(\widetilde{M}, p)$ the universal cover of M and fix a standard action of $\pi_1(M)$ on $\widetilde{M}$. For any compact, orientable surface F, we set

$$\chi\_(F) = \sum_C \max(0, -\chi(C)),$$

where C ranges over the components of F. For any T–surface F, we define the *complexity* $c(F) \in \mathbb{N}^4$ to be $(b(F), \chi\_(F), t(F), s(F))$, where $b(F)$ is the number of components of ∂F, $t(F)$ is the number of components of F that are tori or annuli, and $s(F)$ is the number of closed components of F. We endow the set $\mathbb{N}^4$ with the lexicographical order. It follows from Lemma 3.15 that the set of all T–surfaces in M is non-empty. Hence there is a T–surface

S which has minimal complexity among all T–surfaces. By the definition of a T–surface, we have $S = S_f$ for some transverse equivariant map $f\colon \widetilde{M} \to |T|$. We shall prove Lemma 3.21 by showing that S is essential and that f has no folded boundary-annuli or big folded I–bundles.

Since T is a bipartite $\pi_1(M)$–tree, Proposition 3.10 asserts that there are compact 3–dimensional submanifolds A_0 and A_1 of M such that $A_0 \cap A_1 =$ frontier $A_0 =$ frontier $A_1 = S$.

To show that S is essential, we first observe that since the $\pi_1(M)$–tree T is by hypothesis non-trivial, we have $S \neq \emptyset$ according to Proposition 3.8. Next, we shall show that S is π_1–injective. Assume it is not. Then by a standard consequence of the loop theorem, there is a disk $D \subset M$ such that $D \cap S = \partial D$, and ∂D does not bound a disk in S. Let $X \subset M$ be a ball such that $X \cap S \subset \partial X$, and $R = X \cap S$ is a regular neighborhood of ∂D in S. Then $\partial X - \operatorname{int} R$ is a disjoint union of two disks D_1 and D_2. We must have $X \subset A_j$ for some $j \in \{0, 1\}$. The hypotheses of Lemma 3.18 clearly hold with this choice of X, and with $A = A_j$. Hence 3.18 implies that the surgered surface $S' = (S - R) \cup D_1 \cup D_2$ is a T–surface. We shall reach a contradiction by showing that $c(S') < c(S)$. Note that $b(S') = b(S)$.

Let S_0 denote the component of S containing R. Then $S_0' = (S - R) \cup D_1 \cup D_2$ has either one or two components, and $\chi(S_0') = \chi(S_0) + 2$. We first consider the case in which $\chi(S_0) < 0$. In this case, at most one component of S_0' can be a disk; and since the core curve ∂D of R does not bound a disk in S_0, no component of S_0' can be a sphere. It follows that

$$\chi\_(S_0') \leq -\chi(S_0') + 1 = -\chi(S_0) - 1 = \chi\_(S_0) - 1.$$

Since $\chi\_(S_0') < \chi\_(S_0)$, it is clear that $\chi\_(S') < \chi\_(S)$. Hence $c(S) < c(S')$ in this case. There remains the case in which $\chi(S_0) \geq 0$. Since the core curve of R is homotopically non-trivial in S, the only possibilities are that S_0 is a torus and S_0' is a sphere, or that S_0 is an annulus and S_0' consists of two disks. In both subcases we have $\chi\_(S_0') = \chi\_(S_0) = 0$, so that $\chi\_(S') = \chi\_(S)$, whereas $t(S') < t(S)$. Hence $c(S') < c(S)$, and the proof of π_1–injectivity is complete.

Next we show that no component of S is a 2–sphere. If S does have a 2–sphere component then by irreducibility, any 2–sphere component of S must bound a ball X, and since we have shown that S is π_1–injective, any component of S contained in X must itself be a sphere. Hence if we take X to be minimal with respect to inclusion among all balls in M bounded by components of X, then $X \cap S = \partial X$. We must have $X \subset A_j$ for some $j \in \{0, 1\}$. The hypotheses of Lemma 3.18 clearly hold with this choice of X, and with $A = A_j$. (See Remark

3.17(1).) Hence if S_1 denotes the sphere ∂X, 3.18 implies that $S' = S - S_1$ is a T–surface. But we obviously have $b(S') = b(S)$, $\chi\_(S') = \chi\_(S)$, $t(S') = t(S)$ and $s(S') = s(S) - 1$. Thus $c(S') < c(S)$, a contradiction. This shows that no component of S is a 2–sphere.

To prove that S is essential, it remains to show that S has no boundary-parallel component. If S does have a boundary-parallel component S_2, then S_2 has a *region of boundary-parallelism*, ie, a submanifold X of M such that frontier $X = S_2$ and (X, S_2) is homeomorphic to $(S_2 \times I, S_2 \times \{1\})$. Since we have shown that S is π_1–injective and has no sphere components, it follows from [13] that any component of S contained in X must itself have a region of boundary-parallelism which is contained in X. Hence if we choose S_2 so that X to be minimal with respect to inclusion among all regions of boundary-parallelism for components of S, then $X \cap S = S_2$. We must have $X \subset A_j$ for some $j \in \{0, 1\}$. The hypotheses of Lemma 3.18 clearly hold with this choice of X, and with $A = A_j$. (See Remark 3.17(1).) Hence 3.18 implies that $S' = S - S_2$ is a T–surface. If $\partial S_2 \neq \emptyset$ then $b(S') < b(S)$. If $\partial S_2 = \emptyset$ then $b(S') = b(S)$, $\chi\_(S') \leq \chi\_(S)$, $t(S') \leq t(S)$ and $s(S') = s(S) - 1$. In either case we conclude that $c(S') < c(S)$, a contradiction. This completes the proof that S is essential.

We now turn to the proof that f has no folded boundary-annuli or big folded I–bundles. First suppose that f has a folded boundary-annulus; that is, there is an annulus $R \subset \partial M$ such that $\operatorname{int} R$ is a component of $\partial M - \partial S$, and for some component $\widetilde{R}$ of $p^{-1}(R)$, the components of $\partial\widetilde{R} = \widetilde{R} \cap p^{-1}(E_T)$ are mapped by f to the same point of E_T. We must have $R \subset A_j$ for some $j \in \{0, 1\}$. Let X be a regular neighborhood of R in A_j such that $X \cap S$ is a regular neighborhood of ∂R in S. Thus $X \cap S$ consists of two disjoint annuli R_1 and R_2, while $\operatorname{frontier}_A X$ is an annulus R_3. The hypotheses of Lemma 3.16 clearly hold with this choice of X, and with $A = A_j$. Hence 3.16 implies that the "tubed" surface $S' = (S - (R_1 \cup R_2)) \cup R_3$ is a T–surface. But we have $b(S') < b(S)$ and hence $c(S') < c(S)$, a contradiction.

Finally, suppose that f has a big folded I–bundle; that is, there is a submanifold X of M which is an I–bundle over a compact, connected surface of negative Euler characteristic, such that (i) $Y = X \cap S$ is the associated ∂I–bundle of X, (ii) Y is π_1–injective in S, and (iii) for some component $\widetilde{X}$ of $p^{-1}(X)$, the components of $\widetilde{X} \cap p^{-1}(Y) = \widetilde{X} \cap p^{-1}(E_T)$ are mapped by f to the same point of E_T. We may choose X so that $Y \subset \operatorname{int} S$. The surface $\mathcal{R} = \partial X - \operatorname{int} Y$ is a (possibly empty) disjoint union of annuli. We must have $X \subset A_j$ for some $j \in \{0, 1\}$. The hypotheses of Lemma 3.16 clearly hold with this choice of X, and with $A = A_j$. Hence 3.16 implies that $S' = (S - Y) \cup \mathcal{R}$ is a T–surface.

Since the base of the I–bundle X has negative Euler characteristic, so does every component of Y. Let S_3 denote the union of all components of S that meet Y. (There are at most two such components.) The π_1–injectivity of Y in S implies that each component of S_3 has negative Euler characteristic, so that $\chi\_(S_3) = -\chi(S_3)$. Set $S_3' = (S_3 - Y) \cup \mathcal{R}$. Then each component of S_3' contains a component of ∂Y. (This is vacuously true if Y is closed, since $S_3' = \emptyset$ in that case.) Since Y is π_1–injective in the essential surface S, each component of ∂Y is homotopically non-trivial in M and hence in S_3'. This shows that each component of S_3' is non-simply-connected and hence has non-positive Euler characteristic. Thus

$$\chi\_(S_3') = -\chi(S_3') = -(\chi(S_3) - \chi(Y) + \chi(\mathcal{R})) < -\chi(S_3),$$

since $\chi(\mathcal{R}) = 0$ and $\chi(Y) < 0$. Hence $\chi\_(S_3') < \chi\_(S_3)$, which implies $\chi\_(S') < \chi\_(S)$. As it is clear that $b(S') = b(S)$, it follows that $c(S') < c(S)$, and again we have a contradiction. □

Lemma 3.22 *Suppose that M is an irreducible knot manifold, and that T is a bipartite $\pi_1(M)$–tree. Let $(\widetilde{M}, p)$ denote the universal covering of M and suppose that $f\colon \widetilde{M} \to T$ is a transverse equivariant map which has no folded boundary-annuli. Then for any closed curve c in ∂M we have $\lambda_T(c) = \Delta_{\partial M}(c, \partial S_f)$.*

Proof We fix a standard action of $\pi_1(M)$ on $\widetilde{M}$ that makes f a $\pi_1(M)$–equivariant map.

Since T is a bipartite $\pi_1(M)$–tree, Proposition 3.10 asserts that there are compact 3–dimensional submanifolds A_0 and A_1 of M such that $A_0 \cap A_1 = \text{frontier}\, A_0 = \text{frontier}\, A_1 = S_f$.

Set $\Delta = \Delta_{\partial M}(c, \partial S_f)$. According to Proposition 3.11 we have

$$\lambda_T(c) \le \Delta_M(c, S_f) \le \Delta.$$

It remains to show $\lambda_T(c) \ge \Delta$. We may therefore assume that $\Delta > 0$. In particular, $\partial S_f \neq \emptyset$ and c is homotopically non-trivial. Since $\partial S_f \neq \emptyset$, all the components of $A_0 \cap \partial M$ and $A_1 \cap \partial M$ are annuli. We shall refer to these as *complementary annuli.* We may suppose c to be chosen within its homotopy class so that it is transverse to ∂S_f and $\#(c^{-1}(\partial S_f)) = \Delta$. This implies that for each component α of $c^{-1}(A_0)$ or $c^{-1}(A_1)$, $c|_\alpha$ is a map of the arc α into a complementary annulus R which takes the endpoints of α to different components of ∂R.

After reparametrization we may assume that

$$c^{-1}(\partial S_f) = \{\exp(2k\pi\sqrt{-1}/\Delta) : 0 \le k < \Delta\} \subset S^1.$$

Let $q\colon \mathbb{R} \to S^1$ be the covering map defined by $q(t) = \exp\left(2\pi\left(t - \frac{1}{2}\right)\sqrt{-1}/\Delta\right)$, set $\ell = c \circ q\colon \mathbb{R} \to \partial M$ and let $\widetilde{\ell}\colon \mathbb{R} \to \widetilde{M}$ denote a lift of ℓ. Then for every $t \in \mathbb{R}$ we have $\widetilde{\ell}(t + \Delta) = \gamma \cdot \widetilde{\ell}(t))$, where γ is an element of $[c]$.

Our parametrization of c guarantees that $\widetilde{\ell}^{-1}(p^{-1}(\partial S_f)) = \mathbb{Z} + \frac{1}{2}$. For each $n \in \mathbb{Z}$ we let $\widetilde{S}_n$ denote the component of $p^{-1}(S_f)$ containing $\widetilde{\ell}\left(n + \frac{1}{2}\right)$, and we let $\widetilde{A}_n$ denote the component of $p^{-1}(A_0)$ or $p^{-1}(A_1)$ containing $\widetilde{\ell}\left(\left[n - \frac{1}{2}, n + \frac{1}{2}\right]\right)$. We denote by $\mu_n \in E_T$ the point $f(\widetilde{S}_n)$, by e_n the edge of T whose midpoint is μ_n, and by s_n the unique vertex of T in the component of $|T| - E_T$ containing $f(\widetilde{A}_n)$. The transversality of f to E_T implies that the endpoints of e_n are s_n and s_{n+1} for every $n \in \mathbb{Z}$.

For every $n \in \mathbb{Z}$ we have

$$\mu_{n+\Delta} = f\left(\widetilde{\ell}\left(n + \frac{1}{2} + \Delta\right)\right) = f\left(\gamma \cdot \widetilde{\ell}\left(n + \frac{1}{2}\right)\right) = \gamma \cdot f\left(\widetilde{\ell}\left(n + \frac{1}{2}\right)\right) = \gamma \cdot \mu_n.$$

Hence $e_{n+\Delta} = \gamma \cdot e_n$ for every n.

For each n the interval $\left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ is mapped by ℓ into a complementary annulus R_n, and $\ell\left(n - \frac{1}{2}\right)$ and $\ell\left(n + \frac{1}{2}\right)$ lie in different components of ∂R_n. Hence $\widetilde{\ell}$ maps $\left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ into a component $\widetilde{R}_n \subset \widetilde{A}_n$ of $p^{-1}(R_n)$, and the points $\ell\left(n - \frac{1}{2}\right)$ and $\ell\left(n + \frac{1}{2}\right)$ lie in different components $\widetilde{C}_{n-1} \subset \partial\widetilde{S}_{n-1}$ and $\widetilde{C}_n \subset \partial\widetilde{S}_n$ of $\partial\widetilde{R}_n$. But $\partial\widetilde{R}_n$ must have exactly two components, and the hypothesis that f has no folded boundary-annuli implies that f maps these components to different points of E_T. Hence $\mu_{n-1} \ne \mu_n$, which implies that $e_{n-1} \ne e_n$ for every $n \in \mathbb{Z}$. In particular s_n is the unique common vertex of e_{n-1} and e_n. Since $e_{n+\Delta} = \gamma \cdot e_n$ for every n it now follows that $s_{n+\Delta} = \gamma \cdot s_n$ for every n.

Since T is a tree, and since $e_{n-1} \ne e_n$ for every n, the e_n and s_n make up a subgraph $\mathcal{A}$ of T isomorphic to the real line, triangulated with a vertex at every integer point. In particular, for all $m, n \in \mathbb{Z}$, we have $d_T(s_m, s_n) = |m - n|$, and the arc joining s_m and s_n is contained in $\mathcal{A}$. Hence for any n we have $d_T(s_n, \gamma \cdot s_n) = d_T(s_n, s_{n+\Delta}) = \Delta$. Now consider an arbitrary vertex s of T, let s_n be a vertex of $\mathcal{A}$ for which $d_T(s, s_n)$ is as small as possible, let β denote the arc with endpoints s and s_n, and let $\alpha \subset \mathcal{A}$ denote the arc with endpoints s_n and $\gamma \cdot s_n = s_{n+\Delta}$. Then $\beta \cap \mathcal{A} = \{s_n\}$ and hence $(\gamma \cdot \beta) \cap \mathcal{A} = \gamma \cdot (\beta \cap \mathcal{A}) = \{s_{n+\Delta}\}$. In particular, $\beta \cap \alpha = \{s_n\}$ and $(\gamma \cdot \beta) \cap \alpha = \{s_{n+\Delta}\}$. Since T is a tree it follows that $d(s, \gamma \cdot s) = \Delta + 2d_T(s, s_n) \geq \Delta$. This proves that $\lambda_T(c) \geq \Delta$, as required. □

Lemma 3.23 *Suppose that M is an irreducible knot manifold, and that T is a bipartite $\pi_1(M)$–tree. Let $(\widetilde{M}, p)$ denote the universal covering of M, and suppose that $f\colon \widetilde{M} \to T$ is a transverse equivariant map such that S_f is essential and f has no big folded I–bundle. If $K \subset S_f$ is any π_1–injective, connected, compact polyhedron such that $\chi(K) < 0$, if $\theta \leq t_{S_f}(K)$ is a positive integer (cf 3.12), and if $\Theta \leq \pi_1(M)$ is the subgroup defined up to conjugacy by $\Theta = \operatorname{im}(\pi_1(K) \to \pi_1(M))$, then Θ fixes an arc of length θ in T.*

Proof We choose a base point $\star$ in $K \subset M$ and a standard action of $\pi_1(M) = \pi_1(M, \star)$ on $\widetilde{M}$ that makes f a $\pi_1(M)$–equivariant map.

According to 3.10, there are closed subsets A_0 and A_1 of M which are 3–dimensional submanifolds, such that $A_0 \cap A_1 = \operatorname{frontier} A_0 = \operatorname{frontier} A_1 = S_f$. Since $\theta \leq t_{S_f}(K)$, there is a reduced homotopy $H\colon (K \times I, K \times \partial I) \to (M, S_f)$ of length $\theta - 1$ such that, for some $t \in I$, the map H_t is the inclusion map $K \hookrightarrow S_f \subset M$. By definition H is a composition of essential basic homotopies $H^1, \ldots, H^{\theta-1}$, and by symmetry we may assume that $H^i(K \times I) \subset A_{[i]}$, where $[i]$ denotes the least residue of i modulo 2. Thus there are points $0 = t_0 < t_1 < \cdots < t_{\theta-1} = 1$ of I such that $H|_{K \times [t_{i-1}, t_i]}$ is a reparametrization of H^i for $i = 1, \ldots, \theta - 1$, and there is some $m \in \{0, \ldots, \theta - 1\}$ such that H_{t_m} is the inclusion $K \hookrightarrow M$. We let $\xi\colon \pi_1(K, \star) \to \pi_1(M, \star)$ denote the inclusion homomorphism.

Let $(\widetilde{K}, q)$ denote the universal covering of K, set $h = H \circ (q \times \mathrm{id})\colon \widetilde{K} \times I \to M$, and choose a lift $\widetilde{h}\colon \widetilde{K} \times I \to \widetilde{M}$ of h. Note that with respect to our chosen standard action on $\widetilde{M}$ and some standard action of $\pi_1(K)$ on $\widetilde{K}$, the map $\widetilde{h}$ is ξ–equivariant in the sense that for every $(\widetilde{z}, t) \in \widetilde{K} \times I$ and every $\gamma \in \pi_1(K)$ we have $\widetilde{h}(\gamma \cdot \widetilde{z}, t) = \xi(\gamma) \cdot h(\widetilde{z}, t)$.

For $i = 0, \ldots, \theta - 1$, let $\widetilde{S}_i$ denote the component of $p^{-1}(S_f)$ containing $\widetilde{h}(\widetilde{K} \times \{t_i\})$. Then $f(\widetilde{S}_i)$ is a point of E_T, which means that it is the midpoint of a well-defined edge e_{i+1} of T.

For $i = 0, \ldots, \theta - 1$, since $\widetilde{h}$ maps $\widetilde{K} \times \{t_i\}$ into the component $\widetilde{S}_i$ of the $\pi_1(M)$–invariant set $p^{-1}(S_f) \subset \widetilde{M}$, the ξ–equivariance of $\widetilde{h}$ implies that $\widetilde{S}_i$ is invariant under the subgroup $\Theta = \xi(\pi_1(K))$ of $\pi_1(M)$. The $\pi_1(M)$–equivariance of f then implies that the midpoint of the edge e_{i+1} is fixed by Θ for $i = 0, \ldots, \theta - 1$. Since the bipartite $\pi_1(M)$–tree T is in particular a $\pi_1(M)$–tree without inversions (see 3.3), the edge e_{i+1} is itself fixed by Θ for $i = 0, \ldots, \theta - 1$.

For $i = 1, \ldots, \theta - 1$, let $\widetilde{A}_i$ denote the component of $p^{-1}(A_{[i]})$ containing $\widetilde{h}(\widetilde{K} \times [t_{i-1}, t_i])$. Then $f(\widetilde{A}_i)$ is contained in the closure of a unique component of $T - E_T$. This component contains a unique vertex of T which will be denoted s_i. It is clear that s_i is a common endpoint of e_{i-1} and e_i for $i = 1, \ldots, \theta - 1$. We denote by s_0 the vertex of e_1 which is distinct from s_1, and by s_θ the vertex of e_θ which is distinct from $s_{\theta-1}$. We denote by ω_i the orientation of e_i such that $\mathrm{init}(\omega_i) = s_i$ and $\mathrm{term}(\omega_i) = s_{i+1}$. Then $(\omega_1, \ldots, \omega_{\theta-1})$ is an edge path in T. We claim that this edge path is reduced. This amounts to showing that e_i and e_{i-1} are distinct for any i with $1 \leq i \leq \theta$.

To prove this, we note that since H^i is an essential homotopy in $A_{[i]}$, it follows from [8, "Essential Homotopy Theorem" Chapter III Section 2] that $H^i\colon (K \times I, K \times \partial I) \to (A_{[i]}, S_f)$ is homotopic as a map of pairs to a map J^i such that $J^i(K \times I) \subset X_i$ and $J^i(K \times \partial I) \subset Y_i$, where X_i is a submanifold of $A_{[i]}$, Y_i is a submanifold of $X_i \cap S_f$ which is π_1–injective in S_f, and either (i) $X_i \subset A_{[i]}$ is an I–bundle over a surface and $Y_i \subset S_f$ is the associated ∂I–bundle, or (ii) X_i is a Seifert fibered space and Y_i is a saturated subsurface of ∂X_i. On the other hand, since $H_0\colon K \to M$ is homotopic in M to the inclusion $K \hookrightarrow S_f \subset M$, and since K is π_1–injective in the essential surface S_f and $\chi(K) < 0$, the subgroup $(J^i_0)_\sharp(\pi_1(K)) = (H^i_0)_\sharp(\pi_1(K))$, defined up to conjugacy in $\pi_1(M)$, is non-abelian. Hence Y_i has a component with non-abelian fundamental group. This rules out (ii), and shows that the base of the I–bundle given by (i) must have negative Euler characteristic. Note that since H^i is an essential homotopy, the homotopy $J^i\colon (K \times I, K \times \partial I) \to (X_i, Y_i)$ is also essential.

Since $H|_{K \times [t_{i-1}, t_i]}$ is a reparametrization of the homotopy H_i, it follows that the map $h|_{K \times [t_{i-1}, t_i]}$ is a reparametrization of $h^i = H^i \circ (q \times \mathrm{id})\colon \widetilde{K} \times I \to M$. Hence $\widetilde{h}|_{\widetilde{K} \times [t_{i-1}, t_i]}$ is a reparametrization of a lift $\widetilde{h}^i\colon \widetilde{K} \times I \to \widetilde{M}$ of h^i. Thus $\widetilde{h}^i(\widetilde{K} \times I) \subset \widetilde{A}_i$, $\widetilde{h}^i(\widetilde{K} \times \{0\}) \subset \widetilde{S}_{i-1}$, and $\widetilde{h}^i(\widetilde{K} \times \{1\}) \subset \widetilde{S}_i$. Since $H^i\colon (K \times I, K \times \partial I) \to (A_{[i]}, S_f)$ is homotopic to J^i as a map of pairs, the covering homotopy property of covering spaces implies that $\widetilde{h}^i\colon (\widetilde{K} \times I, \widetilde{K} \times \partial I) \to (\widetilde{A}_i, \widetilde{A}_i \cap p^{-1}(S_f))$ is homotopic as a map of pairs to some lift $\widetilde{j}^i$ of $j^i = J^i \circ (q \times \mathrm{id})$. In particular it follows that $\widetilde{j}^i(\widetilde{K} \times \{0\}) \subset \widetilde{S}_{i-1}$ and that $\widetilde{j}^i(\widetilde{K} \times \{1\}) \subset \widetilde{S}_i$.

Let $\widetilde{X}_i$ denote the component of $p^{-1}(X_i)$ containing $\widetilde{j}^i(\widetilde{K} \times I)$, and set $\widetilde{Y}_i = \widetilde{X}_i \cap p^{-1}(Y_i)$, so that $\widetilde{j}^i(\widetilde{K} \times I) \subset \widetilde{Y}_i$. Since f has no big folded I–bundles, it follows from Remark 3.20 that $\widetilde{X}_i$ is a trivial I–bundle with associated ∂I–bundle $\widetilde{Y}_i$, and that f maps the two components of $\widetilde{Y}_i$ to distinct points of E_T.

Now consider any point $\widetilde{z} \in \widetilde{K}$, and set $z = q(\widetilde{z}) \in K$. Let $w_0^i = \widetilde{j}^i(\widetilde{z}, 0)$ and $w_1^i = \widetilde{j}^i(\widetilde{z}, 1)$. If w_0^i and w_1^i lie in the same component of $\widetilde{Y}_i$, then the path $t \mapsto \widetilde{j}^i(\widetilde{z}, t)$ in the trivial I–bundle $\widetilde{X}_i$ is fixed-endpoint homotopic to a path in $\widetilde{Y}_i$. This implies that the path $t \mapsto J^i(z, t)$ is fixed-endpoint homotopic in X_i to a path in Y_i, which is impossible since $J^i\colon (K \times I, K \times \partial I) \to (X_i, Y_i)$ is an essential homotopy. Hence w_0^i and w_1^i lie in different components of $\widetilde{Y}_i$, and hence $f(w_0^i)$ and $f(w_1^i)$ are distinct points of E_T. On the other hand, we have $w_0^i \in \widetilde{j}^i(\widetilde{K} \times \{0\}) \subset \widetilde{S}_{i-1}$ and $w_1^i \in \widetilde{j}^i(\widetilde{K} \times \{1\}) \subset \widetilde{S}_i$, which implies that $f(w_0^i)$ and $f(w_1^i)$ are the midpoints of e_{i-1} and e_i respectively. This shows that e_{i-1} and e_i are distinct edges of T, and establishes the claim that the edge path $(\omega_1, \ldots, \omega_{\theta-1})$ is reduced.

Since T is a tree, it now follows that the edges $e_1, \ldots, e_\theta$ form an arc of length θ in T. As we have seen that Θ fixes $e_1, \ldots, e_\theta$, we have now produced the required arc of length θ fixed by Θ. □

Proof of Theorem 3.13 The theorem is an immediate consequence of Lemmas 3.21, 3.22, and 3.23. □

4 The tree for GL_2

In this section we record a few facts about the tree for GL_2 over a discretely valued field. Our point of view is close to that of Serre [11], except that to be consistent with the conventions of 1.19 and 3.1 we take the tree to be a 1–connected geometric simplicial 1–complex which realizes the abstract combinatorial structure considered by Serre. We begin by summarizing some results from [11], translated into our geometric setting.

4.1 Suppose that F is a field with a discrete rank–1 valuation v. We always denote the valuation ring associated to v by $\mathcal{O}_v$; it consists of all elements $x \in F$ with $v(x) \geq 0$, where by convention $v(0) = +\infty$. A *lattice* in the 2–dimensional vector space F^2 is a rank–2 $\mathcal{O}$–submodule of F^2.

There is a $\mathrm{GL}_2(F)$–tree, in the sense of Section 3, canonically associated to the valued field F. We shall always denote this tree by T_F, leaving the valuation v implicit in the notation. The vertices of T_F are in bijective correspondence with homothety classes of lattices in F^2. If L is a lattice representing a vertex s of T_F, and if $\pi \in \mathcal{O}$ is a uniformizer (ie, an element such that $v(\pi) = 1$), then any vertex s' can be represented by a lattice $L' \subset L$ which is generated by

e and $\pi^d f$ for some integer $d \geq 0$ and some basis $\{e, f\}$ of L. The integer d, which is uniquely determined by the vertices s and s', is equal to the distance $d_{T_F}(s, s')$; this fact completely characterizes the tree T_F, since two vertices s, s' are joined by an edge if and only if $d_{T_F}(s, s') = 1$. The action of $\mathrm{GL}_2(F)$ on T_F, which is transitive on the vertices, is characterized by the fact that an element $A \in \mathrm{GL}_2(F)$ carries the vertex represented by a lattice L to the vertex represented by $A(L)$.

Proposition 4.2 *If F is a field with a discrete rank–1 valuation v, the $\mathrm{SL}_2(F)$–tree T_F is bipartite. Furthermore, if an element A of $\mathrm{SL}_2(F)$ fixes a vertex of T_F represented by a lattice L, then $A(L) = L$.*

Proof Let s be any vertex of T_F, and L a lattice representing s. Let $A \in \mathrm{SL}_2(F)$ be given, and set $s' = A \cdot s$ and $d = d_{T_F}(s, s')$. Then s' is represented by a lattice $L' \subset L$ which is generated by e and $\pi^d f$ for some basis $\{e, f\}$ of L. As $A(L)$ also represents s', we must have $L' = \pi^k A(L)$ for some $k \in \mathbb{Z}$. Hence if B is the element of $\mathrm{GL}_2(F)$ defined by $B(e) = \pi^{-k} e$ and $B(f) = \pi^{d-k}(f)$, we have $B(L) = A(L)$. Thus $A^{-1}B$ leaves L invariant, and $\det(A^{-1}B) = \det B$ must be a unit in $\mathcal{O}$, ie, $v(\det B) = 0$. But B is conjugate in $\mathrm{GL}_2(F)$ to

$$\begin{pmatrix} \pi^{-k} & 0 \\ 0 & \pi^{d-k} \end{pmatrix},$$

so that $v(\det B) = d - 2k$. Hence $d = 2k$. In particular d is always even, so that T_F is a bipartite $\mathrm{SL}_2(F)$–tree.

Now if $A \cdot s = s$, so that $d = 0$, then $k = 0$ and hence $A(L) = L' = L$. □

Proposition 4.3 *Suppose that F is a field with a discrete rank–1 valuation v. Then for every $A \in \mathrm{SL}_2(F)$ we have*

$$\lambda_{T_F}(A) = 2\max(0, -v(\mathrm{trace}(A))).$$

Proof We set $\mathcal{O} = \mathcal{O}_v$, and we denote by s_0 the homothety class of the standard lattice $\mathcal{O}^2 \subset F^2$. We set $v_0 = \max(0, -v(\mathrm{trace}(A)))$. The proposition asserts that $\lambda_{T_F}(A) = 2v_0$. We first show that $\lambda_{T_F}(A) \geq 2v_0$; for this step we may assume without loss of generality that $v_0 > 0$. We need to prove that for any vertex s of T_F we have $d_{T_F}(s, A \cdot s) \geq 2v_0$. Since $\mathrm{GL}_2(F)$ acts transitively on the vertices of T_F, and since the length function λ_{T_F} is constant on $\mathrm{GL}_2(F)$–conjugacy classes, it suffices to show that for any conjugate B of A in $\mathrm{GL}_2(F)$ we have $d_{T_F}(s_0, B \cdot s_0) \geq 2v_0$.

Since a rank–1 discrete valuation ring is Euclidean, we may reduce the matrix B to a diagonal matrix D using row and column operations over $\mathcal{O}$: a column operation over $\mathcal{O}$ has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b+\alpha a \\ c & d+\alpha c \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a+\alpha b & b \\ c+\alpha d & d \end{pmatrix}$$

for some $\alpha \in \mathcal{O}$, and row operations over $\mathcal{O}$ are defined similarly. In particular we have $D = XBY$ for some $X, Y \in \mathrm{SL}_2(\mathcal{O})$, and hence $D \in \mathrm{SL}_2(F)$. Thus

$$D = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix}$$

for some $\delta \in F$.

Let us define the *height* of an arbitrary matrix $M \in \mathrm{SL}_2(F)$ to be the minimum of $v(\alpha)$, where α ranges over the entries of M. Note that $v(\operatorname{trace} B) = v(\operatorname{trace} A) = -v_0$ (since $v_0 > 0$), and hence $v(\beta) \leq -v_0$ for at least one diagonal entry β of B. Hence B has height at most $-v_0$. It is clear that a row or column operation defined over $\mathcal{O}$ does not affect the height of a matrix, and hence $\operatorname{height} D = \operatorname{height} B \leq -v_0$. This means that either $v(\delta) \leq -v_0$ or that $v(\delta^{-1}) = -v(\delta) \leq -v_0$, so in either case $|v(\delta)| \geq v_0 > 0$. Now $B(\mathcal{O}^2) = X^{-1}DY^{-1}(\mathcal{O}^2) = X^{-1}D(\mathcal{O}^2) = X^{-1}(\delta\mathcal{O} \oplus \delta^{-1}\mathcal{O}) = \delta\mathcal{O}e + \delta^{-1}\mathcal{O}f$, where $\{e, f\}$ is the image of the standard basis for $\mathcal{O}^2$ under X^{-1}, and is itself a basis of $\mathcal{O}^2$ since $X^{-1} \in \mathrm{SL}_2(\mathcal{O})$. We define a lattice $L \subset \mathcal{O}^2$ by $L = \delta^2\mathcal{O}e + \mathcal{O}f$ if $v(\delta) > 0$ and by $L = \{\mathcal{O}e + \delta^{-2}\mathcal{O}f\}$ if $v(\delta) < 0$. In either case L is homothetic to $B(\mathcal{O}^2)$ and hence represents the vertex $B \cdot s_0$, and the definition of distance in T_F implies that $d_{T_F}(s_0, B \cdot s_0) = |v(\delta^2)| = 2|v(\delta)| \geq 2v_0$, as required.

It remains to show that $\lambda_{T_F}(A) \leq 2v_0$. This is trivial if $A = \pm Id$. If $A \neq \pm Id$, we may assume after a conjugation in $\mathrm{GL}_2(F)$ that

$$A = \begin{pmatrix} 0 & -1 \\ 1 & \tau \end{pmatrix}$$

for some $\tau \in F$, and it is apparent that $\tau = \operatorname{trace} A$. If $v(\tau) \geq 0$ then $A \in \mathrm{SL}_2(F)$; hence A fixes s_0, so that $\lambda_{T_F}(A) = 0 \leq 2v_0$. Now suppose that $v(\tau) < 0$. If s_1 denotes the homothety class of the lattice L_1 generated by $(1, 0)$ and $(0, \tau^{-1})$, then $A \cdot s_1$ is represented by the lattice $\tau^{-1}A(L_1) \subset L_1$, which is generated by $(\tau^{-2}, 0)$ and $(0, \tau^{-1})$. The definition of distance in T_F implies that $d_{T_F}(s_1, A \cdot s_1) = v(\tau^{-2}) = -2v(\tau) = 2v_0$. Hence $\lambda_{T_F}(A) \leq 2v_0$. □

Proposition 4.4 *Suppose that F is a field with a discrete rank–1 valuation v. Suppose that J is a subgroup of $\mathrm{SL}_2(F)$ which fixes an arc of length t in T_F.*

Then for every element A of the commutator subgroup $[J,J] \leq J \leq \mathrm{SL}_2(F)$, we have

$$v((\mathrm{trace}(A)) - 2) \geq t.$$

Proof We set $\mathcal{O} = \mathcal{O}_v$. The hypothesis implies that there are vertices s and s' of T_F such that $d_{T_F}(s,s') = t$, $H \cdot s = s$ and $H \cdot s' = s'$. Since $d_{T_F}(s,s') = t$, the vertices s and s' are represented by lattices L and L' which respectively have bases of the forms $\{e,f\}$ and $\{e, \pi^t f\}$. After conjugating by an element of $\mathrm{GL}_2(F)$ we may assume that $\{e,f\}$ is the standard basis for F^2. This implies that $L = \mathcal{O}^2$ and that $L' = C(\mathcal{O}^2)$, where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & \pi^t \end{pmatrix} \in \mathrm{GL}_2(F).$$

Hence $s' = C \cdot s$. It follows that the subgroups J and $C^{-1}JC$ of $\mathrm{SL}_2(F)$ both fix the vertex s. Hence by the second assertion of Proposition 4.2, they are both contained in $\mathrm{SL}_2(\mathcal{O})$.

For any
$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in J,$$

we have
$$C^{-1}XC = \begin{pmatrix} a & \pi^t b \\ \pi^{-t} c & d \end{pmatrix}.$$

As X and $C^{-1}XC$ both belong to $\mathrm{SL}_2(\mathcal{O})$, we have $a,b,d \in \mathcal{O}$ and $c \in \pi^t\mathcal{O}$. It follows that the natural homomorphism $\eta\colon \mathrm{SL}_2(\mathcal{O})$ to $\mathrm{SL}_2(\mathcal{O}/\pi^t\mathcal{O})$ maps J onto a group $\bar{J}$ of upper triangular matrices. Hence $\eta([J,J]) \leq [\bar{J},\bar{J}]$ is a subgroup of $\mathrm{SL}_2(\mathcal{O}/\pi^t\mathcal{O})$ consisting of upper triangular matrices whose diagonal entries are equal to 1. In particular the trace of any element of $\eta([J,J])$ is equal to 2. This means that for any $A \in [J,J]$ we have $\mathrm{trace}\, A \in 2 + \pi^t\mathcal{O}$, so that $v((\mathrm{trace}\, A) - 2) \geq t$, as asserted. □

5 Curves, norms and actions associated to ideal points

5.1 In this subsection we review notation for character varieties as used in [4], and introduce some additional notation that will be needed in this paper.

We begin with some algebraic geometric conventions. Suppose that K denotes the field of rational functions on an irreducible complex projective algebraic curve C and that x is a smooth point of C. For a non-zero element f of K we will write $Z_x(f)$ to denote the order of zero of f at x, or 0 if f does not have a zero at x. Similarly we we will let $\Pi_x(f)$ denote the order of pole of f at x, or 0 if f does not have a pole at x. The function $v_x\colon F^* \to \mathbb{Z}$ defined by $v_x(f) = Z_x(f) - \Pi_x(f)$ is a discrete rank–1 valuation on the field K. We will denote the valued field (K, v_x) by K_x.

Given an irreducible complex affine algebraic curve A we will denote by $\widetilde{A}$ the unique smooth projective curve that admits a birational correspondence $\phi\colon \widetilde{A} \to A$. (The curve $\widetilde{A}$ can be constructed by desingularizing a projective completion of A.) We will say that a point $x \in \widetilde{A}$ is an *ideal point* if it does not correspond to any point of A under ϕ.

Now let Γ be a finitely generated group. We will denote by $R(\Gamma)$ the complex affine algebraic set of representations of Γ in $SL_2(\mathbb{C})$. If R_0 is an irreducible subvariety of $R(\Gamma)$ and if F denotes the field of rational functions on $R(\Gamma)$ then the *tautological representation* $P\colon \Gamma \to \mathrm{SL}_2(F)$ associated to R_0 is defined by

$$P(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where the functions a, b, c, and d satisfy

$$\rho(g) = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix}.$$

We will denote by $X(\Gamma)$ the set of all characters of representations in $R(\Gamma)$, and by $\tau\colon R(\Gamma) \to X(\Gamma)$ the surjective regular map such that $\tau(\rho)$ is the character of ρ. We give $X(\Gamma)$ the structure of an affine algebraic set as in [4].

Next suppose that A is an irreducible affine algebraic curve contained in $X(\Gamma)$ and let $x \in \widetilde{A}$ be an ideal point. Let K denote the field of rational functions on A. We use the birational correspondence between $\widetilde{A}$ and A to identify K with the function field of $\widetilde{A}$, and we regard v_x as a discrete valuation on K. Let R_0 denote an irreducible component of $\tau^{-1}(A)$ which is mapped to a dense subset of A by τ. We use τ to identify K with a subfield of the function field F of R_0. According to [4, Theorem 1.2.3], we may extend the valuation v_x to a discrete valuation on F and we will denote by F_x the resulting discretely valued field. We consider the $\mathrm{GL}_2(F_x)$–tree T_{F_x} and we let T_x denote the Γ–tree which is the pull-back of T_{F_x} under the tautological representation $P\colon \Gamma \to \mathrm{SL}_2(F_x) < \mathrm{GL}_2(F_x)$ associated to R_0.

If γ is an element of Γ then we shall let I_γ denote the rational function on A defined by $I_\gamma(\chi) = \chi(\gamma)$. Using the identifications described above, we shall regard I_γ as an element of F_x. We recall from [4, 1.2.4] that $I_\gamma = \operatorname{trace} P(\gamma)$. If $\mathcal{C}$ is the conjugacy class of $\gamma \in \Gamma$ then $I_\gamma = I_{\gamma'}$ for any element γ' of $\mathcal{C}$. We will write $I_{\mathcal{C}} = I_\gamma$.

Proposition 5.2 *Let Γ be any finitely generated group, let $A \subset X(\Gamma)$ be any curve and let $x \in \widetilde{A}$ be an ideal point. Then for any element γ of Γ we have*

$$2\Pi_x(I_\gamma) = \lambda_{T_x}(\gamma).$$

Proof Using the notation of 5.1, we have

$$\Pi_x(I_\gamma) = \max(0, -v_x(I_\gamma)) = \max(0, -v_x(\mathrm{trace}(P(\gamma)))).$$

It therefore follows from Proposition 4.3 that $2\Pi_x(I_\gamma) = \lambda_{T_{F_x}}(P(\gamma))$. But since the Γ–tree T_x is the pull-back of the $\mathrm{GL}_2(F_x)$–tree T_{F_x} via the representation P we have $\lambda_{T_{F_x}}(P(\gamma)) = \lambda_{T_x}(\gamma)$, and the assertion follows. □

Proposition 5.3 *Let Γ be any finitely generated group, let $A \subset X(\Gamma)$ be any curve and let $x \in \widetilde{A}$ be any ideal point. Then the tree T_x is a non-trivial bipartite Γ–tree.*

Proof We again use the notation of 5.1. Since the $\mathrm{SL}_2(F_x)$–tree T_{F_x} is bipartite by 4.2, it follows that the Γ–tree T_x is also bipartite.

By definition the ideal point x does not correspond to a point of the affine curve A under the birational correspondence between $\widetilde{A}$ and A. Hence there is an element of the coordinate ring of A which, when regarded as a function on $\widetilde{A}$, has a pole at x. Since the functions I_γ generate the coordinate ring there exists $\gamma \in \Gamma$ such that $\Pi_x(I_\gamma) > 0$. Proposition 5.2 implies that γ has no fixed vertex in T_x and hence that T_x is a non-trivial Γ–tree. □

Proposition 5.4 *Let Γ be any finitely generated group, let $A \subset X(\Gamma)$ be any curve, let $x \in \widetilde{A}$ be an ideal point. Suppose that Θ is a subgroup of Γ which fixes an arc of length $t > 0$ in T_x. Then for every element γ of the commutator subgroup $[\Theta, \Theta] \le \Theta \le \Gamma$, we have*

$$Z_x(I_\gamma - 2) \ge t.$$

Proof Using the notation of 5.1, we have

$$Z_x(I_\gamma - 2) = \max(0, v_x(I_\gamma) - 2) = \max(0, v_x((\mathrm{trace}\, P(\gamma)) - 2)) \qquad (5.4.1)$$

for any $\gamma \in \Gamma$. Now suppose that $\gamma \in [\Theta, \Theta]$, where $\Theta \le \Gamma$ fixes an arc of length $t > 0$ in T_x. Set $J = P(\Theta)$, so that $P(\gamma) \in [J, J]$.

Since the Γ–tree T_x is the pull-back of the $\mathrm{GL}_2(F_x)$–tree T_{F_x} via the representation P the group J fixes an arc of length t in T_{F_x}. Hence by Proposition 4.4 we have

$$v_x((\mathrm{trace}(P(\gamma))) - 2) \ge t. \qquad (5.4.2)$$

The conclusion follows from (5.4.1) and (5.4.2). □

5.5 If M is a hyperbolic knot manifold, we define a *principal component* X_0 of the character variety $X(\pi_1(M))$ to be a component that contains the character of a discrete, faithful representation of $\pi_1(M)$.

Lemma 5.6 *Let M be a hyperbolic knot manifold and let X_0 be a principal component of $X(\pi_1(M))$. If γ is a non-trivial peripheral element of $\pi_1(M))$ then the function $I_\gamma|_{X_0}$ is non-constant. Furthermore if γ is any non-trivial element of $\pi_1(M)$ then the function $I_\gamma|_{X_0}$ cannot be identically equal to 2.*

Proof If γ is a non-trivial peripheral element of $\pi_1(M)$ then it follows from [6, Proposition 3.2.1] that $I_\gamma|_{X_0}$ is non-constant.

Now suppose that γ is any non-trivial element of $\pi_1(M)$ and that $I_\gamma|_{X_0}$ is identically equal to 2. Since the principal component X_0 contains the character χ_0 of a discrete, faithful representation ρ_0 of $\pi_1(M)$, we have $\operatorname{trace}\rho_0(\gamma) = I_\gamma(\chi_0) = 2$. This is possible only if γ is a peripheral element. But this contradicts the first part of the statement. □

The following result is a strengthened version of Proposition 1.1.2 from [4].

Proposition 5.7 *Let M be a hyperbolic knot manifold. Let X_0 be a principal component of $X(\pi_1(M))$ and let $x_1, \dots, x_n$ denote the ideal points of $\widetilde{X}_0$. Then there exists a unique norm $\|\cdot\|$ on the vector space $H_1(\partial M;\mathbb{R})$ such that for any element $\alpha \in H_1(\partial M;\mathbb{Z}) \subset H_1(\partial M;\mathbb{R})$ and any closed curve c representing α we have*

$$\|\alpha\| = 2\deg(I_{[c]}|_{X_0}).$$

Moreover there are strict essential surfaces $S_1, \dots, S_n$ in M (some of which may be closed) such that the following conditions hold.

1. *For $i = 1, \dots, n$, the surface S_i is a T_{x_i}–surface, where T_{x_i} is defined as in 5.1, taking $\Gamma = \pi_1(M)$ and $A = X_0$.*
2. *If c is any closed curve in ∂M and if $\alpha \in H_1(\partial M;\mathbb{Z}) \subset H_1(\partial M;\mathbb{R})$ is the homology class represented by c, then $\|\alpha\| = \sum_{i=1}^n \Delta_{\partial M}(c, \partial S_i)$.*
3. *For any $k > 0$ the set $B_k = \{v\colon \|v\| \le k\}$ in $H_1(\partial M;\mathbb{R})$ is a convex polygon. Furthermore, for each vertex v of B_k there is an index $i \le n$ such that $\partial S_i \neq \emptyset$ and v is a scalar multiple of the boundary class of S_i.*
4. *If $i \in \{1, \dots, n\}$ and if K is a π_1–injective, connected, compact subpolyhedron of S_i such that $\chi(K) < 0$, and if $\Theta \le \pi_1(M)$ is the subgroup defined up to conjugacy by $\Theta = \operatorname{im}(\pi_1(K) \to \pi_1(M))$ then, for every element γ of the commutator subgroup $[\Theta, \Theta] \le \Theta \le \pi_1(M)$, we have $Z_{x_i}(I_\gamma - 2) \ge t_{S_i}(K)$.*

Proof According to Proposition 5.3, $T_i = T_{x_i}$ is a non-trivial bipartite $\pi_1(M)$–tree for each $i \in \{1, \ldots, n\}$. Applying Theorem 3.13, we obtain an essential T_{x_i}–surface S_i satisfying conditions 3.13(1) and 3.13(2). (Note that we have not yet shown that the S_i are strict.)

Let $\mathcal{I}$ denote the unique alternating bilinear form on $H_1(\partial M; \mathbb{R})$ which restricts to the homological intersection pairing on $H_1(\partial M; \mathbb{Z})$. For each $i = 1, \ldots, n$ we define an element $\alpha_i \in H_1(\partial M; \mathbb{Z}) \subset H_1(\partial M; \mathbb{R})$ as follows: if $\partial S_i \neq \emptyset$ we take α_i to be a boundary class for S_i, and if S_i is closed we set $\alpha_i = 0$. We define a linear functional l_i on $H_1(\partial M; \mathbb{R})$ by $l_i(x) = \mathcal{I}(x, \alpha_i)$. Note that if S_i has non-empty boundary then the kernel of l_i is spanned by α_i.

If c is an arbitrary closed curve in ∂M then Proposition 5.2 implies for each $i \in \{1, \ldots, n\}$, that $2\Pi_{x_i}(I_{[c]})$ is equal to $\lambda_{T_i}(c)$, which in turn is equal to $\Delta_{\partial M}(c, \partial S_i)$ according to condition 3.13(1). Summing over $i = 1, \ldots, n$ we find that

$$2 \deg I_{[c]} = \sum_{i=1}^{n} 2\Pi_{x_i}(I_{[c]}) = \sum_{i=1}^{n} \Delta_{\partial M}(c, \partial S_i) = \sum_{i=1}^{n} |l_i(\alpha)| \tag{5.7.1}$$

where $\alpha \in H_1(\partial M; \mathbb{R})$ is the class represented by c. By Lemma 5.6, $\deg I_{[c]}|_{X_0}$ is non-zero for any homotopically non-trivial closed curve c in ∂M. Thus the α_i span the vector space $H_1(\partial M; \mathbb{R})$, and we may define a norm on $H_1(\partial M; \mathbb{R})$ by setting

$$\|v\| = \sum_{i=1}^{n} |l_i(v)|. \tag{5.7.2}$$

It follows from 5.7.1 that if $\alpha \in H_1(\partial M; \mathbb{Z}) \subset H_1(\partial M; \mathbb{R})$ is represented by a closed curve c then $\|\alpha\| = 2 \deg(I_{[c]}|_{X_0})$. The uniqueness assertion follows from the observation that, by continuity and homogeneity, any norm on $H_1(\partial M; \mathbb{R})$ is uniquely determined by its restriction to the integer lattice $H_1(\partial M; \mathbb{Z})$.

Conclusion (1) is immediate from the construction of the S_i and conclusion (2) follows from 5.7.1. It follows from 5.7.2 that for each vertex s of the convex polygon $B_k = \{v : \|v\| \leq k\}$, there is an index i such that the linear functional l_i is not identically 0 and $l_i(s) = 0$. Since $l_i \not\equiv 0$ we have $\partial S_i \neq \emptyset$. The kernel of l_i is therefore spanned by the boundary class α_i of S_i. This implies conclusion (3).

To establish conclusion (4), we suppose that we are given an index $i \in \{1, \ldots, n\}$ and a π_1–injective, connected, compact subpolyhedron K of S_i such that $\chi(K) < 0$. We let $\Theta \leq \pi_1(M)$ denote the subgroup defined up to conjugacy by $\Theta = \operatorname{im}(\pi_1(K) \to \pi_1(M))$, and suppose that γ is an element of

$[\Theta,\Theta]$. If t_1 is any positive integer $\le t_{S_i}(K)$, then according to condition 3.13(2), Θ fixes an arc of length t_1 in T_i. By 5.4 it therefore follows that $Z_{x_i}(I_\gamma - 2) \ge t_1$. As this holds for every positive integer $t_1 \le t_{S_i}(K)$ we conclude that $Z_{x_i}(I_\gamma - 2) \ge t_{S_i}(K)$.

Finally, we must show that each S_i is a strict essential surface. Assume to the contrary that S_i is a semi-fiber for some i. According to 3.12 we then have $t_{S_i}(S_i) = +\infty$. On the other hand, according to Proposition 2.3 we have $\chi(S_i) < 0$, and so $\Theta = \mathrm{im}(\pi_1(S_i) \to \pi_1(M))$ is non-abelian. Let us choose a non-trivial element γ of $[\Theta,\Theta]$. Applying condition (4) with $K = S_i$ we deduce that $Z_{x_i}(I_\gamma - 2) = +\infty$, ie, that I_γ must be the constant function 2. But since γ is non-trivial, this contradicts Lemma 5.6. □

Remark 5.8 In [4] the function f_α in $\mathbb{C}(X_0)$ was defined by $f_\alpha = I_{[c]}^2 - 4$ where c is a closed curve in ∂M representing α. The norm referred to in Proposition 1.1.2 of [4] satisfies the condition $\|\alpha\| = \deg f_\alpha$ for all $\alpha \in H_1(\partial M;\mathbb{Z})$. Since the degree of f_α is twice that of $I_{[c]}$, the norm referred to in Proposition 1.1.2 of [4] is the same as that in the given by Proposition 5.7. We may therefore apply Corollary 1.1.4 of [4] to conclude that if $s = \langle\alpha_1\rangle$ is not the boundary slope of any strict essential surface in M, and if the Dehn filled manifold $M(s)$ has cyclic fundamental group, then then $\|\alpha\| \le \|\beta\|$ for any non-zero class $\beta \in H_1(\partial M;\mathbb{Z})$.

Corollary 5.9 *If M is a hyperbolic knot manifold then M has two bounded, strict, connected essential surfaces with distinct boundary slopes.*

Proof It suffices to show that the surfaces S_i given by Proposition 5.7 do not all have the same boundary slope. If they all did have the same boundary slope, there would be a non-zero class $\alpha \in H_1(\partial M;\mathbb{Z})$ which is a boundary class for each S_i. But then the expression given in conclusion (2) would vanish on the subspace spanned by α, contradicting the fact that this expression defines a norm. □

Proposition 5.10 *Suppose that M is a hyperbolic knot manifold. Let X_0 be a principal component of $X(\pi_1(N))$ and let $x \in \widetilde{X}_0$ be an ideal point. If S is an essential T_x–surface then for any closed curve c in M we have $2\Pi_x(I_\gamma) \le \Delta_M(c,S)$.*

Proof We have $2\Pi_x(I_{[c]}) = \lambda_{T_x}(c)$ by Proposition 5.2, and $\lambda_{T_x}(c) \le \Delta_M(c,S)$ by Proposition 3.11. □

6 Manifolds with few essential surfaces

The goal of this section is to prove Theorem 6.7, which gives topological information about an irreducible knot manifold that has at most two isotopy classes of connected, strict essential surfaces. There are a few knot manifolds with this property that arise as exceptions. We will discuss these before stating the theorem.

6.1 The solid torus $S^1 \times D^2$ and the twisted I–bundle K over the Klein bottle are examples of Seifert fibered knot manifolds which have no strict essential surfaces at all. The only connected essential surface in the solid torus is the meridian disk, which is obviously a fiber in a fibration over S^1 and hence not strict. The connected essential surfaces in K are all non-trivial vertical annuli with respect to the I–fibration. Splitting K along an essential separating vertical annulus A results in two twisted I–bundles over Möbius bands for which A is the associated ∂I–bundle, and hence A is a semi-fiber. Similarly any non-separating vertical annulus is a fiber in a fibration of K over S^1.

The only 3–manifolds that fiber over the circle with fiber an annulus are K and $S^1 \times S^1 \times I$. Furthermore K is the only orientable 3–manifold that can be obtained from two twisted I–bundles over Möbius bands by identifying their ∂I–bundles. Hence if M is a 3–manifold not homeomorphic to K or $S^1 \times S^1 \times I$ then any essential annulus in M is a strict essential surface.

A Seifert fibered manifold M with base surface a disk and two singular fibers has exactly one isotopy class of connected essential vertical surfaces, which are annuli. If the two singular fibers are both of order 2 then M is homeomorphic to the twisted I–bundle K. Otherwise a vertical annulus is a strict essential surface; so M gives an example of an irreducible knot manifold with exactly one isotopy class of connected strict essential surfaces.

6.2 We define a *cable space* to be a Seifert fibered manifold over an annulus with one singular fiber. Note that a cable space has exactly three isotopy classes of essential vertical annuli; one has a boundary curve on each boundary torus of the cable space and the other two have both boundary curves on the same boundary torus.

6.3 We will say that an orientable 3–manifold M is an *exceptional graph manifold* if M is not Seifert fibered and M is homeomorphic to either

(1) a manifold obtained from a disjoint union of a cable space C and a twisted I–bundle K over a Klein bottle by gluing ∂K to a component of ∂C via some homeomorphism; or

(2) a manifold obtained from $P \times S^1$, where P is a planar surface with three boundary curves, by gluing two of the boundary tori of $P \times S^1$ to each other via some homeomorphism.

Proposition 6.4 *An exceptional graph manifold has exactly two connected strict essential surfaces up to isotopy. One of these is a torus and the other is an annulus.*

Proof Let M denote an exceptional graph manifold. According to the definition, M is obtained from a manifold M' by identifying two torus boundary components of M'. The image of these two tori under the quotient map $q\colon M' \to M$ is a torus T in M. We denote by M_0 the component of M' which contains $q^{-1}(\partial M)$. In case (1) of the definition M_0 is a cable space and the other component of M', which we shall denote by M_1, is a twisted I–bundle over a Klein bottle. We shall regard M_1 as a Seifert fibration over a disk with two singular fibers of order 2. In case (2) of the definition $M' = M_0$ is homeomorphic to $P \times S^1$, where P is a planar surface with three boundary components. In either case the manifold M' is a Seifert-fibered manifold and, up to isotopy, there is a unique essential annulus in M_0 which has both boundary components in $q^{-1}(\partial M)$. We will let A' denote such an annulus.

Clearly T is an essential surface in M, and T is strict by 1.16. The annulus A' is a strict essential surface in M_0 by 1.16, and hence by Proposition 2.2 the annulus $A = q(A')$ is a strict essential surface in M. We will show that any connected strict essential surface in M is isotopic either to A or to T.

Suppose that F is a connected strict essential surface in M which is not isotopic to T. By Proposition 2.2 we may assume after an isotopy that F is transverse to T, that each component of $F' = q^{-1}(F)$ is essential in the component of M' containing it, and that some component S of F' is a strict essential surface in the component of M' containing it. Since a twisted I–bundle over a Klein bottle has no strict essential surfaces we must have $S \subset M_0$. Thus S is a component of $F_0 = F' \cap M_0$.

We claim that F' is isotopic to a union of vertical annuli in the Seifert-fibered manifold M'. Since the strict essential surface S is a component of F_0, it follows from 1.17 that F_0 cannot be isotopic to a horizontal surface, and hence that it is isotopic to a vertical surface whose components are all vertical annuli in M_0. This proves the claim if M satisfies case (2) of the definition of an exceptional graph manifold. To complete the proof in case (1) it is enough to show that F_1 is also isotopic to a vertical surface. This is true because, in a

Seifert fibration over a disk with two singular fibers of order 2, every essential surface is isotopic to a vertical surface. Thus the claim is proved in both cases.

Next we claim that $\partial F_0 \subset q^{-1}(\partial M)$. Assume to the contrary that $\partial F' \cap q^{-1}(T) \neq \emptyset$. Let T_0 and T_1 denote the two components of $q^{-1}(T)$, where $T_0 \subset M_0$, and let $h\colon T_0 \to T_1$ denote the gluing homeomorphism. Since F' is isotopic to a vertical surface and h maps $\partial F' \cap T_0 \neq \emptyset$ to $\partial F' \cap T_1$, it follows that h is isotopic to a fiber-preserving homeomorphism. Hence M admits a Seifert fibration; this contradicts the definition of an exceptional graph manifold.

We have now shown that F_0 is a vertical surface in M_0 and that $\partial F_0 \subset q^{-1}(\partial M)$. Moreover, F' is connected by hypothesis. Thus F' is isotopic to the annulus A', and F is isotopic to the annulus A, as required for the proof of the proposition. □

Proposition 6.5 *Let M be a compact, irreducible orientable 3–manifold whose boundary components are all tori, and let T_0 be a component of ∂M. If M is not homeomorphic to $S^1 \times D^2$, $S^1 \times S^1 \times I$ or a twisted I–bundle K over the Klein bottle, then M contains a bounded connected strict essential surface S such that $\partial S \subset T_0$. Moreover, if M is the compact core of a complete hyperbolic manifold with finite volume then there are two bounded connected strict essential surfaces in M which have their boundaries contained in T_0 and which have distinct boundary slopes on T_0.*

Proof First consider the case where M is the compact core of a complete hyperbolic manifold with finite volume. If ∂M is connected, then the result follows from Corollary 5.9. If M has more than one boundary component then [6, Theorem 3] implies that M has an essential surface F with ∂F contained in T_0; in this case F is disjoint from at least one component of ∂M and is therefore strict by 1.16.

If M is Seifert-fibered and is not homeomorphic to one of the exceptional manifolds listed in the statement, then we will show that M has a strict essential vertical annulus A whose boundary is contained in T_0. This implies the result in this case. First note that there is an arc α in the base surface B such that α is essential (ie, is not the frontier of a disk disjoint from the image of the singular fibers), and such that α has both its endpoints on the component of ∂B which is the image of T_0. Indeed if such an arc α did not exist are where B would be a disk or an annulus, and there would be no singular fibers in the Seifert fibration of M; this would imply that M is homeomorphic to $S^1 \times D^2$ or $S^1 \times S^1 \times I$, a contradiction.

The inverse image of α under the Seifert fibration is then an essential annulus A which has both boundary components contained in T_0. Since M is not homeomorphic to K or $S^1 \times S^1 \times I$, it follows from 6.1 that the essential annulus A is a strict essential surface.

To prove the proposition in the general case, let $\mathcal{T}$ be a maximal collection of disjoint essential tori in M, no two of which are parallel. We may assume that $\mathcal{T}$ is non-empty since otherwise, by Thurston's Geometrization Theorem, M would either be Seifert-fibered or homeomorphic to the compact core of a complete hyperbolic manifold finite volume. Let R be a regular neighborhood of $\mathcal{T}$ and let N be the closure of the component of $M - R$ which contains T_0. It suffices to show that N contains a bounded strict essential surface which is disjoint from $\partial N - T_0$. Note that N is not homeomorphic to $S^1 \times S^1 \times I$, since the tori in $\mathcal{T}$ are essential. Also, N cannot be homeomorphic to K or $S^1 \times D^2$ since $\mathcal{T}$ is non-empty. Since N contains no essential tori it follows from Thurston's theorem that either N is Seifert-fibered or it is homeomorphic to the compact core of a complete hyperbolic manifold with finite volume. Thus N contains a bounded strict essential surface which is disjoint from $\partial N - T_0$ by the two cases that were handled earlier. This completes the proof. □

Proposition 6.6 *If a knot manifold M has an essential torus then it also has a bounded strict essential surface. Furthermore, a Seifert-fibered knot manifold which contains an essential torus has infinitely many distinct isotopy classes of strict essential surfaces.*

Proof Consider the manifold N obtained by splitting M along a maximal family $\mathcal{T}$ of disjoint, non-parallel essential tori. Let T_0 denote the component of ∂N which corresponds to ∂M and let N_0 be the component of N containing T_0. Since the tori in $\mathcal{T}$ are essential, N_0 is not homeomorphic to $S^1 \times S^1 \times I$. Thus, since N_0 has at least two boundary components, it is not one of the exceptional manifolds listed in Proposition 6.5. Hence N_0 has a bounded strict essential surface F with $\partial F \subset T_0$. Now F is a bounded essential surface in M which is strict since it is disjoint from an essential torus.

For the proof of the second assertion, assume that M is Seifert-fibered and consider an essential vertical torus T. The image of T under the Seifert fibration map is a simple closed curve c in the base surface B. Since T is essential, c does not bound a disk containing fewer than two points which are images of singular fibers. Therefore there exists an arc α in B which meets c transversely in at least one point and has the property that every disk component of the complement of $c \cup \alpha$ contains the image of at least one singular fiber. The images of α under powers of the Dehn twist about c give an infinite family of non-isotopic arcs in B whose inverse images in M are strict essential annuli. □

Theorem 6.7 *Let M be an irreducible knot manifold.*

(1) *If M has no strict essential surface then M is homeomorphic to either a solid torus or a twisted I–bundle over a Klein bottle.*

(2) *If M has exactly one isotopy class of connected strict essential surfaces then M is Seifert-fibered over a disk with two singular fibers.*

(3) *If M has exactly two isotopy classes of connected strict essential surfaces, represented by surfaces F_1 and F_2, then either*

  (3a) *M is an exceptional graph manifold; or*

  (3b) *M is a hyperbolic knot manifold, F_1 and F_2 are bounded surfaces of negative Euler characteristic, and the boundary slopes of F_1 and F_2 are distinct.*

The proof of Theorem 6.7 depends on the following lemma, which is contained in [9, Lemma 2.3].

Lemma 6.8 *Suppose that N is a cable space (cf 6.2) with boundary tori T_1 and T_2. Then there is a bijection ϕ from the set of slopes on T_1 to the set of slopes on T_2 such that for each slope s on T_1 there exists a connected essential surface in N, having nonempty intersection with both T_1 and T_2 and having s and $\phi(s)$ as boundary slopes.*

Proof of Theorem 6.7 If M has no strict essential surface then Proposition 6.5 implies that M is homeomorphic to either a solid torus, a twisted I–bundle over a Klein bottle or a product of a torus and an interval. Since the latter is not a knot manifold, assertion (1) of the theorem follows.

As a preliminary to proving assertions (2) and (3) we observe that if M has no essential torus then, by Thurston's Geometrization Theorem, M is either a hyperbolic knot manifold or a Seifert-fibered manifold. Moreover, all Seifert-fibered knot manifolds with no essential tori are Seifert-fibered over a disk with two singular fibers.

To prove assertion (2), suppose that M contains exactly one strict essential surface S. Proposition 6.6 implies that S cannot be a torus, and Proposition 6.5 implies that M cannot be a hyperbolic knot manifold. Thus the observation above implies the conclusion of (2).

We now turn to the proof of assertion (3). Assume that, up to isotopy, M contains exactly two strict essential surfaces F_1 and F_2. According to 6.1 there is only one isotopy class of strict essential surfaces in a Seifert-fibered manifold over a disk with at most two singular fibers. Thus by the observation above we have two cases: either M is hyperbolic, or M has an essential torus.

We first consider the case that M is hyperbolic. Proposition 6.5 implies that F_1 and F_2 are bounded and have distinct boundary slopes. Proposition 2.3 implies that $\chi(F_1)$ and $\chi(F_2)$ are strictly negative. Thus conclusion (3b) holds in this case.

The remaining case is that M contains an essential torus T. It follows from 1.16 that any closed essential surface in M is a strict essential surface. In particular T is strict. Proposition 6.6 implies that there is also a bounded strict essential surface A in M, and by the hypothesis of assertion (3) any strict essential surface in M is isotopic to either T or A. In particular, any closed essential surface in M is isotopic to T. Note that Proposition 6.6 also implies that M is not Seifert-fibered.

Let M' be the manifold obtained by splitting M along the torus T, and let M_0 denote the component of M' which contains $q^{-1}(\partial M)$ where $q\colon M' \to M$ is the quotient map. Note that M' cannot contain a closed essential surface since the image of such a surface in M would be a closed essential surface but would not be isotopic to T. In particular, no component of M' contains an essential torus.

We claim that M_0 cannot be homeomorphic to the compact core of a complete hyperbolic manifold with finite volume. Otherwise by Proposition 6.5, M_0 would contain two non-isotopic bounded strict essential surfaces which are disjoint from T. These would be strict essential surfaces in M by 1.16. Since A is the only bounded strict essential surface in M up to isotopy this is a contradiction.

It now follows from Thurston's Geometrization Theorem that M_0 is a Seifert fibered manifold. Moreover, since T is not boundary-parallel, M_0 is not homeomorphic to $S^1 \times S^1 \times I$ and must therefore contain an essential annulus with its boundary contained in $q^{-1}(\partial M)$. By 1.16 any such annulus must be a strict essential surface in M. It follows that A is an annulus and that, up to isotopy, A is the only essential annulus in the Seifert fibered manifold M_0.

If T is non-separating, then M_0 is a Seifert-fibered manifold with three boundary components and contains only one essential annulus up to isotopy. It follows that M_0 is homeomorphic to $P \times S^1$ where P is a planar surface with three boundary curves. Since M is not Seifert-fibered, M is an exceptional graph manifold by 6.3(2). Thus (3a) holds.

Now consider the case where T is separating. We let M_1 denote the component of M' which does not contain $q^{-1}(\partial M)$, and we identify M_0 and M_1 with submanifolds of M. In this case M_0 has two boundary components and is a Seifert-fibered manifold which contains only one essential annulus up to isotopy. Hence M_0 is a cable space. It follows from Proposition 6.5 that either M_1 is a twisted I–bundle over a Klein bottle, or M_1 has a bounded strict essential surface. If M_1 is a twisted I–bundle over a Klein bottle then, since M is not Seifert-fibered, M is an exceptional graph manifold by 6.3(2), so (3a) holds.

Finally suppose that M_1 contains a bounded strict essential surface F, and let s denote the boundary slope of F in M_1. By Lemma 6.8 there exists a connected essential surface G in the cable space M_0, having boundary slope s on T and boundary slope $\phi(s)$ on ∂M_0. Thus, for suitably chosen positive integers m and n, the surface consisting of m parallel copies of F in M_1 and the surface consisting of n parallel copies of G in M_0 have isotopic intersections with T. Hence there exists a connected surface $\widehat{F}$ in M which meets M_1 in parallel surfaces isotopic to F and meets M_0 in parallel surfaces isotopic to G. According to Proposition 2.2, $\widehat{F}$ is a bounded strict essential surface in M. Since $\widehat{F}$ is not isotopic to the annulus A, this is a contradiction. Thus M_1 cannot contain a bounded strict essential surface, and the proof is complete. □

Definition 6.9 We will say that M is a *two-surface knot manifold* provided that M is an irreducible knot manifold and that M has at most two distinct isotopy classes of strict essential surfaces. We say that a two-surface knot manifold M is an *exceptional two-surface knot manifold* if M is Seifert fibered or if M is an exceptional graph manifold.

6.10 According to Theorem 6.7, if M is a non-exceptional two-surface knot manifold then M is a hyperbolic knot manifold and M has exactly two distinct isotopy classes of connected strict essential surfaces. Moreover, if F_1 and F_2 are representatives of these two isotopy classes then they have distinct boundary slopes and both F_1 and F_2 have negative Euler characteristic. These properties of non-exceptional two-surface knot manifolds will be used throughout the sequel.

Definition 6.11 A knot K in a closed orientable 3–manifold Σ will be said to be a *non-exceptional two-surface knot* provided that the knot manifold $M = \Sigma(K)$ is a non-exceptional two-surface knot manifold.

7 General principles about two-surface knot manifolds

As we explained in the introduction, there is a combinatorially defined quantity $\kappa(F_1, F_2)$, associated to two bounded essential surfaces, which plays a central role in our estimates. We begin with the definition.

Definition 7.1 Suppose that F_1 and F_2 are bounded connected essential surfaces in an irreducible knot manifold M. Let m_i denote the number of boundary components of F_i for $i = 1, 2$. We define an element $\kappa(F_1, F_2)$ of the interval $[0, \infty]$ of the extended real line by

$$\kappa(F_1, F_2) = \inf_K \frac{m_2 \cdot \#(K \cap F_1)}{m_1 \cdot t_{F_2'}(K)},$$

where F_2' ranges over all surfaces that are isotopic to F_2 and meet F_1 transversally, while K ranges over all compact connected π_1–injective 1–dimensional polyhedra of Betti number 2 which are contained in F_2' and meet F_1 transversally. (To say that K meets F_1 transversally means in particular that $K \cap F_1$ consists entirely of points at which K is locally Euclidean. We interpret the quotient in the definition as being 0 if $t_{F_2'} = \infty$.) Note that $\kappa(F_1, F_2) < \infty$ if and only if F_2 contains a π_1–injective connected 1–dimensional polyhedron of Betti number 2, ie, if and only if $\chi(F_2) < 0$.

Theorem 7.2 *Suppose that M is a non-exceptional two-surface knot manifold (so that M is hyperbolic by 6.10). Let X_0 be a principal component of $X(\pi_1(M))$ and let $\|\cdot\|$ denote the norm on $H_1(\partial M, \mathbb{R})$ given by Proposition 5.7. Let F_1 and F_2 be representatives of the two isotopy classes of connected strict essential surfaces in M, and for $i = 1, 2$ let α_i denote a boundary class of F_i (which is a bounded surface by 6.10). Then we have*

$$\frac{\|\alpha_1\|}{\|\alpha_2\|} \leq \kappa(F_1, F_2).$$

Proof Suppose that F_1 and F_2 satisfy the hypotheses and also meet transversally. Let K be a compact connected π_1–injective 1–dimensional polyhedron of Betti number 2 contained in F_2 and meeting F_1 transversally. Set $t = t_{F_2}(K)$ and $\ell = \#(K \cap F_1)$. In this setting we will show that

$$\frac{\|\alpha_1\|}{\|\alpha_2\|} \leq \frac{m_2 \ell}{m_1 t}.$$

In view of the definition of $\kappa(F_1, F_2)$, this will establish the theorem.

Let $x_1, \ldots, x_n$ be the ideal points of $\widetilde{X}_0$. We fix strict essential surfaces $S_1, \ldots, S_n$ in M satisfying conditions (1)–(4) of Proposition 5.7. It follows from the hypotheses that each component of each S_i is isotopic to either F_1 or F_2. For $i = 1, 2$ let m_i denote the number of boundary components of F_i and let $s_i = \langle \alpha_i \rangle$ denote its boundary slope. Since $s_1 \neq s_2$ by 6.10, there cannot exist disjoint surfaces isotopic to F_1 and F_2. Hence for each $i \in \{1, \ldots, n\}$, either every component of S_i is isotopic to F_1 or every component of S_i is isotopic to F_2. We may therefore suppose the S_i (and the x_i) to be indexed in such a way that all components of S_i isotopic to F_1 when $1 \le i \le k$, and all components of S_i are isotopic to F_2 when $k < i \le n$. Here k is *a priori* an integer with $0 \le k \le n$. However, if k were equal to 0 or n then the S_i would all have the same boundary slope. It would then follow from part (2) of Proposition 5.7 that $\|\alpha\| = 0$, contradicting the definition of a norm. Hence $0 < k < n$.

We let ν_i denote the number of components of S_i for $i = 1, \ldots, n$, and we set $N_1 = \sum_{i=1}^{k} \nu_i$ and $N_2 = \sum_{i=k+1}^{n} \nu_i$. For $1 \le i \le k$, the boundary of S_i consists of $\nu_i m_1$ simple closed curves of slope s_1, and for $k < i \le n$, the boundary of S_i consists of $\nu_i m_2$ simple closed curves of slope s_2. Thus if $C \subset \partial M$ is a non-trivial simple closed curve and s denotes its slope, we have

$$\Delta_{\partial M}(C, \partial S_i) = \nu_i m_1 \Delta(s, s_1) \qquad \text{for } 1 \le i \le k$$

and
$$\Delta_{\partial M}(C, \partial S_i) = \nu_i m_2 \Delta(s, s_2) \qquad \text{for } k < i \le n.$$

Hence if β is the homology class in $H_1(\partial M; \mathbb{Z}) \subset H_1(\partial M; \mathbb{R})$ represented by some orientation of C then 5.7(2) gives

$$\begin{aligned} \|\beta\| = \sum_{i=1}^{n} \Delta_{\partial M}(C, \partial S_i) &= \sum_{i=1}^{k} \nu_i m_1 \Delta(s, s_1) + \sum_{i=k+1}^{n} \nu_i m_2 \Delta(s, s_2) \\ &= N_1 m_1 \Delta(s, s_1) + N_2 m_2 \Delta(s, s_2). \end{aligned}$$

In particular, taking C to be a simple closed curve with slope s_1 or s_2, setting $\Delta = \Delta(s_1, s_2) = \Delta(s_2, s_1)$ and observing that $\Delta(s_1, s_1) = \Delta(s_2, s_2) = 0$, we find that

$$\|\alpha_1\| = N_2 m_2 \Delta \qquad \text{and} \qquad \|\alpha_2\| = N_1 m_1 \Delta,$$

so that

$$\frac{\|\alpha_1\|}{\|\alpha_2\|} = \frac{N_2 m_2}{N_1 m_1}. \tag{7.2.1}$$

Let us fix collarings h_1 and h_2 of F_1 and F_2 in M. Since each component of S_i is isotopic to F_1 if $i \leq k$ and to F_2 if $i > k$, it follows from Proposition 2.4 that after modifying each S_i within its isotopy class we may assume that S_i has the form $h_1(F_1 \times Y_i)$ (if $i \leq k$) or $h_2(F_2 \times Y_i)$ (if $i > k$) where $Y_i \subset [-1, 1]$ is a set of cardinality ν_i. Note also that since, by 5.7(1), S_i is a T_{x_i}–surface, and since the T_{x_i} are non-trivial $\pi_1(M)$–trees by Proposition 5.3, we have $S_i \neq \emptyset$ for each $i \in \{1, \ldots, n\}$, and hence $\nu_i > 0$ for each i. We may therefore assume the isotopic modifications of the S_i to have been made in such a way that $0 \in Y_i$ for each i. Hence $S_i \supset F_1$ for $i \leq k$, and $S_i \supset F_2$ for $i > k$.

There exist generators x and y of the rank–2 free group $\pi_1(K)$ such that the conjugacy class of the commutator $[x, y]$ is represented by a map $c\colon S^1 \to K$ which has the property that $\#(c^{-1}(p)) \leq 2$ for all points p at which K is locally Euclidean. We regard c as a map of S^1 to M, ie, a closed curve in M, and denote by γ an element of the conjugacy class in $\pi_1(M)$ which is represented by c. Since K is π_1–injective in the essential surface $F \subset M$ we have $\gamma \neq 1$.

We consider the function $I_\gamma\colon X_0 \to \mathbb{C}$. Since $K \subset F_2 \subset S_i$ for each $i > k$, and since γ is a commutator in $\mathrm{im}(\pi_1(K) \to \pi_1(M))$, it follows from 5.7(4) that $Z_{x_i}(I_\gamma - 2) \geq t_{S_i}(K)$ for $i = k+1, \ldots, n$. But since for $i > k$ we have $S_i = h_2(F_2 \times Y_i)$, where $0 \in Y_i \subset [-1, 1]$ and $\#(Y_i) = \nu_i$, it follows from 2.5 that $t_{S_i}(K) \geq \nu_i \cdot t_{F_2}(K) = \nu_i \cdot t$. Hence

$$Z_{x_i}(I_\gamma - 2) \geq \nu_i \cdot t \tag{7.2.2}$$

for $i = k+1, \ldots, n$.

The function I_γ is non-constant by Lemma 5.6. We can therefore estimate its degree by using (7.2.2): we have

$$\deg I_\gamma = \deg(I_\gamma - 2) \geq \sum_{i=k+1}^{n} Z_{x_i}(I_\gamma - 2) \geq \sum_{i=k+1}^{n} \nu_i \cdot t,$$

and hence

$$\deg I_\gamma \geq N_2 t. \tag{7.2.3}$$

We shall compare the lower bound (7.2.3) for $\deg I_\gamma$ with an upper bound calculated in terms of poles. The definition of I_γ shows that it has no poles on the affine curve X_0. For an ideal point x_i with $i > k$, it follows from (7.2.2) that I_γ takes the value 2 at x_i and hence does not have a pole. Hence

$$\deg I_\gamma = \sum_{i=1}^{k} \Pi_{x_i}(I_\gamma). \tag{7.2.4}$$

For $i \leq k$, it follows from 5.7(1) and 5.10 that

$$2\Pi_{x_i}(I_\gamma) \leq \Delta_M(c, S_i). \tag{7.2.5}$$

We shall give an upper bound for $\Delta_M(c, S_i)$. Since $S_i = h_1(F_1 \times Y_i)$, where $\#(Y_i) = \nu_i$, and since the polyhedron K meets F_1 transversally, it is clear that K can be isotoped in M to a polyhedron K_i for which $\#(K_i \cap S_i) = \nu_i \cdot \#(K_i \cap F_1) = \nu_i \ell$. If we write $K_i = \eta_i(K)$, where $\eta_i\colon M \to M$ is isotopic to the identity, then $c\colon S^1 \to M$ is homotopic to $c_i = \eta_i \circ c$. Since $\#(c^{-1}(p)) \leq 2$ for every non-vertex point $p \in K$, we have

$$\#(c_i^{-1}(S_i)) \leq 2\nu_i \ell. \tag{7.2.6}$$

But the definition of geometric intersection number implies that

$$\Delta_M(c, S_i) \leq \#(c_i^{-1}(S_i)). \tag{7.2.7}$$

By combining the inequalities (7.2.5)–(7.2.7), summing, and comparing with (7.2.4), we find that

$$2 \deg I_\gamma \leq \sum_{i=1}^{k} 2\nu_i \ell$$

ie,

$$\deg I_\gamma \leq N_1 \ell. \tag{7.2.8}$$

From (7.2.3) and (7.2.8) it follows that

$$N_2 t \leq N_1 \ell. \tag{7.2.9}$$

Combining (7.2.9) with (7.2.1) we obtain

$$\frac{\|\alpha_1\|}{\|\alpha_2\|} \leq \frac{m_2 \ell}{m_1 t}$$

as required. □

Remark 7.3 In the proof of Theorem 7.2 we fixed strict essential surfaces $S_1, \ldots, S_n$ in M satisfying conditions (1)–(4) of Proposition 5.7 and then concluded from the hypotheses that each component of each S_i is isotopic to either F_1 or F_2. The proof remains valid as long as M contains two surfaces F_1 and F_2 such that each component of each S_i is isotopic to either F_1 or F_2. The results in the rest of this section and the subsequent sections, would also remain valid under this much weaker, but much more technical hypothesis.

Nathan Dunfield's computations of A–polynomials suggest that there are many examples of knot manifolds in lens spaces that have such a pair of surfaces.

(See `http://www.its.caltech.edu/~dunfield/snappea/tables/A-polys`.)

Theorem 7.4 *Suppose that M is a non-exceptional two-surface knot manifold. Let F_1 and F_2 be representatives of the two isotopy classes of connected strict essential surfaces. Suppose that F_1 and F_2 intersect transversally and that no component of $\mathcal{A} = F_1 \cap F_2$ is a homotopically trivial simple closed curve. Then for $i = 1, 2$, every component of $\mathrm{int}(F_i - \mathcal{A})$ is an open disk or an open annulus.*

Remark 7.5 It is a standard observation that if two transversally intersecting essential surfaces F_1 and F_2 are chosen within their rel-boundary isotopy classes so as to minimize the number of components of $F_1 \cap F_2$, then no component of $F_1 \cap F_2$ is a homotopically trivial simple closed curve.

Proof of Theorem 7.4 By symmetry it suffices to prove that every component of $\mathrm{int}(F_2 - \mathcal{A})$ is a disk or annulus. The hypothesis that no component of $F_1 \cap F_2$ is a homotopically trivial simple closed curve implies that every component of $\mathrm{int}(F_2 - \mathcal{A})$ is π_1–injective in F_2, and hence in M since F_2 is essential.

By Theorem 7.2, the surfaces F_1 and F_2 have non-empty boundaries and, if we let α_1 and α_2 denote boundary classes of F_1 and F_2, we have $\frac{\|\alpha_1\|}{\|\alpha_2\|} \leq \kappa(F_1, F_2)$. Suppose that some component C of $\mathrm{int}(F_2 - \mathcal{A})$ is not an open disk or annulus. Then $\chi(C) < 0$. Hence C contains a connected 1–dimensional polyhedron K of Betti number 2 which is π_1–injective in C and hence in M. Since $K \subset \mathrm{int}(F_2 - \mathcal{A})$, we have $\#(K \cap F_1) = 0$. By the definition of $\kappa(F_1, F_2)$ it follows that $\kappa(F_1, F_2) = 0$, and hence that $\frac{\|\alpha_1\|}{\|\alpha_2\|} \leq 0$. This is impossible, because $\|\alpha_1\|$ and $\|\alpha_2\|$ are norms of non-zero elements of $H_1(M; \partial R)$ and are therefore strictly positive real numbers. □

Corollary 7.6 *Suppose that M is a non-exceptional two-surface knot manifold. Let F_1 and F_2 be representatives of the two isotopy classes of connected strict essential surfaces. Let s_i denote the boundary slope of F_i and let M_i denote the number of boundary components of F_i. Then for $i = 1, 2$ we have*

$$\chi(F_i) \geq \frac{-m_1 m_2 \Delta(s_1, s_2)}{2}.$$

Proof Since the number of arc components of $F_1 \cap F_2$ is $m_1 m_2 \Delta(s_1, s_2)/2$, and since each component of $F_i - F_1 \cap F_2$ has non-negative Euler characteristic by Theorem 7.4, we have

$$\chi(F_i) = \chi(F_i - F_1 \cap F_2) - \frac{m_1 m_2 \Delta(s_1, s_2)}{2} \geq \frac{-m_1 m_2 \Delta(s_1, s_2)}{2}. \qquad \square$$

Theorem 7.7 *Suppose that K is a non-exceptional two-surface knot in a closed, orientable 3–manifold Σ with $\pi_1(\Sigma)$ cyclic. Set $M = M(K)$ and let F_1 and F_2 be representatives of the two isotopy classes of connected strict essential surfaces in the non-exceptional two-surface knot manifold M. Let $\mathfrak{m}$ denote the meridian slope of M, let s_i denote the boundary slope of F_i (which is well-defined by 6.10) and assume that $s_2 \neq \mathfrak{m}$. Set $q_i = \Delta(s_i, \mathfrak{m})$ (so that q_i is the denominator of s_i in the sense of 1.13), and set $\Delta = \Delta(s_1, s_2)$ (so that $\Delta \neq 0$ by 6.10). Then*

$$\frac{q_1^2}{\Delta} \leq 2\kappa(F_1, F_2).$$

Proof The inequality $\frac{q_1^2}{\Delta} \leq 2\kappa(F_1, F_2)$ holds trivially if $s_1 = \mathfrak{m}$, since the left hand side is 0 in this case. Thus we may assume that $s_1 \neq \mathfrak{m}$, and by hypothesis we have $s_2 \neq \mathfrak{m}$. Hence $\mathfrak{m}$ is not the boundary slope of any strict essential surface. We choose a meridian class μ (in the sense of 1.13). Thus $\langle \mu \rangle = \mathfrak{m}$, and μ is not a boundary class of any strict essential surface.

If $\|\cdot\|$ denotes the norm on $H_1(\partial M, \mathbb{R})$ associated to a principal component X_0 of $X(\pi_1(M))$, we have

$$\frac{\|\alpha_1\|}{\|\alpha_2\|} \leq \kappa(F_1, F_2),$$

where α_i is a boundary class of F_i, so $s_i = \langle \alpha_i \rangle$.

Let L denote the lattice $H_1(\partial M; \mathbb{Z})$ in the vector space $V = H_1(\partial M; \mathbb{R})$. The homological intersection pairing $L \times L \to \mathbb{Z}$ has a unique extension to an alternating bilinear form $\omega\colon V \times V \to \mathbb{R}$. Thus for all $\alpha, \beta \in L$ we have $\Delta(\langle\alpha\rangle, \langle\beta\rangle) = |\omega(\alpha, \beta)|$.

The alternating form ω determines an area element on V. If v and v' are linear independent vectors in V then the parallelogram with vertices $\{0, v, v', v + v'\}$ has area $|\omega(v, v')|$; in particular, a fundamental parallelogram for L has area 1. Furthermore, the parallelogram with vertices $\{v, v', -v, -v'\}$ has area $2|\omega(v, v')|$. (If e_1 and e_2 form a basis of L then $|\omega(e_1, e_2)| = 1$ and $|\omega(xe_1 + ye_2, ze_1 + we_2)| = |zw - yz|$, so if we use the basis (e_1, e_2) to identify L with $\mathbb{Z}^2$ and V with $\mathbb{R}^2$ then we recover the standard area element on $\mathbb{R}^2$.)

We set $r = \min_{0 \neq \lambda \in L} \|\alpha\|$. According to 5.7(3) the set $B_r = \{v \in V | \|v\| \leq r\}$ is a convex polygon. Since $\|\cdot\|$ is a norm, B_r is *balanced*, ie, invariant under the involution $x \mapsto -x$. The definition of r implies that $\operatorname{int} B_r$ contains no non-zero points of the lattice L. It therefore follows from [10, Theorem 6.21] that the area of B_r is at most 4.

According to Proposition 5.7(3), each vertex of B_r, regarded as a vector, is a scalar multiple of a boundary class of a strict essential surface. As s_1 and s_2 are the only slopes that arise as boundary slopes of strict essential surfaces, there are at most two lines through the origin that contain vertices of B_r. As B_r is a balanced convex polygon with non-empty interior, it must be a parallelogram, in which two opposite vertices are multiples of an element $\alpha_1 \in L$ such that $\langle\alpha_1\rangle = s_1$, and the other two vertices are multiples of $\alpha_2 \in L$ with $\langle\alpha_2\rangle = s_2$.

The Dehn-filled manifold $M(\mathfrak{m})$ is homeomorphic to Σ, and hence $\pi_1(M(\mathfrak{m}))$ is cyclic. As $\mathfrak{m} = \langle\mu\rangle$ is not the boundary slope of a strict essential surface, it then follows from Corollary 1.1.4 of [4] (see Remark 5.8) that $\|\mu\| = r$, so that μ lies on the boundary of B_r. Since μ is a primitive element of L, but is not a boundary class of any strict essential surface, it cannot be a vertex of B_r. Thus μ lies on an edge of B_r, whose endpoints v_1 and v_2 must respectively be scalar multiples of α_1 and α_2. We may suppose the signs of the α_i to be chosen in such a way that $\alpha_i = \frac{\|\alpha_i\|}{\|\mu\|}v_i$ for $i = 1, 2$.

As the parallelogram B_r has vertices $\pm v_1$ and $\pm v_2$, its area is $2|\omega(v_1, v_2)|$. Hence

$$|\omega(v_1, v_2)| \leq 2.$$

We have $\mu = (1-t)v_1 + tv_2$ for some $t \in (0, 1)$. Hence $\omega(v_1, \mu) = t\omega(v_1, v_2)$. This gives

$$q_1 = |\omega(\alpha_1, \mu)| = \|\alpha_1\||\omega(v_1, \mu)| = t\|\alpha_1\||\omega(v_1, v_2)|.$$

On the other hand,

$$\Delta = |\omega(\alpha_1, \alpha_2)| = \|\alpha_1\|\|\alpha_2\||\omega(v_1, v_2)|.$$

Hence
$$\frac{q_1^2}{\Delta} = t^2\frac{\|\alpha_1\|}{\|\alpha_2\|}|\omega(v_1, v_2)| \leq 2\frac{\|\alpha_1\|}{\|\alpha_2\|} \leq 2\kappa(F_1, F_2),$$

and the proof is complete. □

8 Short subgraphs I

This section provides the graph-theoretical background needed for Theorem 9.5, which is the first of our main concrete results about two-surface knots in manifolds with cyclic fundamental group.

Notation 8.1 If Γ_0 is a subgraph of a graph Γ, and r is a non-negative integer, we shall denote by $N_r(\Gamma_0)$ the union of Γ_0 with the tracks of all edge paths of length at most r whose initial vertices lie in Γ_0. Thus $N_r(\Gamma_0)$ is again a subgraph of Γ for each $r \geq 0$, and $N_0(\Gamma_0) = \Gamma_0$. If v is a vertex of Γ we shall set $B_r(v) = N_r(\{v\})$.

Lemma 8.2 *Suppose that Γ is a finite graph in which every vertex has valence at least 3, and that Γ_0 is a subgraph of Γ. Set $\Gamma_1 = N_1(\Gamma_0)$. For $i = 0, 1$, let n_i denote the number of valence–1 vertices of Γ_i. Assume that $n_1 < 2n_0$. Then there is a subset t of $|\Gamma_1| - |\Gamma_0|$ with the following properties.*

(1) *The set t is a union of vertices and (open) edges of Γ_1, and is closed in the subspace topology of $|\Gamma_1| - |\Gamma_0|$.*

(2) *If E_t and V_t denote respectively the number of edges and the number of vertices contained in t, then we have $1 \leq E_t - V_t \leq 2$. Furthermore, if $n_1 < 2n_0 - 2$ then $E_t - V_t = 2$.*

(3) *If w denotes the number of vertices in $\bar{t} - t \subset |\Gamma_0|$, where $\bar{t}$ denotes the closure of t in $|\Gamma|$, we have $\max(w, E_t) \leq 2(E_t - V_t)$.*

Proof Since $\Gamma_1 = N_1(\Gamma_0)$, each edge of Γ_1 has at least one endpoint in Γ_0. Hence if v is a vertex in $|\Gamma_1| - |\Gamma_0|$, no loop based at v can lie in $|\Gamma_1|$.

We first consider the case in which some vertex $v_0 \in |\Gamma_1| - |\Gamma_0|$ has valence at least 3 in Γ_1. Since no loop based at v_0 is contained in $|\Gamma_1|$ we can choose three distinct edges e_1, e_2 and e_3 having v_0 as an endpoint, and the other endpoints of the e_i must lie in $|\Gamma_0|$. It follows that $t = \{v_0\} \cup e_1 \cup e_2 \cup e_3$ satisfies (1). Furthermore, in the notation of (2) we have $E_t = 3$ and $V_t = 1$, so that $E_t - V_t = 2$, and both assertions of (2) are automatically true. If w is defined as in (3), we have $w \leq 3$ since each e_i has $v \in |\Gamma_0|$ as an endpoint, and so (3) holds as well.

For the rest of the argument we assume that every vertex in $|\Gamma_1| - |\Gamma_0|$ has valence at most 2 in Γ_1.

We denote by $\mathcal{T}$ the set of all connected components of $|\Gamma_1| - |\Gamma_0|$ which do not contain valence–1 vertices of Γ_1. We denote by $\mathcal{T}_0 \subset \mathcal{T}$ the set of all connected components of $|\Gamma_1| - |\Gamma_0|$ which contain no vertices whatever, and we set $\mathcal{T}_1 = \mathcal{T} - \mathcal{T}_0$. It is clear that each element τ of $\mathcal{T}_0$ consists of a single edge of Γ_1 (possibly a loop) whose endpoints lie in $|\Gamma_0|$.

Now if τ is an element of $\mathcal{T}_1$, and $v \in \tau$ is a vertex, then in view of the definition of $\mathcal{T}$, and the fact that every vertex in $|\Gamma_1| - |\Gamma_0|$ has valence at most 2 in

Γ_1, the valence of v in Γ_1 must be exactly 2. Since no loop based at v can lie in $|\Gamma_1|$, there are exactly two edges e_1 and e_2 having v as an endpoint, and the other endpoints of the e_i (which may or may not coincide with each other) must lie in $|\Gamma_0|$. As $\{v\} \cup e_1 \cup e_2$ is clearly open and closed in the subspace topology of $|\Gamma_1| - |\Gamma_0|$, we must have $\tau = \{v\} \cup e_1 \cup e_2$.

To summarize, we have shown that each component $\tau \in \mathcal{T}_0$ consists of a single edge whose endpoint lies in $|\Gamma_0|$, and that each $\tau \in \mathcal{T}_1$ consists of a single vertex v and two edges, each of which has one endpoint at v and one in $|\Gamma_0|$. It follows that for each $\tau \in \mathcal{T}$, if we denote by E_τ the number of edges in τ, by V_τ the number of vertices in τ, and by w_τ the number of vertices in $\bar{\tau} - \tau \subset |\Gamma_0|$, then we have

$$E_\tau - V_\tau = 1 \tag{8.2.1}$$

and

$$\max(w_\tau, E_\tau) \leq 2. \tag{8.2.2}$$

We wish to estimate $m = \#(\mathcal{T})$. If τ is any element of $\mathcal{T}_0$ then τ consists of a single edge e, and each of the oriented edges with underlying edge e has its initial vertex contained in $|\Gamma_0|$. If τ is any element of $\mathcal{T}_1$ then τ contains two edges e_1 and e_2, and each e_i has a unique orientation ω_i such that $\mathrm{init}(\omega_i)$ lies in $|\Gamma_0|$. Hence for every $\tau \in \mathcal{T}$ there are exactly two oriented edges whose underlying edges lie in τ and whose initial points lie in τ. Thus if we denote by Ω_0 the set of all oriented edges ω such that ω lies in $|\Gamma_1| - |\Gamma_0|$ and such that no endpoint of $|\omega|$ has valence 1 in $|\Gamma_1|$, then $\#(\Omega_0) = 2m$.

On the other hand, since every vertex of Γ has valence at least 3, every valence–1 vertex of Γ_0 is the initial vertex of at least two oriented edges ω_1, ω_2 in $|\Gamma| - |\Gamma_0|$; since $\Gamma_1 = N_1(\Gamma_0)$, the edge $|\omega_i|$ lies in $|\Gamma_1| - |\Gamma_0|$ for $i = 1, 2$. Hence if Ω denotes the set of all oriented edges in $|\Gamma_1| - |\Gamma_0|$ with initial vertices in Γ_0, we have $\#(\Omega_0) \geq 2n_0$. Now $\Omega - \Omega_0$ consists of all oriented edges whose initial points lie in Γ_0 and whose terminal points are valence–1 vertices of Γ_1. Since each edge of Γ_1 has at least one endpoint in $|\Gamma_0|$, we have $\#(\Omega - \Omega_0) = n_1$. Hence $\#(\Omega_0) \geq 2n_0 - n_1$, ie,

$$2m \geq 2n_0 - n_1. \tag{8.2.3}$$

Since by hypothesis we have $2n_0 - n_1 > 0$, it follows from (8.2.3) that $m > 0$. We set $\beta = \min(2, m)$, so that $1 \leq \beta \leq 2$. We choose a subset $\mathcal{T}' \subset \mathcal{T}$ of cardinality β. We set $t = \bigcup_{\tau \in \mathcal{T}'} \tau$. It is clear that t satisfies (1). If we define E_t and V_t as in (2), we have $E_t = \sum_{\tau \in \mathcal{T}'} E_\tau$ and $V_t = \sum_{\tau \in \mathcal{T}'} V_\tau$. Hence it follows from (8.2.1) that

$$E_t - V_t = \beta.$$

Since $1 \leq \beta \leq 2$ it follows that $1 \leq E_t - V_t \leq 2$. Furthermore, if $n_1 < 2n_0 - 2$ then (8.2.3) implies that $m > 1$ and hence that $\beta = \max(m, 2) = 2$. This proves condition (2) for our choice of t. Finally, using (8.2.2), we find

$$\max(w, E_t) = \max(\sum_{\tau \in \mathcal{T}'} w_\tau, \sum_{\tau \in \mathcal{T}'} E_t) \leq \sum_{\tau \in \mathcal{T}'} \max(w_\tau, E_\tau) \leq 2\beta = 2(E_t - V_t)$$

This proves condition (3). □

Definition 8.3 The *bigirth* of a graph Γ is the infimum of the lengths of all finite connected subgraphs K of Γ with $\chi(|K|) < 0$. Thus the bigirth of Γ is finite if and only if Γ has a component Γ_0 with $\chi(|\Gamma_0|) < 0$.

Lemma 8.4 *Suppose that Γ is a graph in which each vertex has valence at least 3, and that v_0 is a vertex of Γ. For each $r \geq 1$, let m_r denote the number of valence–1 vertices of $B_r(v_0)$, Suppose that s is a positive integer.*

(1) *If $m_{s+1} < 2m_s - 2$, then* $\text{bigirth}(\Gamma) \leq 4s + 4$.

(2) *If $m_{s+1} < 2m_s$ then either* $\text{bigirth}(\Gamma) \leq 4s + 4$, *or $B_{s+1}(v_0)$ contains a connected subgraph H such that $\chi(|H|) \leq 0$,* $\text{length}(H) \leq 2s + 2$, *and $v_0 \in |H|$.*

(3) *If $m_{s+1} < 2m_s$ and $B_s(v_0)$ contains a connected subgraph H such that $\chi(|H|) \leq 0$,* $\text{length}(H) \leq 2s + 2$, *and $v_0 \in |H|$, then* $\text{bigirth}(\Gamma) \leq 4s + 4$.

Proof Note that $B_{s+1}(v_0) = N_1(B_s(v_0))$. If the hypothesis of any of the assertions (1)–(3) holds, then in particular $m_{s+1} < 2m_s$. Thus the hypotheses of Lemma 8.2 hold if we set $\Gamma_0 = B_s(v_0)$, $n_0 = m_s$, $\Gamma_1 = B_{s+1}(v_0)$ and $n_1 = m_{s+1}$. Let $t \subset |B_{s+1}(v_0)| - |B_s(v_0)|$ be a subset satisfying conditions (1)–(3) of Lemma 8.2. As in Lemma 8.2, we let E_t and V_t denote respectively the number of edges and the number of vertices contained in t. We set $W = \bar{t} - t \subset |B_s(v_0)|$, where $\bar{t}$ denotes the closure of t in $|\Gamma|$, and as in 8.2(3) we let w denote the number of vertices in W.

For every vertex $v \in W \subset B_s(v_0)$ we choose an arc A_v in the graph $B_s(v_0)$ having length $\leq s$ and endpoints v_0 and v. Let K denote the connected subgraph of $B_s(v_0)$ with $|K| = \bigcup_{v \in W} |A_v|$. Since by 8.2(3) we have $w \leq 2(E_t - V_t)$, the length of K is at most $2s(E_t - V_t)$. Note that the sets $|K|$ and t are disjoint since $|K| \subset |N_s(v_0)|$.

Now let H denote the subgraph of $N_{s+1}(v_0)$ with $|H| = |K| \cup t$. Since the closure of each component of t meets $|K|$, it follows that $|H|$ is connected. We have $\text{length}(H) = \text{length}\, K + E_t \leq 2s(E_t - V_t) + E_t$, and since $E_t \leq 2(E_t - V_t)$ by 8.2(3), we conclude that

$$\text{length}(H) \leq 2(s+1)(E_t - V_t). \tag{8.4.1}$$

Since $|K|$ and t are disjoint, we have $\chi(|H|) = \chi(|K|) - (E_t - V_t)$, where $\chi(|K|) \leq 1$ since K is a connected graph. Hence

$$\chi(|H|) \leq 1 - (E_t - V_t). \tag{8.4.2}$$

We now prove assertion (1) of Lemma 8.4. If $m_{s+1} < 2m_s - 2$, then according to 8.2(2) we have $E_t - V_t = 2$. Hence (8.4.1) and (8.4.2) give $\text{length}(H) \leq 4s+4$ and $\chi(|H|) < 0$. From the definition of the bigirth it follows that $\text{bigirth}(\Gamma) \leq 4s+4$, and assertion (1) is proved.

To prove assertion (2) we recall that $E_t - V_t$ is equal to 1 or 2 by 8.2(2). If $E_t - V_t = 2$ then (8.4.1) and (8.4.2) again give $\text{length}(H) \leq 4s+4$ and $\chi(|H|) < 0$, so that $\text{bigirth}(\Gamma) \leq 4s+4$. If $E_t - V_t = 1$ then (8.4.1) and (8.4.2) give $\text{length}(H) \leq 2s+2$ and $\chi(|H|) \leq 0$. As the construction of H guarantees that $v_0 \in |H|$, this completes the proof of assertion (2).

To prove assertion (3) we assume that $B_s(v_0)$ has a subgraph H_0 such that $\chi(|H_0|) \leq 0$, $\text{length}(H_0) \leq 2s+2$, and $v_0 \in |H_0|$. We consider the subgraph H' with $|H'| = |H| \cup |H_0|$. Since H has been shown to have length at most $2s+2$, we have

$$\text{length}(H') \leq \text{length}(H) + \text{length}(H_0) \leq 4s+4.$$

On the other hand, we may write $|H'| = |K'| \cup t$, where K' is the subgraph of $B_s(v_0)$ such that $|K'| = |K| \cup |H_0|$. Since K and H_0 are both connected and both have v_0, as a vertex, the graph K' is connected, and hence $\chi(|K'|) \leq \chi(|K|) \leq 0$. Now the sets $|K'|$ and t are disjoint since $|K'| \subset |N_s(v_0)|$. Since $E_t - V_t > 1$ by 8.2.2, it follows that $\chi(|H'|) = \chi(|K'|) - (E_t - V_t) \leq \chi(|K'|) - 1 < 0$. We have shown that K' has length at most $4s-4$ and that $\chi(|K'|) < 0$; in view of the definition it follows that $\text{bigirth}(\Gamma) \leq 4s+4$. This proves (3). ◻

Proposition 8.5 *Suppose that Γ is a non-empty finite graph which has at least two vertices, and in which every vertex has valence at least 3. Then*

$$\text{bigirth}(\Gamma) \leq 4 \log_2 V,$$

where V denotes the number of vertices of Γ.

Proof We first dispose of some degenerate cases. If some component of Γ has a unique vertex v, then since the valence of v is at least 3 there must be at least two loops based at v. Hence $\text{bigirth}(\Gamma) \leq 2$, and since by hypothesis we have $V \geq 2$, the conclusion holds in this case. If some component contains exactly two vertices v_1 and v_2, then there is at least one edge joining v_1 and v_2. Furthermore, since each v_i has valence at least 3, either there are at least three edges joining v_1 and v_2, or there are loops based both at v_1 and at v_2. In either case it follows that $\text{bigirth}(\Gamma) \leq 3$, and again the conclusion holds. Next suppose that some component of Γ has at least three vertices, but that every vertex of Γ lies in a circuit of length at most 2. Then there are distinct vertices v, v_1, v_2 such that v is joined to v_i by an edge e_i for $i = 1, 2$. For $i = 1, 2$, choose a circuit C_i of length ≤ 2 containing v_i, and let H denote the subgraph with $|H| = |C_1| \cup |C_2| \cup \bar{e}_1 \cup \bar{e}_2$, so that $\text{length}(H) \leq 6$. If $C_1 \neq C_2$ then there are two distinct circuits in H, and hence $\chi(|H|) < 0$. If $C_1 = C_2$, then in H we have the circuit C_1 and the arc A with $|A| = \bar{e}_1 \cup \bar{e}_2$, which has its endpoints in C_1 but contains the vertex $v \notin C_1$. Hence we have $\chi(|H|) < 0$ in this subcase as well. Hence $\text{bigirth}(\Gamma) \leq 6$, and since $V \geq 3$ in this case, the conclusion again holds.

Hence we may assume that there is a vertex v of Γ which has valence at least 3 and does not lie in any circuit of length ≤ 2. It follows that the subgraph $B_1(v)$ has at least three vertices of valence 1.

For each $r \geq 1$, we let X_r denote the set of all valence–1 vertices of $B_r(v)$, and we set $m_r = \#(X_r)$. Thus $m_1 \geq 3$. Note that a valence–1 vertex of $B_r(v)$ has valence at least 3 in $B_{r'}(v)$ for every $r' > r$. In particular, the X_r are pairwise disjoint for $r \geq 1$. Since none of the X_r contains v, the number V of vertices of Γ is at least $1 + \sum_{r=1}^{\infty} m_r$, where $m_r = 0$ for large enough r.

Since $m_r = 0$ for large r, there is a smallest non-negative integer s such that $m_{s+1} < 2^{s+1}$. Since $m_1 \geq 3$, we have $s \geq 1$. The minimality of s then implies that $m_r \geq 2^r$ for $r = 2, \ldots, s$. Hence

$$V \geq 1 + \sum_{r=1}^{s} m_r \geq 1 + 3 + \sum_{r=2}^{s} 2^r = 2^{s+1}.$$

This means that $s + 1 \leq \log_2 V$. Hence to prove the proposition it suffices to show that $\text{bigirth}(\Gamma) \leq 4s + 4$.

We distinguish two cases according to whether $m_s \geq 2^s + 1$ or $m_s = 2^s$. First suppose that $m_s \geq 2^s + 1$. Then since $m_{s+1} < 2^{s+1}$, we have $m_{s+1} < 2m_s - 2$. It therefore follows from Lemma 8.4(1) that $\text{bigirth}(\Gamma) \leq 4s + 4$, as required.

Now suppose that $m_s = 2^s$. Note that in this case there must be an integer s', with $1 \leq s' < s$, such that $m_{s'+1} < 2m_{s'}$. Indeed, if we had $m_{r+1} \geq 2m_r$

for $r = 1, \dots, s-1$, we would have $m_s \geq 2^{s-1}m_1 = 3 \cdot 2^{s-1}$, a contradiction to $m_s = 2^s$.

If s' satisfies $1 \leq s' < s$ and $m_{s'+1} < 2m_s$, then by Lemma 8.4(2), there is a subgraph H of $B_{s'+1}(v) \subset B_s(v)$ such that $\chi(|H|) \leq 0$, $\mathrm{length}(H) \leq 2s'+2 < 2s+2$, and $v \in |H|$. On the other hand, we have $2m_s - m_{s+1} > 2$ since $m_s = 2^s$ and $m_{s+1} < 2^{s+1}$. It therefore follows from Lemma 8.4(3) that $\mathrm{bigirth}(\Gamma) \leq 4s+4$ in this case as well. □

Lemma 8.6 *Suppose that Γ is a finite graph with $\chi(|\Gamma|) < 0$. Set $\alpha = \mathrm{length}(\Gamma)/|\chi(\Gamma)|$. Then Γ has a subgraph Γ_0 such that*

1. $\chi(|\Gamma_0|) < 0$,
2. *every vertex of Γ_0 has valence at least 2,*
3. *every component of $|\Gamma_0|$ contains a vertex whose valence in Γ_0 is at least 3,*
4. *if $\mathcal{V}$ denotes the set of all vertices of valence ≥ 3 in Γ_0, every component of $|\Gamma_0| - \mathcal{V}$ is homeomorphic to an open interval and contains at most $\lfloor \alpha \rfloor$ edges.*

Proof Let $\mathcal{G}$ denote the set of all subgraphs G of Γ such that (a) $\chi(|G|) < 0$ and (b) $\mathrm{length}(G)/|\chi(|G|)| \leq \alpha$. Note that $\Gamma \in \mathcal{G}$. Let Γ_0 be a subgraph in $\mathcal{G}$ which is minimal with respect to inclusion. Then Γ_0 satisfies (1), and we shall complete the proof by showing that it satisfies (2)–(4) as well.

If Γ_0 has an isolated vertex v, then the subgraph G defined by $|G| = \Gamma_0 - \{v\}$ satisfies $\chi(|G|) = \chi(|\Gamma|) - 1$ and $\mathrm{length}(G) = \mathrm{length}(\Gamma)$. Hence $G \in \mathcal{G}$, and the minimality of Γ_0 is contradicted. If Γ_0 has a vertex v of valence 1, and if e is the edge of Γ_0 incident to v, then the subgraph G defined by $|G| = |\Gamma_0| - (\{v\} \cup e)$ satisfies $\chi(|G|) = \chi(|\Gamma|)$ and $\mathrm{length}(G) = \mathrm{length}(\Gamma) - 1$. Again it follows that $G \in \mathcal{G}$, in contradiction to the minimality of Γ_0. This proves (2). If (3) fails, it now follows that there is a component C of Γ_0 whose vertices all have valence 2. In this case the subgraph G defined by $|G| = |\Gamma_0| - |C|$ satisfies $\chi(|G|) = \chi(|\Gamma|)$ and $\mathrm{length}(G) < \mathrm{length}(\Gamma)$, and again we have a contradiction to the minimality of Γ_0. This proves (3).

To prove (4), we first note that every vertex in $|\Gamma_0| - \mathcal{V}$ has valence 2. Since every component of Γ_0 has a vertex of valence 3 it follows that every connected component of $|\Gamma_0| - \mathcal{V}$ is homeomorphic to an open interval. Now suppose that some component C of $|\Gamma_0| - \mathcal{V}$ contains $m > \alpha$ edges, and we consider the subgraph G defined by $|G| = |\Gamma_0| - C$. Note that since $\Gamma_0 \in \mathcal{G}$ we have

$\chi(|\Gamma_0|) < 0$; this implies that Γ_0 has at least one vertex of valence ≥ 3, and hence that $\mathrm{length}(G) > 0$.

Now set $c = -\chi(|\Gamma_0|)$, so that $c - 1 = -\chi(|G|)$. We have $c - 1 \geq 0$ since $\chi(|\Gamma_0|) < 0$. Since $\Gamma_0 \in \mathcal{G}$ we have $\mathrm{length}(\Gamma_0)/|\chi(|\Gamma_0|)| \leq \alpha$, ie, $\mathrm{length}(\Gamma_0) \leq c\alpha$. Hence

$$\mathrm{length}(G) = \mathrm{length}(\Gamma_0) - m \leq c\alpha - m < \alpha(c-1). \tag{8.6.1}$$

Since $\mathrm{length}(G) > 0$ it follows from (8.6.1) that $-\chi(|G|) = c - 1 > 0$. From (8.6.1) we then conclude that

$$\mathrm{length}(G) < \alpha|\chi(|G|)|.$$

It now follows that $G \in \mathcal{G}$, and again we have a contradiction to the minimality of Γ_0. This establishes (4). □

Lemma 8.7 *Let Γ be a finite graph with no simply connected components. Then we have $\chi(|\Gamma_0|) \geq \chi(|\Gamma|)$ for any subgraph Γ_0 of Γ.*

Proof In this proof we shall denote the set of vertices of a finite graph G by $\mathcal{V}(G)$, and the set of its edges by $\mathcal{E}(G)$; thus $\chi(G) = \#(\mathcal{V}(G)) - \#(\mathcal{E}(G))$. The conclusion of the lemma is equivalent to the assertion that $\#(\mathcal{V}(\Gamma) - \mathcal{V}(\Gamma_0)) \leq \#(\mathcal{E}(\Gamma) - \mathcal{E}(\Gamma_0))$.

We first consider the special case in which every vertex in the set $\mathcal{V}(\Gamma) - \mathcal{V}(\Gamma_0)$ has valence at least 2 in Γ. Since every edge having an endpoint in the set $\mathcal{V}(\Gamma) - \mathcal{V}(\Gamma_0)$ must belong to the set $\mathcal{E}(\Gamma) - \mathcal{E}(\Gamma_0)$, we have

$$2\#(\mathcal{V}(\Gamma) - \mathcal{V}(\Gamma_0)) \leq \sum_{v \in \mathcal{V}(\Gamma) - \mathcal{V}(\Gamma_0)} \mathrm{valence}(v) \leq 2\#(\mathcal{E}(\Gamma) - \mathcal{E}(\Gamma_0)),$$

which implies the conclusion.

To prove the lemma in general, we use induction on $\#(\mathcal{V}(\Gamma))$. If $\#(\mathcal{V}(\Gamma)) = 0$ the assertion is trivial. Suppose that $n > 0$, that the assertion is true for graphs with $n - 1$ vertices, that Γ is a graph with $\#(\mathcal{V}(\Gamma)) = n$ and having no simply-connected components, and that Γ_0 is a subgraph of Γ. If every vertex in $\mathcal{V}(\Gamma) - \mathcal{V}(\Gamma_0)$ has valence at least 2 in Γ we are in the special case already proved. No vertex of Γ has valence 0 because no component of Γ is simply connected. Hence we may assume that some vertex $v_1 \in \mathcal{V}(\Gamma) - \mathcal{V}(\Gamma_0)$ has valence 1 in Γ. If Γ' denotes the graph obtained from Γ by removing v_1 and the unique edge incident to v_1, then Γ_0 is a subgraph of Γ'. Since $\#(\mathcal{V}(\Gamma)) = n - 1$, the induction hypothesis gives $\chi(\Gamma_0) \geq \chi(\Gamma') = \chi(\Gamma)$. □

Proposition 8.8 *Suppose that Γ is a finite graph such that no component of $|\Gamma|$ is simply connected and $\chi(|\Gamma|) < 0$. Then*

$$\text{bigirth}(\Gamma) \leq 4(\log_2 |2\chi(|\Gamma|)|) \left\lfloor \frac{\text{length}(\Gamma)}{|\chi(|\Gamma|)|} \right\rfloor.$$

Proof Set $\alpha = \text{length}(\Gamma)/|\chi(|\Gamma|)|$. Let Γ_0 be a subgraph of Γ satisfying conditions (1)–(4) of Lemma 8.6. Conditions (2)–(4) imply that there exists a graph Γ_0^* such that $|\Gamma_0^*| = |\Gamma_0|$ and every vertex of Γ_0^* has valence at least 3. (If, as in the statement of condition (4), we let $\mathcal{V}$ denote the set of all vertices of valence at least 3 in Γ_0, then the vertices of Γ_0^* are the vertices in $\mathcal{V}$ and the edges of Γ_0^* are the connected components of $|\Gamma_0| - \mathcal{V}$.)

By condition (1) we have $\chi(|\Gamma_0^*|) < 0$, and in particular $|\Gamma_0^*| \neq \emptyset$. By Proposition 8.5, if V^* denotes the number of vertices of Γ_0^*, we have either $V = 1$ or

$$\text{bigirth}(\Gamma_0^*) \leq 4 \log_2 V^*.$$

Since Γ_0^* has V^* vertices, all of valence at least 3, it must have at least $3V^*/2$ edges. Hence $\chi(|\Gamma_0^*|) \leq V^* - (3V^*/2) = -V^*/2$. On the other hand, $|\Gamma_0^*|$ is homeomorphic to $|\Gamma_0| \subset |\Gamma|$, and by Lemma 8.7 $0 > \chi(|\Gamma_0^*|) = \chi(|\Gamma_0|) \geq \chi(|\Gamma|)$. Hence $V^* \leq |2\chi(|\Gamma|)|$, so that

$$\text{bigirth}(\Gamma_0^*) \leq 4 \log_2 |2\chi(|\Gamma|)|, \tag{8.8.1}$$

provided that $V \neq 1$. However, if $V = 1$, then since the valence of the unique vertex v is at least 3 there must be at least two loops based at v; hence $\text{bigirth}(\Gamma) \leq 2$, whereas $\chi(|\Gamma|) \leq -1$. Hence (8.8.1) holds in all cases. By definition this means that there is a subgraph H^* of Γ_0^* with $\chi(|H|) < 0$ and $\text{length}(H) \leq 4 \log_2 |2\chi(|\Gamma|)|$.

According to condition (4) of Lemma 8.6, every edge of Γ_0^* contains at most $\lfloor \alpha \rfloor$ edges of the subgraph Γ_0 of Γ. Hence if H denotes the subgraph of $\Gamma_0 \subset \Gamma$ with $|H| = |H^*|$, then

$$\text{length}(H^*) \leq \lfloor \alpha \rfloor \, \text{length}(H) \leq 4 \lfloor \alpha \rfloor \log_2 |2\chi(|\Gamma|)|.$$

Since H is in particular a subgraph of Γ with $\chi(|H|) < 0$, it follows that

$$\text{bigirth}(\Gamma) \leq 4 \lfloor \alpha \rfloor \log_2 2|\chi(|\Gamma|)|.$$

This is the conclusion of the proposition. □

9 Slopes and genera I

The goal of this section is to prove Theorem 9.5.

Definition 9.1 Suppose that $\mathcal{A}$ is a properly embedded 1–manifold in a compact orientable 2–manifold F. A finite graph Γ with $|\Gamma| \subset \operatorname{int} F$ will be called a *dual graph* of $\mathcal{A}$ in F if it has the following properties.

(1) Every edge of Γ meets $\mathcal{A}$ transversally.

(2) Every component of $F - \mathcal{A}$ contains a unique vertex of Γ.

(3) There is a bijective correspondence $A \mapsto e_A$ between the components of $\mathcal{A}$ and the edges of Γ, such that for each component A of $\mathcal{A}$ we have $A \cap |\Gamma| = \{m_A\}$, where m_A denotes the midpoint of e_A.

Note that every properly embedded 1–manifold $\mathcal{A}$ in F has a dual graph Γ, and it is unique up to ambient isotopy. Note also that $|\Gamma|$ is a retract of F, so that in particular $|\Gamma|$ is π_1–injective. Furthermore, $|\Gamma|$ has the same number of connected components as F.

Lemma 9.2 *Let Γ denote a dual graph of a properly embedded 1–manifold $\mathcal{A}$ in a compact, connected, orientable 2–manifold F. Suppose that every component of $F - \mathcal{A}$ is a planar surface. Then the first Betti number of $|\Gamma|$ is greater than or equal to the genus of F.*

Proof We shall argue by induction on the number of components of $\mathcal{A}$. If $\mathcal{A} = \emptyset$ the assertion is true because a planar surface has genus 0 by definition. Now suppose that $\mathcal{A}$ has $\nu > 0$ components, and that the lemma is true for all properly embedded 1–manifolds with fewer than ν components in compact, connected, orientable 2–manifolds. Let b denote the first Betti number of Γ, and let g denote the genus of F. Choose a component A of $\mathcal{A}$, let V denote a collar neighborhood of A in F, and let F' denote the closure of $F - V$, let e denote the unique edge of Γ which meets A, and let Γ' denote the subgraph of Γ with $|\Gamma'| = |\Gamma| - e$. Then $\mathcal{A}' = \mathcal{A} - A$ is a properly embedded 1–manifold in F', and Γ' is a dual graph of $\mathcal{A}'$ in F'.

If F' is connected then $|\Gamma'|$ is connected and has first Betti number $b - 1$. Furthermore, in this case the genus of F' is at least $g - 1$. Hence the induction hypothesis implies that $b - 1 \geq g - 1$, so that $b \geq g$. Now suppose that F' is disconnected, let F_1' and F_2' denote its components. For $i = 1, 2$, let Γ_i' denote the component of Γ' such that $|\Gamma_i'| \subset F_i'$, and let g_i' and b_i' denote respectively the genus of F_i' and the first Betti number of $|\Gamma_i'|$. Then $g = g_1' + g_2'$ and

$b = b_1' + b_2'$. But the induction hypothesis gives $b_i' \geq g_i'$ for $i = 1, 2$, and hence $b \geq g$. □

Lemma 9.3 *Suppose that $\mathcal{A}$ is a properly embedded 1–manifold in a compact, connected, orientable 2–manifold F. Let $\mathcal{A}_0$ denote the union of all those components of $\mathcal{A}$ that are arcs. Assume that a dual graph Γ of $\mathcal{A}_0$ in F has finite bigirth. Then there is a compact π_1–injective 1–dimensional polyhedron $K \subset F$ of Betti number 2 such that $\#(K \cap \mathcal{A}) \leq \mathrm{bigirth}(\Gamma)$.*

Proof First of all, observe that if $\mathcal{A}^* \supset \mathcal{A}_0$ denotes the union of all components of $\mathcal{A}$ that are arcs or homotopically non-trivial simple closed curves, then $\mathcal{A} - \mathcal{A}^*$ is contained in a union of disjoint disks $D_1, \ldots, D_k$ which are disjoint from $\mathcal{A}^*$. If $K^* \subset F$ is a compact π_1–injective 1–dimensional polyhedron of Betti number 2 such that $\#(K^* \cap \mathcal{A}^*) \leq \mathrm{bigirth}(\Gamma)$, there is an isotopy of F supported on $D_1 \cup \cdots \cup D_k$ which carries K^* onto a polyhedron K disjoint from $\mathcal{A} - \mathcal{A}^*$, and it follows that $\#(K \cap \mathcal{A}) \leq \mathrm{bigirth}(\Gamma)$. Hence we may assume without loss of generality that $\mathcal{A}$ contains no homotopically trivial simple closed curves.

Since $\mathrm{bigirth}(\Gamma) < \infty$, there is a connected subgraph $\mathcal{H}$ of Γ with Betti number 2 such that $\mathrm{length}(\mathcal{H}) = \mathrm{bigirth}(\Gamma)$. Since the polyhedron $|\Gamma| \subset F$ is π_1–injective, $|\mathcal{H}|$ is also π_1–injective in F. It follows that $\pi_1(F)$ is non-abelian and hence that F is not a torus. Note also that the definition of bigirth implies that $\mathcal{H}$ has minimal length among all connected subgraphs of Γ with Betti number 2; hence $\mathcal{H}$ has no valence–1 vertices. Since $\chi(|\mathcal{H}|) = -1$ it follows that every vertex of $\mathcal{H}$ has valence at least 2, and that no vertex of $\mathcal{H}$ has valence greater than 4.

Since $\mathrm{length}(\mathcal{H}) = \mathrm{bigirth}(\Gamma)$, it follows from the definition of the dual graph that $\#(|\mathcal{H}| \cap \mathcal{A}_0) = \mathrm{bigirth}(\Gamma)$.

Let $C = \mathcal{A} - \mathcal{A}_0$ denote the union of all components of $\mathcal{A}$ that are simple closed curves. Since F is not a torus, any two components of C can cobound at most one annulus. Hence if Z denotes the union of C with all annuli whose boundaries are contained in C, the components of a regular neighborhood of Z are themselves annuli. Clearly Z has a regular neighborhood which is disjoint from $\mathcal{A}_0$. Define the complexity of a regular neighborhood N of Z to be the pair (p, v) where $p = \#(\partial N \cap |\mathcal{H}|)$ and v is the number of vertices of $\mathcal{H}$ which lie in $F - N$. Among all regular neighborhoods of Z that are disjoint from $\mathcal{A}_0$ choose one, $\mathcal{Z}$, which has minimal complexity with respect to lexicographical order. In particular $\mathcal{H}$ and $\partial\mathcal{Z}$ intersect transversally.

It follows from the construction of $\mathcal{Z}$ that $\mathcal{A} \cap \partial\mathcal{Z} = \emptyset$. Another consequence of the construction is that any annulus cobounded by two curves in $\partial\mathcal{Z}$ is itself a component of $\mathcal{Z}$. Note also that since $\mathcal{A}$ contains no homotopically trivial simple closed curves, the annuli that make up $\mathcal{Z}$ are homotopically non-trivial.

Since $|\mathcal{H}|$ is π_1–injective and the components of $\mathcal{Z}$ are annuli, $|\mathcal{H}|$ cannot be contained in $\mathcal{Z}$. If $|\mathcal{H}| \cap \mathcal{Z} = \emptyset$ then $\#(|\mathcal{H}| \cap \mathcal{A}) = \#(|\mathcal{H}| \cap \mathcal{A}_0) = \text{bigirth}(\Gamma)$, and the conclusion of the lemma follows if we set $K = |\mathcal{H}|$. Hence we may assume that there is a component X of $|\mathcal{H}| \cap \overline{F - \mathcal{Z}}$ such that $X \cap \partial\mathcal{Z} \neq \emptyset$.

Consider the case in which $X \supset |E|$ for some circuit E of $\mathcal{H}$. Let $\beta \subset X$ be an arc having one endpoint in $|E|$ and one in $\partial\mathcal{Z}$. Let c denote the component of $\partial\mathcal{Z}$ containing an endpoint of β. In this case we shall show that the polyhedron $K = |E| \cup \beta \cup c$ has the properties stated in the lemma.

Since E is a circuit in the dual graph Γ of $\mathcal{A}_0$, at least one edge of Γ is contained in $|E|$; hence there is a component α of $\mathcal{A}_0$ that meets $|E|$ in exactly one point. Since $c \cap \alpha \subset \mathcal{Z} \cap \mathcal{A}_0 = \emptyset$ it follows that the homology class $[|E|] \in H_1(F; \mathbb{Z}/2\mathbb{Z})$ is not a multiple of $[c]$. But c is homotopically non-trivial in F since $\mathcal{Z}$ is made up of non-trivial annuli, and hence $K = |E| \cup \beta \cup c$ is π_1–injective. On the other hand we have $K \cap \mathcal{A} \subset |\mathcal{H}| \cap \mathcal{A}_0$ since $\partial\mathcal{Z} \cap \mathcal{A} = \emptyset$ and $\mathcal{A} - \mathcal{A}_0 \subset \text{int}(\mathcal{Z})$. Hence $\#(K \cap \mathcal{A}) \leq \#(|\mathcal{H}| \cap \mathcal{A}_0) = \text{bigirth}(\Gamma)$, as required.

There remains the case in which there is no circuit E of $\mathcal{H}$ such that $X \supset |E|$. Then X is homeomorphic to a finite tree. In this case we let β denote an arc in X that joins two endpoints of X. If $X \cap \mathcal{A}_0$ happens to be non-empty, we choose β to contain a point of $X \cap \mathcal{A}_0$; this is possible because every point in a finite tree is contained in an arc joining two endpoints of the tree.

We let c denote the union of all components (there are at most two) of $\partial\mathcal{Z}$ that contain endpoints of β. In this case we shall show that the polyhedron $K = \beta \cup c$ has the properties stated in the lemma. We have $K \cap \mathcal{A} \subset |\mathcal{H}| \cap \mathcal{A}_0$ since $\partial\mathcal{Z} \cap \mathcal{A} = \emptyset$ and $\mathcal{A} - \mathcal{A}_0 \subset \text{int}(\mathcal{Z})$. Hence $\#(K \cap \mathcal{A}) \leq \#(|\mathcal{H}| \cap \mathcal{A}_0) = \text{bigirth}(\Gamma)$, so we need only prove that K is π_1–injective in F.

There are several subcases. If c has two components which do not cobound an annulus in F then K is automatically π_1–injective. If c has two components c_1 and c_2 which do cobound an annulus in F, it follows from the construction of C that c_1 and c_2 are the two boundary components of a single annulus component of $\mathcal{Z}$, say $\mathcal{Z}_0$. Thus K is the union of $\partial\mathcal{Z}_0$ with the properly embedded arc β in $F - \text{int}\, \mathcal{Z}_0$. Since F is not a torus, K is π_1–injective.

In the remaining subcases, c will be connected, and since ∂C is made up of non-trivial annuli, c is a homotopically non-trivial curve. Hence in order to establish π_1–injectivity it suffices to show that there is no arc $\delta \subset C$ such that

$\partial\delta = \partial\beta$ and such that $\delta \cup \beta$ bounds a $D \subset F - \operatorname{int} \mathcal{Z}$. In these subcases we shall assume such an arc δ and disk D exist, and derive a contradiction.

Consider the subcase in which $X \cap \mathcal{A}_0 \neq \emptyset$. It then follows from our choice of β that $\beta \cap \mathcal{A}_0 \neq \emptyset$. Let α denote a component of $\mathcal{A}_0$ that meets β. Since β is contained in the dual graph Γ, it must meet α transversally in exactly one point. But $\delta \cap \alpha \subset \mathcal{Z} \cap \mathcal{A}_0 = \emptyset$, so that $\delta \cup \beta$ meets α transversally in exactly one point. This gives a contradiction if $\delta \cup \beta$ bounds a disk D.

Now suppose that $X \cap \mathcal{A}_0 = \emptyset$. Since $\beta \cup \mathcal{A}_0 \subset X \cap \mathcal{A}_0 = \emptyset$ and $\delta \cup \mathcal{A}_0 \subset \mathcal{Z} \cap \mathcal{A}_0 = \emptyset$, we have $D \cap \mathcal{A}_0 = \emptyset$.

On the other hand, since $X \cap \mathcal{A}_0 = \emptyset$, it follows in particular that X cannot contain any edge of the subgraph $\mathcal{H}$ of Γ. Hence either X is an interior arc of an edge of $\mathcal{H}$, or X is a connected subset of the open star of some vertex v of $\mathcal{H}$, and $v \in X$. In any event, since every vertex of $\mathcal{H}$ has valence 2, 3 or 4, X is either a topological arc, or a cone over a three- or four-point set. If X is an arc we must have $X = \beta$. If X is a cone over a three- or four-point set then the cone point is a vertex v of $\mathcal{H}$ which is an interior point of β. Furthermore either $X = \beta \cup \epsilon_1$ or $X = \beta \cup \epsilon_1 \cup \epsilon_2$, where each ϵ_i is an arc having v as an endpoint and $\epsilon_i \cap \beta = \{v\}$. Note that for each i either ϵ_i is a properly embedded arc in the disk D, or $\epsilon_i \cap D = \{v\}$.

Let $\mathcal{Z}'$ denote a small regular neighborhood of $\mathcal{Z} \cup D$ in F. Then $\mathcal{Z}'$ is also a regular neighborhood of Z, and it is disjoint from $\mathcal{A}_0$ since $\mathcal{Z}$ and D are both disjoint from $\mathcal{A}_0$. If $X = \beta$ we have $\#(\partial\mathcal{Z}' \cap |\mathcal{H}|) = \#(\partial\mathcal{Z} \cap |\mathcal{H}|) - 2$. If X is a cone and one of the arcs ϵ_i is contained in D, then $\#(\partial\mathcal{Z}' \cap |\mathcal{H}|) \leq \#(\partial\mathcal{Z} \cap |\mathcal{H}|) - 2$. If X is a cone over a three-point set and ϵ_1 meets D only at the vertex v, we may choose the regular neighborhood $\mathcal{Z}'$ of $\mathcal{Z} \cup D$ so that $\#(\partial\mathcal{Z}' \cap |\mathcal{H}|) = \#(\partial\mathcal{Z} \cap |\mathcal{H}|) - 1$. If X is a cone over a four-point set and both of the arcs ϵ_1 and ϵ_2 meet D only at the vertex v, then we may choose the regular neighborhood $\mathcal{Z}'$ of $\mathcal{Z} \cup D$ so that $\#(\partial\mathcal{Z}' \cap |\mathcal{H}|) = \#(\partial\mathcal{Z} \cap |\mathcal{H}|)$ but $\mathcal{Z}'$ contains strictly more vertices of $\mathcal{H}$ than $\mathcal{Z}$ does. In all cases $\mathcal{Z}'$ would have lower complexity than $\mathcal{Z}$, contradiction. □

Proposition 9.4 *Suppose that F_1 and F_2 are connected essential surfaces in an irreducible knot manifold M. Suppose that F_1 and F_2 intersect transversally, and that ∂F_1 and ∂F_2 are non-empty and intersect minimally in the sense of 1.8. Assume that every component of $F_2 - (F_1 \cap F_2)$ is a disk or an annulus. For $i = 1, 2$, let g_i, s_i and m_i denote, respectively, the genus, boundary slope and number of boundary components of F_i. Assume that $g_2 \geq 2$. Then there* *is a compact π_1–injective 1–dimensional polyhedron $K \subset F_2$ of Betti number 2 such that*

$$\#(K \cap F_1) \leq \frac{2m_1 m_2 \Delta(s_1, s_2) \log_2(2g_2 - 2)}{g_2 - 1}.$$

In particular, according to Definition 7.1, we have

$$\kappa(F_1, F_2) \leq \frac{2m_2^2 \Delta(s_1, s_2) \log_2(2g_2 - 2)}{g_2 - 1}.$$

Proof Set $\mathcal{A} = F_1 \cap F_2$, so that $\mathcal{A}$ is a properly embedded 1–manifold in F_2. Let $\mathcal{A}_0$ denote the union of all those components of $\mathcal{A}$ that are arcs.

We claim that the components of $F_2 - \mathcal{A}_0$ are disks and annuli. Indeed, if Z is any component of $F_2 - \mathcal{A}_0$, then $Z \cap \mathcal{A}$ is a union of simple closed curve components of $\mathcal{A}$. Each component of $Z - (Z \cap \mathcal{A})$ is a component of $F_2 - \mathcal{A}$, and is therefore a disk or annulus according to the hypothesis. It follows that $\chi(Z) \geq 0$. Since $\partial F_2 \neq \emptyset$, it follows that Z is a disk or an annulus.

In particular the components of $F_2 - \mathcal{A}_0$ are planar surfaces. Hence if Γ denotes a dual graph of $\mathcal{A}_0$ in F, it follows from Lemma 9.2 that the first Betti number of $|\Gamma|$ is greater than or equal $g_2 \geq 2$. We therefore have $\chi(|\Gamma|) < 0$ and $|\chi(|\Gamma|)| \geq g_2 - 1$. Since $g_2 - 1$ and $\chi(|\Gamma|)$ are positive integers, it follows that

$$\frac{\log_2 |2\chi(|\Gamma|)|}{\chi(|\Gamma|)} \leq \frac{\log_2(2(g_2 - 1))}{g_2 - 1}.$$

Moreover, Γ is connected since F_2 is connected.

The length of Γ is equal to the number of components of $\mathcal{A}_0$. Since ∂F_1 and ∂F_2 intersect minimally, it follows that

$$2\,\text{length}(\Gamma) = \#((\partial F_1) \cap (\partial F_2)) = m_1 m_2 \Delta(s_1, s_2).$$

Since $\chi(\Gamma) < 0$ and Γ is connected, it follows from Proposition 8.8 that

$$\text{bigirth}(\Gamma) \leq 4(\log_2 |2\chi(\Gamma)|)\frac{\text{length}(\Gamma)}{|\chi(\Gamma)|} \leq \frac{2\log_2(2(g_2 - 1))}{g_2 - 1} \cdot m_1 m_2 \Delta(s_1, s_2). \tag{9.4.1}$$

But according to Lemma 9.3, there is a compact π_1–injective 1–dimensional polyhedron $K \subset F$ of Betti number 2 such that $\#(K \cap \mathcal{A}) \leq \text{bigirth}(\Gamma)$, and the first assertion of the proposition therefore follows by (9.4.1). Since we must have $t_{F_2}(K) \geq 1$, the second assertion follows from the first one together with Definition 7.1. ☐

Theorem 9.5 *Suppose that K is a non-exceptional two-surface knot in a closed, orientable 3–manifold Σ such that $\pi_1(\Sigma)$ is cyclic. Let $\mathfrak{m}$ denote the meridian slope of K and let F_1 and F_2 be representatives of the two isotopy classes of connected strict essential surfaces in $M(K)$. Let s_i, g_i and m_i denote, respectively, the boundary slope (well-defined by 6.10), the genus and the number of boundary components of F_i. Assume that $s_2 \neq \mathfrak{m}$ and that $g_2 \geq 2$. Set $q_i = \Delta(s_i, \mathfrak{m})$ (so that q_i is the denominator of s_i in the sense of 1.13), and set $\Delta = \Delta(s_1, s_2)$ (so that $\Delta \neq 0$ by 6.10). Then*

$$\left(\frac{q_1}{\Delta}\right)^2 \leq \frac{4m_2^2 \log_2(2g_2 - 2)}{g_2 - 1}.$$

Proof We may assume after an isotopy that F_1 and F_2 intersect transversally, and that ∂F_1 and ∂F_2 intersect minimally in the sense of 1.8. Furthermore, F_1 and F_2 may be assumed to be chosen within their rel-boundary isotopy classes so as to minimize the number of components of $F_1 \cap F_2$. Then no component of $F_1 \cap F_2$ is a homotopically trivial simple closed curve (cf Remark 7.5). Set $\mathcal{A} = F_1 \cap F_2$. It now follows from Theorem 7.4 that every component of $(\operatorname{int} F_i) - \mathcal{A}$ is an open disk or an open annulus. Hence by Proposition 9.4 we have

$$\kappa(F_1, F_2) \leq \frac{2m_2^2 \Delta \log_2(2g_2 - 2)}{g_2 - 1}. \tag{9.5.1}$$

On the other hand, according to Theorem 7.7 we have

$$\frac{q_1^2}{\Delta} \leq 2\kappa(F_1, F_2). \tag{9.5.2}$$

The inequality in the conclusion of Theorem 9.5 follows from (9.5.1) and (9.5.2). □

If K is a knot in a homology 3–sphere with $M(K)$ irreducible then it follows from Remark 3.9 that $M(K)$ has an essential surface whose numerical boundary slope with respect to a standard framing is 0.

Corollary 9.6 *Suppose that K is a knot in a homotopy 3–sphere Σ such that $M(K)$ is irreducible and has only two essential surfaces up to isotopy. Then, with respect to a standard framing, one of these surfaces has numerical boundary slope 0 and the other has numerical boundary slope $r \neq 0$. If $r \neq \infty$ and if the essential surface with boundary slope 0 has genus $g \geq 2$ then*

$$\frac{g - 1}{4 \log_2(2g - 2)} \leq r^2.$$

Proof Any knot in a homology 3–sphere has an essential spanning surface with connected boundary and numerical boundary slope 0. Hence we may denote the two non-isotopic essential surfaces in $M(K)$ by F_1 and F_2 where F_2 is a spanning surface.

The hypotheses imply in particular that $M(K)$ has at most two strict essential surfaces, so one of the conclusions of Theorem 6.7 must hold. Since Σ is a homology sphere, it cannot contain a non-separating torus or a Klein bottle. Moreover $M(K)$ is not a solid torus since it has two non-isotopic essential surfaces. This rules out conclusions (1) and (3a) of Theorem 6.7. Therefore $M(K)$ is either Seifert-fibered over the disk with two singular fibers or a non-exceptional two-surface knot manifold. In the first case Σ is homeomorphic to S^3 and K is a torus knot. If K is the (m,n)–torus knot then $M(K)$ contains an essential annulus of slope $mn \neq 0$ and an essential spanning surface of genus $(m-1)(n-1)/2$. These must be isotopic to F_1 and F_2 respectively. Thus the conclusions hold in this case.

If conclusion (3b) of Theorem 6.7 holds then F_1 and F_2 must be the two non-isotopic strict essential surfaces. If we set $r = p/q$, where p and q are relatively prime, then in the notation of Theorem 9.5 we have $\Delta = p$, $q_1 = q$, $g_2 = g$ and $m_2 = 1$. In particular $r \neq 0$. Furthermore if $r \neq \infty$ then $s_2 \neq \mathfrak{m}$, and the inequality in the statement of the corollary is equivalent to the inequality in the conclusion of Theorem 9.5. □

For future reference (see 11.18), we record the following qualitative consequence of Theorem 9.5.

Corollary 9.7 *There is a positive-valued function $f_0(x)$ of a positive real variable x with the following properties.*

(1) *For every $\epsilon > 0$ we have*

$$\lim_{x\to\infty} x^{1-\epsilon} f_0(x) = 0.$$

(2) *If K is any non-exceptional two-surface knot in a closed, orientable 3–manifold Σ such that $\pi_1(\Sigma)$ is cyclic, and if $\mathfrak{m}$, F_i, g_i, s_i, m_i, q_i and Δ are defined as in the statement of Theorem 9.5, and if $g_2 \geq 2$, then*

$$\left(\frac{q_1}{\Delta}\right)^2 \leq m_2^2 f_0(g_2).$$

10 Short subgraphs II

This section presents the more subtle combinatorial ideas that are needed in the proof of Theorem 11.16, our second main concrete result about two-surface knots in manifolds with cyclic fundamental group.

Definition 10.1 By an *essential arc system* in a compact, orientable surface S we mean a non-empty, properly embedded 1–manifold $\mathcal{A} \subset S$ such that every component of $\mathcal{A}$ is a non-boundary-parallel arc in S. If $\mathcal{A}$ is an essential arc system we shall denote by $\partial_{\mathcal{A}} S$ the union of all boundary components of S which meet $\mathcal{A}$. We shall denote by $\Gamma_{\mathcal{A}}$ the trivalent graph with $|\Gamma_{\mathcal{A}}| = \mathcal{A} \cup \partial_{\mathcal{A}} S$, in which the vertex set is $\partial\mathcal{A}$. The edges of $\Gamma_{\mathcal{A}}$ are the components of $\operatorname{int} \mathcal{A}$, which we call *interior edges*, and the components of $(\partial_{\mathcal{A}} S) - (\partial\mathcal{A})$, which we call *boundary edges*. Note that every vertex of $\Gamma_{\mathcal{A}}$ is an endpoint of a unique interior edge.

In this section we shall often consider subgraphs of $\Gamma_{\mathcal{A}}$. Such a subgraph need not be connected, and it may have vertices of valence 0, 1, 2 and 3. If Γ is a subgraph of $\Gamma_{\mathcal{A}}$ we define the *interior edges* and *boundary edges* of Γ to be, respectively, the interior edges and boundary edges of $\Gamma_{\mathcal{A}}$ that are contained in $|\Gamma|$.

By a *reduced arc system* in a compact, orientable surface S we mean an essential arc system in S such that no two components of $\mathcal{A}$ are parallel.

Proposition 10.2 *Suppose that S is a compact, connected, orientable surface which is not an annulus, that $\mathcal{A}$ is a reduced arc system in S and that Γ_0 is a subgraph of $\Gamma_{\mathcal{A}}$. Let ν denote the number of interior edges of Γ_0. Then Γ_0 has a subgraph Γ_1 such that*

(1) *$|\Gamma_1|$ is π_1–injective in S,*

(2) *$|\Gamma_1|$ contains every vertex of Γ_0 and every boundary edge of Γ_0, and*

(3) *the number of interior edges of Γ_1 is at least $\nu/3$.*

Proof We may assume $\nu > 0$. Let $\mathcal{A}_0$ denote the union of those components of $\mathcal{A}$ that are contained in Γ_0, so that ν is the number of components of $\mathcal{A}_0$. In particular $\mathcal{A}_0 \neq \emptyset$. We let μ denote the number of components of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_0)$ that are topological open disks. We begin by showing that $\mu \leq 2\nu/3$. If N is a regular neighborhood of $\mathcal{A}_0$ in S, if $D_1, \ldots, D_\mu$ are the components of $\overline{S - N}$ that are closed disks, and if c_i is the number of frontier components of D_i then, since each frontier component of $D_1 \cup \cdots \cup D_\mu$ is a component of the frontier of N, we have $\sum_{i=1}^{\mu} c_i \leq 2\nu$. We have $c_i \neq 0$ for every i because S is connected and $\mathcal{A}_0 \neq \emptyset$. We have $c_i \neq 1$ for every i because no component of $\mathcal{A}_0 \subset \mathcal{A}$ is a boundary-parallel arc. We have $c_i \neq 2$ for every i because S is not an annulus and no two components of $\mathcal{A}_0 \subset \mathcal{A}$ are parallel arcs. Hence $c_i \geq 3$ for every i, and so $3\mu \leq \sum_{i=1}^{\mu} c_i \leq 2\nu$. This proves that $\mu \leq 2\nu/3$.

To prove the proposition, it suffices to find a submanifold $\mathcal{A}_1$ which is a union of components of $\mathcal{A}_0$, has at least $\nu/3$ components, and has the property that no component of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_1)$ is an open disk. Indeed, if $\mathcal{A}_1$ has these properties, and if we define Γ_1 to be the subgraph of Γ_0 made up of all vertices and boundary edges of Γ_0 and of those interior edges which are components of $\operatorname{int} \mathcal{A}_1$, then Γ_1 clearly satisfies conditions (1)–(3) of the proposition.

For any properly embedded 1–manifold $\mathcal{B} \subset S$ let $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ denote respectively the number of components of $\mathcal{B}$ and the number of components of $(\operatorname{int} S) - (\operatorname{int} \mathcal{B})$ that are topological open disks. We shall say that $\mathcal{B}$ is *admissible* if it a union of components of $\mathcal{A}_0$ and if $\nu_{\mathcal{B}} - \mu_{\mathcal{B}} = \nu - \mu$. Clearly $\mathcal{A}_0$ is itself admissible. If $\mathcal{B}$ is any admissible submanifold, then

$$\nu_{\mathcal{B}} \geq \nu_{\mathcal{B}} - \mu_{\mathcal{B}} = \nu - \mu \geq \nu - \frac{2\nu}{3} = \nu/3,$$

ie, $\mathcal{B}$ has at least $\nu/3$ components.

Among all admissible submanifolds choose one, $\mathcal{A}_1$, for which the number of components is as small as possible. We shall complete the proof by showing that no component of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_1)$ is an open disk.

Suppose that some component D of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_1)$ is an open disk. Choose any component A of the frontier of D in S. Then A is also a component of $\mathcal{A}_1$. Set $\mathcal{B} = \mathcal{A}_1 - A$. If D is the only component of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_1)$ whose closure contains A, then $D \cup A$ is a component of $(\operatorname{int} S) - (\operatorname{int} \mathcal{B})$ homeomorphic to an open annulus, and the other components of $(\operatorname{int} S) - (\operatorname{int} \mathcal{B})$ are the components $\neq D$ of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_1)$. If C is a second component of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_1)$ whose closure contains A, then $D \cup A \cup C$ is a component of $(\operatorname{int} S) - (\operatorname{int} \mathcal{B})$ homeomorphic to C, and the other components of $(\operatorname{int} S) - (\operatorname{int} \mathcal{B})$ are the components $\neq C, D$ of $(\operatorname{int} S) - (\operatorname{int} \mathcal{A}_1)$. In either case it follows that $\mu_{\mathcal{B}} = \mu_{\mathcal{A}_1} - 1$. Since $\nu_{\mathcal{B}} = \nu_{\mathcal{A}_1} - 1$, we have $\nu_{\mathcal{B}} - \mu_{\mathcal{B}} = \nu_{\mathcal{A}_1} - \mu_{\mathcal{A}_1} = \nu - \mu$, so that $\mathcal{B}$ is admissible. This contradicts the minimality of $\mathcal{A}_1$, and the proof is complete. □

Definition 10.3 Let S be a compact, orientable surface. By a *labeling* of an essential arc system $\mathcal{A}$ we mean a surjective map ι from the set of interior edges of $\mathcal{A}$ to some finite set I, called the *label set* of ι. If e is an interior edge of $\Gamma_{\mathcal{A}}$, we shall refer to $\iota(e)$ as the *label* of e. For every vertex v of $\Gamma_{\mathcal{A}}$ we define the *label* of v, denoted $\iota(v)$, to be the label of the unique interior edge having v as an endpoint. For every $i \in I$ we define the *multiplicity of* i with respect to the labeling of ι, denoted θ^ι_i, to be the number of arcs in $\mathcal{A}$ with label i. Thus $\sum_{i\in I}\theta^\iota_i = \#(\mathcal{A})$. For every vertex v and every edge e we shall write $\theta^\iota(v) = \theta^\iota(\iota(v))$ and $\theta^\iota(e) = \theta^\iota(\iota(e))$.

If ι is a labeling of an essential arc system $\mathcal{A}$, then for every subgraph Γ of $\Gamma_{\mathcal{A}}$ we define

$$\theta^\iota(\Gamma) = \min_e \theta^{\mathcal{A}}_e$$

where e ranges over the interior edges of Γ.

Let ι be a labelling of an essential arc system $\mathcal{A}$, with label set I. We define a *system of weights* for $\mathcal{A}$ with respect to ι to be a positive real-valued function λ on the set I. If λ is a system of weights, and Γ is a subgraph of $\Gamma_{\mathcal{A}}$, we shall denote by $\lambda(\Gamma)$ the quantity $\sum_{v\in\mathcal{V}}\lambda(\iota(v))$, where $\mathcal{V}$ denotes the set of vertices of Γ.

Definition 10.4 For every real number $\tau > 1$, we define a positive real-valued function ϕ_τ on the set of all positive integers by

$$\phi_\tau(n) = \frac{1}{\tau - 1}\min_m \tau^{2m+2} n^{1/m},$$

where m ranges over all positive integers. Note that ϕ_τ is monotone increasing, and that for every $\epsilon > 0$ we have

$$\lim_{n\to\infty}\frac{\phi_\tau(n)}{n^\epsilon} = 0.$$

Remark 10.5 One can show using calculus that for a given $\tau > 1$ the function ϕ_τ grows roughly like $e^{c\sqrt{\ln x}}$ for some constant c. We will not give a precise statement or proof of this here, but we will prove and use a more technical result along these lines, Lemma 11.13.

Lemma 10.6 *Suppose that S is a compact, connected, orientable surface which is not an annulus, that $\mathcal{A}$ is a reduced arc system in S, and that ι is a labeling for $\mathcal{A}$ with label set I. Set $\theta_i = \theta^\iota_i$ for every $i \in I$. Set $\Theta = \#(\mathcal{A}) = \sum_{i\in I}\theta^\iota_i$ and $\theta_\infty = \max_{i\in I}\theta_i$. Let a real number $q > 1$ be given, and set*

$$\tau = \frac{7q-1}{q-1}.$$

Suppose that E^* is a set of interior edges of $\Gamma_{\mathcal{A}}$. For every $i \in I$ set $E_i^* = E^* \cap E_i$ and $\theta_i^* = \#(E_i^*)$, and suppose that $\theta_i^* \le \theta_i/q$ for every $i \in I$. Suppose that λ is a system of weights for $\mathcal{A}$ with respect to ι. Then there is a subgraph Γ_1 of $\Gamma_{\mathcal{A}}$ such that

(1) *$|\Gamma_1|$ is π_1–injective in S,*

(2) *$|\Gamma_1|$ contains no edge in E^*,*

(3) *the Euler characteristic $\chi(|\Gamma_1|)$ is strictly negative, and*

(4) $$\frac{\lambda(\Gamma_1)}{\theta^\iota(\Gamma_1)\cdot|\chi(|\Gamma_1|)|} < \phi_\tau(\theta_\infty)\frac{\sum_{i\in I}\lambda(i)}{\Theta}.$$

Proof For every subset J of I, let us denote by E_J the set of all interior edges e of $\Gamma_{\mathcal{A}}$ such that $\iota(e) \in J$. We set $\Theta_J = \#(E_J) = \sum_{i\in J}\theta_i$. We also set $E_J^* = E_J \cap E^*$ and $\Theta_J^* = \#(E_J) = \sum_{i\in J}\theta_i^*$.

For every $J \subset I$, let us denote by V_J the set of all vertices v of $\Gamma_{\mathcal{A}}$ such that $\iota(v) \in J$. If a vertex v is an endpoint of an interior edge e, the definition of $\iota(v)$ implies that $\iota(v) = \iota(e)$. Since each vertex of $\Gamma_{\mathcal{A}}$ is an endpoint of a unique interior edge, and each edge has two endpoints, it follows that

$$\#(V_J) = 2\Theta_J \tag{10.6.1}$$

for every $J \subset I$.

According to the definition of ϕ_τ (10.4), we may fix a positive integer m such that

$$\phi_\tau(\theta_\infty) = \frac{\tau^{2m+2}}{\tau-1}\theta_\infty^{1/m}.$$

We set $A = \tau^{m+1}/(\tau-1)$, $\alpha = A^{-1}$, and $\omega = (\sum_{i\in I}\lambda(i))/\Theta$.

By hypothesis we have $\theta_i^* \le \theta/q$ for every $i \in I$. Summing this over the labels in any given subset J of I, we find

$$\Theta_J^* \le \Theta_J/q. \tag{10.6.2}$$

We decompose the label set I as a disjoint union

$$I = I_0 \sqcup \cdots \sqcup I_m,$$

where

$I_0 = \{i \in I : \lambda(i) > A\omega\theta_i\}$,
$I_j = \{i \in I - I_0 : \theta_\infty^{(m-j)/m} < \theta_i \le \theta_\infty^{(m-j+1)/m}\}$ for $j = 1, \dots, m-1$, and
$I_m = \{i \in I - I_0 : 1 \le \theta_i \le \theta_\infty^{1/m}\}$.

We set $\Theta_j = \Theta_{I_j}$ and $\Theta_j^* = \Theta_{I_j}^*$ for $j = 0, \ldots, m$. Thus we have $\Theta_j = \sum_{i \in I_j} \theta_i$ and $\Theta = \sum_{j=0}^{m} \Theta_j$; similarly, $\Theta_j^* = \sum_{i \in I_j} \theta_i^*$ and $\Theta^* = \sum_{j=0}^{m} \Theta_j^*$.

Using the definitions of I_0 and ω we find that

$$\begin{aligned} A\omega\Theta_0 = \sum_{i \in I_0} A\omega\theta_i &\le \sum_{i \in I_0} \lambda(i) \\ &\le \sum_{i \in I} \lambda(i) = \omega\Theta, \end{aligned}$$

so that

$$\Theta_0 \le \alpha\Theta. \tag{10.6.3}$$

On the other hand, since

$$\sum_{j=0}^{m} \tau^j = \frac{\tau^{m+1} - 1}{\tau - 1} < A,$$

we have

$$\sum_{j=0}^{m} \Theta_j = \Theta = A\alpha\Theta > \sum_{j=0}^{m} \tau^j \alpha\Theta,$$

and hence $\Theta_j > \tau^j \alpha\Theta$ for some $j \in \{0, \ldots, m\}$. Let k denote the smallest index for which $\Theta_k > \tau^k \alpha\Theta$. It follows from (10.6.3) that $k > 0$.

We set $I^+ = I_k \cup \cdots \cup I_m$ and $I^- = I_0 \cup \cdots \cup I_{k-1}$, so that $I = I^+ \sqcup I^-$. We define a subgraph Γ_0 of $\Gamma_{\mathcal{A}}$ to consist of all vertices in V_{I^+}, all interior edges in $E_{I_k} - E_{I_k}^*$, and all boundary edges whose endpoints both lie in V_{I^+}. Since an interior edge in $E_{I_k} - E_{I_k}^*$ has its endpoints in $V_{I_k}* \subset V_{I^+}$, it follows that Γ_0 is indeed a subgraph of $\Gamma_{\mathcal{A}}$.

The number of interior edges of Γ_0 is $\Theta_k - \Theta_k^*$. It therefore follows from Proposition 10.2 that there is a subgraph Γ_1 of Γ_0 such that $|\Gamma_1|$ is π_1–injective in S, contains every vertex of Γ_0 and every boundary edge of Γ_0, and contains at least $(\Theta_k - \Theta_k^*)/3$ interior edges. In particular Γ_1 satisfies condition (1) of Lemma 10.6. Since the edge set of $\Gamma_0 \supset \Gamma_1$ is $E_{I_k} - E_{I_k}^*$, the subgraph Γ_1 satisfies condition (2) as well. We shall complete the proof by showing that it also satisfies conditions (3) and (4).

For any interior edge e of Γ_1 we have $e \in E_{I_k}$ and hence $\iota(e) \in I_k$; since we have shown that $k > 0$, the definition of the I_j then implies that $\theta_e = \theta_{\iota(e)} \ge \theta_\infty^{(m-k)/m}$. Hence

$$\theta^\iota(\Gamma_1) \ge \theta_\infty^{(m-k)/m}. \tag{10.6.4}$$

We next turn to the estimation of $\lambda(\Gamma_1)$. Since $\mathcal{V}_{I+}$ is the vertex set of Γ_1, we have

$$\begin{aligned}\lambda(\Gamma_1) &= \sum_{v\in\mathcal{V}_{I+}} \lambda(\iota(v)) \\ &= \sum_{i\in I^+} \lambda(i)\cdot\#(V_i) = 2\sum_{i\in I^+}\lambda(i)\theta_i,\end{aligned}$$

where the last step follows by applying (10.6.1) with $J = I^+$. Since we have shown that $k > 0$, we have $I^+ \cap I_0 = \emptyset$. In view of the definition of I_0 it follows that $\lambda(i) \le A\omega\theta_i$ for every $i \in I^+$. Hence

$$\lambda(\Gamma_1) \le 2A\omega \sum_{i\in I^+} \theta_i^2.$$

On the other hand, the definition of the I_j shows that for every $i \in I^+$ we have $\theta_i \le \theta_\infty^{(m-k+1)/m}$. Hence

$$\lambda(\Gamma_1) \le 2A\omega\theta_\infty^{(m-k+1)/m} \sum_{i\in I^+} \theta_i \le 2A\omega\theta_\infty^{(m-k+1)/m} \sum_{i\in I} \theta_i,$$

ie,

$$\lambda(\Gamma_1) \le 2A\omega\theta_\infty^{(m-k+1)/m}\Theta. \qquad (10.6.5)$$

Now we turn to the estimation of $\chi(|\Gamma_1|)$. First note that

$$\chi(|\Gamma_1|) = \beta - \gamma, \qquad (10.6.6)$$

where β denotes the number of simply connected components of the set $\mathcal{B} = |\Gamma_1| \cap \partial S = |\Gamma_0| \cap \partial S$ and γ denotes the number of interior edges of Γ_1. According to the construction of Γ_1 we have $\gamma \ge (\Theta_k - \Theta_k^*)/3$. By taking $J = I_k$ in (10.6.2) we find that $\Theta_k^* \le \Theta_k/q$, and so $\gamma \ge (q-1)\Theta_k/3q$. But by our choice of k we have $\Theta_k > \tau^k\alpha\Theta$. Hence

$$\gamma > \frac{(q-1)\tau^k}{3q}\alpha\Theta. \qquad (10.6.7)$$

To estimate β, we let β' denote the number of components of the set $\mathcal{B}' = (\partial S) - \mathcal{B}$, and note that $\beta \le \beta'$ (with equality holding unless some component of ∂S is contained in $\mathcal{B}'$). According to our construction of Γ_0, the set $\mathcal{B}$ consists of the vertices in the set V_{I+} and the boundary edges of $\Gamma_{\mathcal{A}}$ that have both endpoints in V_{I+}. Thus every component of $\mathcal{B}'$ contains at least one vertex of V_{I-}, and therefore

$$\beta \le \beta' \le \#(V_{I-}) = 2\Theta_{I^-},$$

where the last step follows by applying (10.6.1) with $J = I^+$. But since we defined k to be the smallest index for which $\Theta_k > \tau^k\alpha\Theta$, we have $\Theta_j \le \tau^j\alpha\Theta$ for $j = 0, \dots, k-1$. Hence

$$\Theta_{I^-} = \sum_{j=0}^{k-1} \Theta_j \le \sum_{j=0}^{k-1} \tau^j \alpha\Theta = \frac{\tau^k - 1}{\tau - 1}\alpha\Theta,$$

and so
$$\beta \le \frac{2(\tau^k - 1)}{\tau - 1}\alpha\Theta.$$

Combining this with (10.6.6) and (10.6.7) we find that

$$\chi(|\Gamma_1|) = \beta - \gamma < \frac{2(\tau^k - 1)}{\tau - 1}\alpha\Theta - \frac{(q-1)\tau^k}{3q}\alpha\Theta.$$

Since $\tau = (7q-1)/(q-1)$, this last inequality simplifies to

$$\chi(|\Gamma_1|) < -\frac{2\alpha\Theta}{\tau - 1}. \tag{10.6.8}$$

It follows from (10.6.8) that Γ_1 satisfies condition (3) of Lemma 10.6. Furthermore, it follows from (10.6.4), (10.6.5) and (10.6.8) that

$$\begin{aligned}\frac{\lambda(\Gamma_1)}{\theta^{\mathcal{A}}(\Gamma_1) \cdot |\chi(|\Gamma_1|)|} &< 2A\frac{\omega\theta_\infty^{(m-k+1)/m}\Theta}{\theta_\infty^{(m-k)/m}(2\alpha\Theta/(\tau-1))}\\ &= (\tau - 1)A^2\omega\theta_\infty^{1/m}\\ &= \frac{\tau^{2m+2}}{\tau - 1}\theta_\infty^{1/m}\omega = \phi_\tau(\theta_\infty)\omega,\end{aligned}$$

which in view of the definition of ω gives condition (4) of Lemma 10.6. ☐

Remark 10.7 If a graph Γ_1 satisfies the conclusions of Lemma 10.6, and if Γ_1' denotes the subgraph of Γ_1 such that $|\Gamma_1'|$ is union of all non-simply-connected components of $|\Gamma_1|$, then conclusions (1)–(4) also hold when Γ_1 replaced by Γ_1'. (To verify condition (4) observe that $\lambda(\Gamma_1') \le \lambda(\Gamma_1)$, $\chi(|\Gamma_1'|) \le \chi(|\Gamma_1|)$ and $\theta^\iota(\Gamma_1') \ge \theta^\iota(\Gamma_1)$.) Thus Lemma 10.6 remains true if we add the condition

(5) *No component of* $|\Gamma_1|$ *is simply-connected.*

Definition 10.8 If $\mathcal{A}^0$ is any essential arc system in a compact, orientable surface S, we may define an equivalence relation on the set of components of $\mathcal{A}^0$ in which two components are equivalent if and only if they are parallel in S. The equivalence classes for this relation will be called $\mathcal{A}^0$*–parallelism classes.* An essential arc system $\mathcal{A}$ will be called a *reduction* of $\mathcal{A}^0$ if $\mathcal{A} \subset \mathcal{A}^0$ and if every $\mathcal{A}^0$–parallelism class contains exactly one component of $\mathcal{A}$. It is clear that a reduction always exists, that it is unique up to isotopy, and that it is itself a reduced arc system.

If $\mathcal{A}$ is a reduction of an essential arc system $\mathcal{A}^0$, we define the $\mathcal{A}^0$*–width* of any interior edge e of $\Gamma_{\mathcal{A}}$ to be the cardinality of the $\mathcal{A}^0$–parallelism class containing the component $\bar{e}$ of $\mathcal{A}^0$.

If $\mathcal{A}$ is a reduction of an essential arc system $\mathcal{A}^0$, we have $|\Gamma_{\mathcal{A}}| \subset |\Gamma_{\mathcal{A}^0}|$. Indeed, $\Gamma_{\mathcal{A}^0}$ has a subgraph $\Gamma'_{\mathcal{A}}$ which is a subdivision of a subgraph of $\Gamma_{\mathcal{A}}$, in the sense that every vertex of $\Gamma_{\mathcal{A}}$ is a vertex of $\Gamma'_{\mathcal{A}}$, and every edge of $\Gamma_{\mathcal{A}}$ is a union of edges and vertices of $\Gamma'_{\mathcal{A}}$. It follows that for every subgraph Γ of $\Gamma_{\mathcal{A}}$ there is a unique subgraph Γ^0 of $\Gamma_{\mathcal{A}^0}$ such that $|\Gamma^0| = |\Gamma|$. We shall refer to Γ^0 as the subgraph of $\Gamma_{\mathcal{A}^0}$ *associated to* Γ.

Lemma 10.9 *Suppose that S is a compact, connected, orientable surface which is not an annulus, that $\mathcal{A}^0$ is an essential arc system in S, and that $\mathcal{A}$ is a reduction of $\mathcal{A}^0$. Suppose that ι is a labeling for $\mathcal{A}$ with label set I. For every interior edge e of $\Gamma_{\mathcal{A}}$, let $w(e)$ denote the $\mathcal{A}^0$–width of e. Define a weight system for $\mathcal{A}$ with respect to ι as follows: for each label $i \in I$, set $\lambda(i) = \max_e w(e)$, where e ranges over all interior edges of $\Gamma_{\mathcal{A}}$ with label i. Suppose that Γ is a subgraph of $\Gamma_{\mathcal{A}}$, and let Γ^0 denote the subgraph of $\Gamma_{\mathcal{A}^0}$ associated to Γ. Then we have*

$$\text{length}(\Gamma^0) \leq \frac{3}{2}\lambda(\Gamma).$$

Proof For each vertex v of $\Gamma_{\mathcal{A}}$ we set $w(v) = w(e)$, where e is the unique edge of $\Gamma_{\mathcal{A}}$ having v as an endpoint. The definition of the $\lambda(i)$ implies that $w(v) \leq \lambda(\iota(v))$ for every vertex v.

Since $\mathcal{A}$ is a reduction of $\mathcal{A}^0$, each arc in $\mathcal{A}^0$ is parallel to $\bar{e}$ for a unique interior edge e of $\Gamma_{\mathcal{A}}$. Thus if for each interior edge e of $\Gamma_{\mathcal{A}}$ we let $\mathcal{F}_e$ denote the union of all arcs in $\mathcal{A}^0$ that are parallel to $\bar{e}$, then $\mathcal{A}^0$ is the disjoint union of the sets $\mathcal{F}_e$ as e ranges over the interior edges of $\Gamma_{\mathcal{A}}$. By definition the number of components of $\mathcal{F}_e$ is the width $w(e)$. For each e, since $\mathcal{F}_e$ is a family of parallel arcs and S is not an annulus, there is a topological disk or arc $R_e \subset S$ such that $\mathcal{F}_e \subset R_e$ and $\partial R_e \subset \mathcal{F}_e \cup \partial S$. Furthermore, $R_e \cap \partial S$ consists of two possibly degenerate arcs. If e and e' are distinct interior edges we have $R_e \cap R_{e'} = \emptyset$. We set $\mathcal{R} = \bigcup_e R_e$, where e ranges over the interior edges of $\Gamma_{\mathcal{A}}$. We have $\mathcal{A}^0 \subset \mathcal{R}$.

If e is an interior edge of $\Gamma_{\mathcal{A}}$, each of the two arcs that make up $R_e \cap \partial S$ contains exactly one endpoint of e. Hence if we set $\mathcal{B} = \mathcal{R} \cap \partial S$, each component of $\mathcal{B}$

contains a unique vertex of $\Gamma_{\mathcal{A}}$. We shall denote by B_v the component of $\mathcal{B}$ containing a given vertex v of $\Gamma_{\mathcal{A}}$. Note that since $\mathcal{A}^0 \subset \mathcal{R}$, every vertex of $\Gamma_{\mathcal{A}^0}$ lies in $\mathcal{R}$. Note also that since $\mathcal{F}_e$ consists of $w(e)$ arcs for each interior edge e of $\Gamma_{\mathcal{A}}$, it follows that for each vertex v of $\Gamma_{\mathcal{A}}$ the arc B_v contains exactly $w(v)$ vertices of $\Gamma_{\mathcal{A}^0}$, and hence contains exactly $w(v)-1$ edges of $\Gamma_{\mathcal{A}^0}$.

Let us fix an orientation of each component of ∂S. For every vertex v of $\Gamma_{\mathcal{A}}$ we may write B_v in a unique way as a union of arcs B_v^+ and B_v^-, one or both of which may be degenerate, such that and v is the negative endpoint of B_v^+, and the positive endpoint of B_v^-, with respect to the orientation of the component of ∂S containing g. It follows that $B_v^+ \cap B_v^- = \{v\}$. In particular, if b_v^+ and b_v^- denote the number of edges of $\Gamma_{\mathcal{A}^0}$ contained in B_v^+ and B_v^- respectively, we have

$$b_v^+ + b_v^- = w(v) - 1.$$

For every boundary edge g of $\Gamma_{\mathcal{A}}$, let us denote by $v_+(g)$ and $v_-(g)$ the positive and negative endpoints of g with respect to the orientation of the component of ∂S containing g. (It may happen that $v_+ = v_-$.) We have $\bar{g} \cap \mathcal{B} = B^-_{v^+(g)} \cup B^+_{v^-(g)}$. In particular, $B^-_{v^+(g)} \cup B^+_{v^-(g)}$ contains all the vertices of $\Gamma_{\mathcal{A}^0}$ in $\bar{g}$. Since each component of $\mathcal{B}$ is a (possibly degenerate) topological arc containing exactly one vertex of $\Gamma_{\mathcal{A}}$, the arcs $B^-_{v^+(g)}$ and $B^+_{v^-(g)}$ are distinct, and hence $g - (B^-_{v^+(g)} \cup B^+_{v^-(g)})$ is an open topological arc; as it contains no vertices of $\Gamma_{\mathcal{A}^0}$, it must be an edge of $\Gamma_{\mathcal{A}^0}$. Hence the number of edges of $\Gamma_{\mathcal{A}^0}$ contained in g is $b^-_{v^+(g)} + b^+_{v^-(g)} + 1$.

It now follows that if G and G^0 respectively denote the sets of boundary edges of the subgraph Γ of $\Gamma_{\mathcal{A}}$ and of the associated subgraph Γ^0 of $\Gamma_{\mathcal{A}^0}$, then

$$\#(G^0) \leq \sum_{g\in G}(b^-_{v^+(g)} + b^+_{v^-(g)} + 1) = \#(G) + \sum_{g\in G} b^-_{v^+(g)} + \sum_{g\in G} b^+_{v^-(g)}.$$

Note that $v^+(g)$ and $v^-(g)$ are vertices of Γ for every $g \in G$, and that for a given vertex v of Γ there is at most one edge g such that $v = v^-(g)$, and at most one edge g' such that $v = v^-(g')$. Hence if V denotes the vertex set of Γ, we have $\sum_{g\in G} b^-_{v^+(g)} \leq \sum_{v\in V} b_v^-$ and $\sum_{g\in G} b^+_{v^-(g)} \leq \sum_{v\in V} b_v^+$, and therefore

$$\begin{aligned}\#(G^0) &\leq \#(G) + \sum_{v\in V} b_v^- + \sum_{v\in V} b_v^+ = \#(G) + \sum_{v\in V}(b_v^- + b_v^+)\\ &= \#(G) + \sum_{v\in V}(w(v) - 1) = (\#(G) - \#(V)) + \sum_{v\in V} w(v).\end{aligned}$$

But we have observed that $w(v) \leq \lambda(\iota(v))$, and since $\Gamma \cap \partial S$ is a subgraph of a triangulated 1–manifold we have $\#(G) \leq \#(V)$. Hence

$$\#(G^0) \leq \sum_{v \in V} \lambda(\iota(v)) = \lambda(\Gamma).$$

Finally, if E denotes the set of interior edges of Γ, then since no two edges in E can have a common endpoint, we have

$$\#(E) \leq \frac{1}{2}\#(V) \leq \frac{1}{2}\sum_{v \in V} \lambda(\iota(v)) = \frac{1}{2}\lambda(\Gamma),$$

so that

$$\mathrm{length}(\Gamma^0) = \#(G^0) + \#(E) \leq \frac{3}{2}\lambda(\Gamma).$$

□

Proposition 10.10 *Suppose that S is a compact, connected, orientable surface which is not an annulus, that $\mathcal{A}^0$ is an essential arc system in S, and that $\mathcal{A}$ is a reduction of $\mathcal{A}^0$. Let a real number $q > 1$ be given, and set*

$$\tau = \frac{7q-1}{q-1}.$$

Suppose that ι is a labeling for $\mathcal{A}$ with label set I. Set $\theta_i = \theta_i^\iota$ for every $i \in I$. Set $\Theta = \#(\mathcal{A}') = \sum_{i \in I} \theta_i^\iota$ and $\theta_\infty = \max_{i \in I} \theta_i$. Suppose that E^* is a set of interior edges of $\Gamma_{\mathcal{A}}$. For every $i \in I$ set $E_i^* = E^* \cap E_i$ and $\theta_i^* = \#(E_i^*)$, and suppose that $\theta_i^* \leq \theta_i/q$ for every $i \in I$. For every interior edge e of $\Gamma_{\mathcal{A}}$, let $w(e)$ denote the $\mathcal{A}^0$–width of e. For each label $i \in I$, set $\lambda(i) = \max_e w(e)$, where e ranges over all interior edges of $\Gamma_{\mathcal{A}}$ with label i. Then $\Gamma_{\mathcal{A}}$ has a subgraph K such that

1. *$|K|$ is π_1–injective in S,*
2. *$|K|$ contains no edge in E^*,*
3. *the first Betti number of $|K|$ is equal to 2,*
4. *K has no vertices of valence ≤ 1, and*
5. *if K^0 is the subgraph of $\Gamma_{\mathcal{A}^0}$ associated to K, we have*

$$\frac{\mathrm{length}(K^0)}{\theta^\iota(K)} < 6\phi_\tau(\theta_\infty)(\log_2(2\Theta))\frac{\sum_{i \in I} \lambda_i}{\Theta}.$$

Proof If for every $i \in I$ we define $\lambda(i)$ as in the statement of Proposition 10.10, then λ is a weight system for $\mathcal{A}$ with respect to ι. Applying Proposition 10.6 with this choice of the weight system λ, we fix a subgraph Γ_1 of $\Gamma_{\mathcal{A}}$ such that conditions (1)–(5) of 10.6 and 10.7 hold. We denote by Γ_1^0 the subgraph of $\Gamma_{\mathcal{A}^0}$ associated to Γ_1.

According to Lemma 10.9 we have $\mathrm{length}(\Gamma_1^0) \le \frac{3}{2}\lambda(\Gamma_1)$. Furthermore, it is apparent that $\chi(\Gamma_1^0) = \chi(\Gamma_1)$. Hence condition (4) of 10.6 implies that

$$\frac{\mathrm{length}(\Gamma_1^0)}{\theta^\iota(\Gamma_1)\cdot|\chi(|\Gamma_1^0|)|} < \frac{3}{2}\phi_\tau(\theta_\infty)\frac{\sum_{i\in I}\lambda(i)}{\Theta}. \tag{10.10.1}$$

On the other hand, since $\chi(|\Gamma_1^0|) = \chi(|\Gamma_1|) < 0$ by condition (3) of 10.6, and since Γ_1^0 has no simply connected components by condition (5) of 10.7, it follows from Proposition 8.8 that

$$\mathrm{bigirth}(\Gamma_1^0) \le 4(\log_2|2\chi(\Gamma_1)|)\left\lfloor\frac{\mathrm{length}(\Gamma_1^0)}{|\chi(|\Gamma_1|)|}\right\rfloor.$$

Hence Γ_1^0 has a subgraph H such that $\chi(|H|) < 0$ and

$$\mathrm{length}(H) \le 4(\log_2|2\chi(|\Gamma_1|)|)\left\lfloor\frac{\mathrm{length}(\Gamma_1^0)}{|\chi(|\Gamma_1|)|}\right\rfloor. \tag{10.10.2}$$

After possibly replacing H by a smaller subgraph (which cannot increase its length), we may assume that H is connected, has Betti number 2 and has no valence–1 vertices. Since Γ_1^0 is a subdivision of Γ_1, the absence of vertices of valence ≤ 1 in H implies that $|H| = |K|$ for some subgraph K of Γ_1. Thus H is the subgraph of $\Gamma_{\mathcal{A}^0}$ associated to K; for consistency with the statement of Proposition 10.10 we shall write $K^0 = H$. Since Γ_1 satisfies conditions (1) and (2) of 10.6, it now follows that K satisfies conditions (1) and (2) of 10.10. Our choice of K also guarantees that it satisfies conditions (3) and (4) of 10.10.

Since K is a subgraph of Γ_1, it follows from the definitions that $\theta^\iota(K) \ge \theta^\iota(\Gamma_1)$. Combining this observation with the inequalities (10.10.1) and (10.10.2), we deduce that

$$\frac{\mathrm{length}(K_0)}{\theta^\iota(\Gamma_1)} < 6\phi_\tau(\theta_\infty)(\log_2|2\chi(|\Gamma_1|)|)\frac{\sum_{i\in I}\lambda_i}{\Theta}. \tag{10.10.3}$$

To estimate the factor $\log_2|\chi(|\Gamma_1|)|$ in (10.10.3), note that since Γ_1 is a subgraph of $\Gamma_{\mathcal{A}}$ it has at most Θ interior edges. Hence $\chi(|\Gamma_1|) \ge \chi(|\Gamma_1|\cap\partial S) - \Theta$. But $|\Gamma_1 \cap \partial S|$ is a subpolyhedron of a triangulated 1–manifold and must therefore have non-negative Euler characteristic, so that

$$|\chi(|\Gamma_1|)| \le \Theta. \tag{10.10.4}$$

Condition (5) of 10.10 follows immediately from (10.10.3) and (10.10.4). □

11 Slopes and genera II

The goal of this section is to prove Theorem 11.16.

Definitions 11.1 If M is a manifold of arbitrary dimension, we define a *proper path* in M to be a map $\alpha\colon I \to M$ such that $\alpha(\partial I) \subset \partial M$. A *proper homotopy (of paths)* in M is a homotopy $H\colon (I \times I) \to M$ such that $H((\partial I) \times I) \subset \partial M$. Two proper paths α and β in M are *properly homotopic* in M, or *properly M–homotopic*, if there is a proper homotopy $H\colon I \times I \to M$ such that $H_0 = \alpha$ and $H_1 = \beta$.

If A is a properly embedded arc in a manifold M, a *parametrization* of A, ie, a homeomorphism $\alpha\colon I \to A$, is a proper arc in M. We shall say that two properly embedded arcs are *properly homotopic* if they admit properly homotopic parametrizations.

Now suppose that F is an essential surface in a compact, orientable, irreducible 3–manifold M. A reduced homotopy $H\colon (I \times I, I \times \partial I) \to (M, F)$ will be termed *proper* if the homotopy $H\colon (I \times I) \to M$ is proper. A proper reduced homotopy may be regarded as a map of triples $H\colon (I \times I, I \times \partial I, \partial I \times I) \to (M, F, \partial M)$. By a *proper reduced homotopy of length 0* we will mean a homotopy $H\colon (I \times I, I \times \partial I) \to (F, \partial F)$.

Lemma 11.2 *Suppose that F is an essential surface in an irreducible knot manifold M, and suppose that α and β are proper paths in F which are properly M–homotopic. Then there is a proper reduced homotopy H in (M, F) such that $H_0 = \alpha$ and $H_1 = \beta$.*

Proof If M is a solid torus then F must be a disk, so that α and β are homotopic in F. Thus in this case we may take H to be a length–0 homotopy. We may therefore restrict attention to the case in which ∂M is π_1–injective.

Let $J\colon I \times I \to M$ be a proper homotopy such that $J_0 = \alpha$ and $J_1 = \beta$. For $i = 0, 1$, let $\gamma_i\colon I \to \partial M$ be the path defined by $\gamma_i(t) = J(i, t)$. We first consider the case in which γ_1 is fixed-endpoint homotopic in ∂M to a path in ∂F. In this case we may assume that $\gamma_1(I) \subset \partial F$. The path γ_0 is fixed-endpoint homotopic to a composition γ_0' of the paths α, γ_1 and the inverse path $\bar\beta$ of β. We have $\gamma_0'(I) \subset F$ and $\gamma_0(I) \subset \partial M$. Applying Lemma 2.1, with γ_0' in place of α, we conclude that γ_0' is fixed-endpoint homotopic in F to a path in ∂F. Hence γ_0 is fixed-endpoint homotopic in M to a path in ∂F, and so after modifying the map J, without changing its values on $(I \times \partial I) \cup (\{1\} \times I)$, we may assume that $J(\partial(I \times I)) \subset F$. Since F is π_1–injective in M, there is a map $H\colon I \times I \to F$ which agrees with J on the boundary of the disk $I \times I$. Then H is a length–0 homotopy from α to β, and the conclusion holds in this case. If we assume that γ_0 is fixed-endpoint homotopic in ∂M to a path in ∂F, the argument is precisely similar.

Now suppose that neither γ_0 nor γ_1 is fixed-endpoint homotopic in ∂M to a path in ∂F. After modifying the γ_i within their fixed-endpoint homotopy classes in ∂M, we may assume that each γ_i is transverse to ∂F, so that $\gamma_i^{-1}(F)$ is a finite set $\{t_{i,0} \dots, t_{i,n_i}\}$, where $0 = t_{i,0} < \cdots < t_{i,n_i} = 1$. For $i = 0, 1$ and for $j = 1, \dots, n_i$, let $\delta_{i,j} \colon I \to \partial M$ denote a path which is a reparametrization of $\gamma_i|_{[t_{i,j-1}, t_{i,j}]}$. Since ∂M is a torus, and since in particular no γ_i is fixed-endpoint homotopic in ∂M to a path in ∂F, we may assume the γ_i to have been chosen within their fixed-endpoint homotopy classes in ∂M in such a way that no $\delta_{i,j}$ is fixed-endpoint homotopic in ∂M to a path in ∂F.

If some $\delta_{i,j}$ is fixed-endpoint homotopic in M to a path δ' in F, then by applying Lemma 2.1, with δ' in place of α, we conclude that δ' is fixed-endpoint homotopic in F to a path in ∂F. Hence $\delta_{i,j}$ is fixed-endpoint homotopic in M to a path in ∂F. Since ∂M is π_1–injective, it follows that $\delta_{i,j}$ is fixed-endpoint homotopic in ∂M to a path in ∂F, a contradiction. Hence no $\delta_{i,j}$ is fixed-endpoint homotopic in M to a path in F.

After further modifications of the map J, which do not change its values on $\partial(I \times I)$, we may assume that that $J|_{(I \times \operatorname{int} I)}$ is transverse to F. If some component C of $J^{-1}(F)$ is a simple closed curve, then the π_1–injectivity of F implies that $J|_C$ is homotopically trivial in F. Hence if $D \subset I \times I$ is the disk bounded by C, we may modify J on a small neighborhood of D to obtain a map $J' \colon I \times I \to M$, agreeing with J on $\partial(I \times I)$, such that $J|_{(I \times \operatorname{int} I)}$ is transverse to F and such that $(J')^{-1}(F)$ has fewer components than $J^{-1}(F)$. After a finite number of such modifications we may assume that every component of $J^{-1}(F)$ is an arc.

If some component of $J^{-1}(F)$ has both its endpoints in the same component $\{i\} \times I$ of $(\partial I) \times I$, then there is a disk $D \subset I \times I$ such that $D \cap \partial(I \times I) \subset \{i\} \times I$ and $\operatorname{frontier} D$ is a component of $J^{-1}(F)$. Among all disks with these properties, we may suppose D to be chosen so as to be minimal with respect to inclusion. If we set $A = \operatorname{frontier} D$, we then have $\partial A = \{t_{i,j-1}, t_{i,j}\}$ for some j with $1 < j < n_i$. The map $J|_D$ defines a fixed-endpoint homotopy from $\delta_{i,j}$ to a path in F which is a reparametrization of $J|_A$. This is a contradiction. Hence every component of $J^{-1}(F)$ is an arc which has one endpoint in $\{0\} \times I$ and one in $\{1\} \times I$.

It follows that by pre-composing J with a self-homeomorphism of $I \times I$ which is the identity on $I \times \partial I$, we obtain a homotopy $H \colon I \times I \to M$ such that $H^{-1}(F)$ has the form $I \times Y$ for some finite set $Y \subset I$. Hence H is a composition of basic homotopies $H^{(1)}, \dots, H^{(n)}$. The fact that no $\delta_{i,j}$ is fixed-endpoint homotopic in M to a path in F implies that $H^{(1)}, \dots, H^{(n)}$ are all essential. The fact that

$J|_(I \times \operatorname{int} I)$ is transverse to F implies that, given a transverse orientation of F, for each $i \in \{1, \ldots, n-1\}$ there is an element ω of $\{-1, +1\}$ such that H^i ends on the ω side and H^{i+1} starts on the $-\omega$ side. Thus H is the required reduced homotopy in (M, F) from α to β. □

Lemma 11.3 *Suppose that F is a transversally oriented essential surface in an irreducible knot manifold M. Then there exist a finite set $Y \subset S^1$ and a homeomorphism $J\colon S^1 \times S^1 \to \partial M$ such that $J^{-1}(\partial F) = S^1 \times Y$. Furthermore, if J and Y have this property, and if $H\colon (I \times I, I \times \partial I, \partial I \times I) \to (M, F, \partial M)$ is a proper reduced homotopy of length $k > 0$, then there is a proper reduced homotopy $H'\colon (I \times I, I \times \partial I, \partial I \times I) \to (M, F, \partial M)$ such that*

(1) *H' has length k, starts on the same side as H and ends on the same side as H,*
(2) *$H'_0 = H_0$,*
(3) *H'_1 is properly F–homotopic to H_1, and*
(4) *for each $s \in \{0, 1\}$, the path $t \mapsto H(s, t)$ is an immersion of I in $J(\{c_s\} \times S^1)$ for some $c_s \in S^1$.*

Proof The first assertion of the lemma, about the existence of J, follows from the fact that the components of ∂F are disjoint homotopically non-trivial simple closed curves on the torus ∂M. To prove the second assertion, about the reduced proper homotopy H, we argue by induction on the length k of H. If $k = 1$ then H is an essential basic homotopy in (M, F); hence for $s = 0, 1$ the path $t \mapsto H(s, t)$ is a basic essential path (1.7) in the pair $(\partial M, \partial F)$, which is homeomorphic via J to $(S^1 \times S^1, S^1 \times Y)$. But for any basic essential path α in $(S^1 \times S^1, S^1 \times Y)$, there is a homotopy $A\colon I \times I \to M$, constant on $0 \in I$, such that for every $s \in I$ the path A_s is a basic essential path in $(\partial M, \partial F)$, and such that $A_0 = \alpha$ and A_1 is an immersion of I in $J(c \times S^1)$ for some $c \in S^1$. The existence of the required homotopy H' therefore follows from the homotopy extension property for polyhedra.

Now assume that $k > 1$ and that the assertion is true for reduced proper homotopies of length $k - 1$. We write H as a composition of k essential basic homotopies $H^1, \ldots, H^k$ in such a way that, for each $j \in \{1, \ldots, n-1\}$ there is an element ω_j of $\{-1, +1\}$ such that H^j ends on the ω_j side and H^{j+1} starts on the $-\omega_j$ side. A composition H^* of $H^1, \ldots, H^{k-1}$ is a reduced proper homotopy of length $k - 1$. After applying the induction hypothesis to H^* and using the homotopy extension property, we may assume that for each $s \in \{0, 1\}$, the path $t \mapsto H^*(s, t)$ is an immersion of I in $J(\{c_s\} \times S^1)$ for some $c_s \in S^1$.

According to the case $k=1$ of our assertion, which has already been proved, there is a basic essential homotopy $(H^k)'$, starting on the $-\omega_{k-1}$ side and ending on the same side as H^k, such that $(H^k)'_0 = H^k_0 = H{*}_1$, $(H^k)'_1$ is properly F–homotopic to $H^k_1 = H_1$, and $t \mapsto (H^k)'(s,t)$ is an immersion of I in $J(\{c_s\} \times S^1)$. We define H' to be a composition of H^* and $(H^k)'$. Since $t \mapsto H^*(s,t)$ and $t \mapsto (H^k)'(s,t)$ are immersions of I in $J(\{c_s\} \times S^1)$, and since H^* ends on the ω_{k-1} side and H^k starts on the ω_{k-1} side, it follows that $t \mapsto H'(s,t)$ is also an immersion of I in $J(\{c_s\} \times S^1)$, and the induction is complete. □

Proposition 11.4 *Suppose that F is a transversally oriented essential surface in an irreducible knot manifold M. Suppose that $H, H'\colon (I\times I, I\times\partial I, \partial I\times I) \to (M, F, \partial M)$ are proper reduced homotopies with* $\mathrm{length}(H) = \mathrm{length}(H') > 0$. *Suppose that H_0 and H'_0 are properly F–homotopic and that H and H' start on the same side (in the sense of 1.7). Then H and H' end on the same side, and H_1 and H'_1 are properly F–homotopic.*

Proof We may assume without loss of generality that $H_0 = H'_0$. According to the first assertion of Lemma 11.3, we may fix a finite set $Y \subset S^1$ and a homeomorphism $J\colon S^1 \times S^1 \to \partial M$ such that $J^{-1}(\partial F) = S^1 \times Y$. According to the second assertion of 11.3, the proof of the proposition reduces to the case in which, for each $s \in \{0,1\}$, the paths $t \mapsto H(s,t)$ and $t \mapsto H'(s,t)$ are immersions of I into submanifolds $J(\{c_s\} \times S^1)$ and $J(\{c'_s\} \times S^1)$ of ∂M for some $c_s, c'_s \in S^1$. Since $H_0 = H'_0$, we have $c_s = c'_s$ for $s = 0,1$. In this case, we shall prove that H_1 and H'_1 are properly F–homotopic, which includes the conclusion of the proposition.

By hypothesis the proper reduced homotopies H and H' start on the same side and have the same length. Since H and H' are now assumed to be immersions of I into $J(\{c_i\} \times S^1)$, it follows that for each $s \in \{0,1\}$ the path $\gamma'_s\colon t \mapsto H'(s,t)$ is a reparametrization of $\gamma_s\colon t \mapsto H(s,t)$.

Let $\bar\gamma_0$ and $\bar\gamma'_0$ denote the inverses of the paths $\bar\gamma_0$ and $\bar\gamma'_0$. Let α denote a composition of the paths $\bar\gamma_0$, H_0 and γ_1; likewise, let α' denote a composition of $\bar\gamma'_0$, $H'_0 = H_0$ and γ'_1. Then α' is a reparametrization of α. But the existence of the homotopies H and H' imply that the path H_1 is fixed-endpoint homotopic to α in M, and that the path H'_1 is fixed-endpoint homotopic to α' in M. Hence the paths H_1 and H'_1 are fixed-endpoint homotopic to each other in M. Since these paths lie in the π_1–injective surface $F \subset M$, they are in fact fixed-endpoint homotopic in F, as required. □

Definition 11.5 Let F denote a transversally oriented essential surface in an irreducible knot manifold M. For any proper path α in F and any $\omega \in \{-1,+1\}$, we define the ω *height* of α, denoted $\text{height}_\omega(\alpha)$, to be the supremum of all integers $k \geq 0$ for which there exists a proper reduced homotopy $H\colon (I \times I, I \times \partial I, \partial I \times I) \to (M, F, \partial M)$ which starts on the ω side, has length k, and satisfies $H_0 = \alpha$. Thus $\text{height}_\omega(\alpha)$ is either a non-negative integer or $+\infty$. We define the *minheight* of α, denoted $\text{minheight}(\alpha)$, to be $\min(\text{height}_{+1}(\alpha), \text{height}_{-1}(\alpha))$. It is clear that $\text{height}_{+1}(\alpha)$, $\text{height}_{-1}(\alpha)$ and $\text{minheight}(\alpha)$ depend only on the proper homotopy class of α in F. Furthermore, $\text{minheight}(\alpha)$ is independent of the choice of a transverse orientation of F in M.

Lemma 11.6 *Suppose that F is a transversally oriented essential surface in an irreducible knot manifold M. For any proper path α in F, any integer $k > 0$, and any $\omega \in \{-1,+1\}$, the following conditions are equivalent:*

(i) $\text{height}_\omega(\alpha) = k$*; and*

(ii) *there exists a proper reduced homotopy $H\colon (I \times I, I \times \partial I, \partial I \times I) \to (M, F, \partial M)$ which starts on the ω side and has length k, and such that $H_0 = \alpha$ and* $\text{minheight}(H_1) = 0$.

Proof First suppose that (i) holds. It follows from the definition of ω–height that there is a proper reduced homotopy H of length k, starting on the ω side, such that $H_0 = \alpha$. Set $\beta = H_1$. Define an element ϵ of $\{-1,+1\}$ by the condition that H ends on the ϵ side, and set $k^* = \text{height}_{-\epsilon}(\beta) > 0$. If $\text{minheight}(\beta) > 0$, then in particular $k^* > 0$. Again from the definition, there is a proper reduced homotopy H^* of length k^*, starting on the $-\epsilon$ side, such that $H^*_1 = \beta$. Then a composition of H and H^* is a proper reduced homotopy of length $k + k^*$, starting on the ω side and having length $k + k^* > k$. This is a contradiction since $\text{height}_\omega(\alpha) = k$. Hence we must have $\text{minheight}(\beta) = 0$, and (ii) is established.

Conversely, if (ii) holds, then the ω–height ℓ of α is by definition $\geq k$. Assume that $\ell > k$, and fix a proper reduced homotopy J of length ℓ, starting on the ω side, such that $J_0 = \alpha$. Set $k^* = \ell - k > 0$. It follows from the definition of a reduced homotopy that we may write J as a composition of a reduced homotopy H' of length k and a reduced homotopy H^* of length k^*, and that for some $\epsilon \in \{-1,+1\}$, the homotopy H' ends on the ϵ side while H^* starts on the $-\epsilon$ side. Set $\beta = H_1$ and $\beta' = H'_1 = H^*_0$. Since H' and the homotopy H given by (ii) both start on the ω side, have length k and satisfy $H_0 = H'_0 = \alpha$, it follows from Lemma 11.4 that β and β' are properly F–homotopic.

Since $\beta' = H_0^*$ we have $\text{height}_{-\epsilon}(\beta') \geq k^* > 0$. On the other hand, the inverse homotopy $\bar{H}'$ of H' starts on the ϵ side and has length k; hence $\text{height}_{\epsilon}(\beta') \geq k > 0$. It follows that $\text{minheight}(\beta) > 0$. But $\beta = H_1$ has minheight 0 according to (ii), and since β and β' are properly F–homotopic they must have the same minheight. This is a contradiction. It follows that $\ell = k$, so that (i) holds. □

Proposition 11.7 *Suppose that F is an essential surface in an irreducible knot manifold M, that m is a non-negative integer, and that $\mathcal{C}$ is a proper M–homotopy class of proper paths in M. Then $\mathcal{C}$ contains at most $2m+2$ proper F–homotopy classes of proper paths in F which are of minheight at most m.*

Proof We first give the proof in the special case $m = 0$. Suppose that α, β and β' are proper arcs of minheight 0 in F, and that no two of them are properly F–homotopic. Since $\text{minheight}(\alpha) = 0$, we may assume by symmetry that $\text{height}_{-1}(\alpha) = 0$. Since β and β' are properly M–homotopic to α, it follows from Lemma 11.2 that there are proper reduced homotopies H and H' such that $H_0 = H'_0 = \alpha$, $H_1 = \beta$, and $H'_1 = \beta'$. Since neither β nor β' is properly F–homotopic to α, both k and ℓ are of strictly positive length. Since $\text{height}_{-1}(\alpha) = 0$, it follows that H and H' must both start on the $+1$ side. Since β and β' have minheight 0, it follows from Lemma 11.6 that $\text{length}(H) = \text{height}_{+1}(\alpha) = \text{length}(K)$. Hence according to Lemma 11.4, the proper paths β and β' are properly F–homotopic, a contradiction. This completes the proof in the case $m = 0$.

Now suppose that $m > 0$. For each proper path α in F, let $[\alpha]$ denote the proper F–homotopy class of α. Let Z denote the set of all classes $[\alpha]$ such that $\text{minheight}(\alpha) = 0$; by the case of the proposition already proved, we have $\#(Z) \leq 2$. For each $[\alpha] \in Z$, and each $k \in \{0, 1, \ldots, m\}$, let $\mathcal{H}_{([\alpha],k)}$ denote the set of all classes $[\gamma]$ for which there exists a reduced homotopy H of length k such that $H_0 = \gamma$ and such that H_1 is properly F–homotopic to α. It follows from Lemma 11.6 that the set

$$\mathcal{H} = \bigcup_{([\alpha],k) \in Z \times \{0,\ldots,m\}} \mathcal{H}_{([\alpha],k)}$$

contains all classes $[\gamma]$ for which $0 < \text{height}_{+1}(\gamma) \leq m$ or $0 < \text{height}_{-1}(\gamma) \leq m$. Since $\mathcal{H}$ obviously also contains all classes $[\gamma]$ for which $\text{minheight}(\gamma) = 0$, it therefore contains all classes $[\gamma]$ for which $\text{minheight}(\gamma) \leq m$. Thus it suffices to prove that $\#(\mathcal{H}) \leq 2m+2$; and since $\#(Z) \leq 2$, it suffices to prove that

$\#(\mathcal{H}_{([\alpha],k)}) \leq 1$ for each $([\alpha],k) \in Z \times \{0,\ldots,m\}$. This is clear for $k = 0$, because $\mathcal{H}_{([\alpha],0)} = \{[\alpha]\}$ in view of the definition of a length–0 reduced homotopy.

If $k > 0$, and if $[\gamma]$ and $[\gamma']$ are elements of $\mathcal{H}_{(\alpha,k)}$, we have proper reduced homotopies $H, H'\colon (I \times I, I \times \partial I, \partial I \times I) \to (M, F, \partial M)$ which have length k, and such that $H_0 = \gamma$, $H'_0 = \gamma'$, and both H_1 and H'_1 are properly F–homotopic to α. The inverse homotopies $\bar{H}$ and $\bar{H}'$ are reduced homotopies of length k such that $\bar{H}_0$ and $\bar{H}'_0$ are properly F–homotopic to α, while $H_1 = \gamma$ and $H'_1 = \gamma'$. If $\bar{H}$ and $\bar{H}'$ start on different sides then the definition of height implies that both $\text{height}_{-1}(\alpha)$ and $\text{height}_{+1}(\alpha)$ are strictly positive, a contradiction since $\text{minheight}(\alpha) = 0$. Hence $\bar{H}$ and $\bar{H}'$ start on the same side, and it follows from Lemma 11.4 that $\gamma = H_1$ and $\gamma' = H'_1$ are properly F–homotopic, ie, $[\gamma] = [\gamma']$. This shows that $\#(\mathcal{H}_{([\alpha],k)}) \leq 1$, as required. □

Definition 11.8 Let F be an essential surface in a compact, orientable, irreducible 3–manifold M, and let A be a properly embedded arc in F. By the *minheight* of A we mean the minheight of any parametrization of A. It is clear that all parametrizations have the same minheight.

Lemma 11.9 *Suppose that F is an essential surface in an irreducible knot manifold M. Suppose that k is a non-negative integer, that $\mathcal{A}$ is an essential arc system in F (in the sense of 10.1), and that K is a subgraph of $\Gamma_{\mathcal{A}}$ such that for every interior edge e of K, the arc $\bar{e}$ has minheight at least k. Then the thickness of $|K|$ (in the sense of 3.12) is at least $2k+1$.*

Proof We may assume that $k > 0$, as the case $k = 0$ is trivial. We fix a transverse orientation of $F \subset M$.

Let $r\colon |K| \to F$ denote the inclusion map. To establish the conclusion of the lemma, it suffices to show that there exist reduced homotopies $H^+\colon (|K| \times I, |K| \times \partial I) \to (M, F)$ of length k such that H^+ starting on the $+1$ side, H^- starts on the -1 side, and $H_0^{+1} = H_0^{-1} = r$. Indeed, if H^+ and H^- are such homotopies, then by composing the inverse $\bar{H}^+$ of H^+ with H^-, we obtain a reduced homotopy H of length $2k$ such that $H_t = r$ for some $t \in I$; according to the definition, this implies that $|K|$ has thickness at least $2k+1$. By symmetry, it suffices to construct H^+.

Let m denote the number of components of ∂F, and set $\zeta = e^{2\pi\sqrt{-1}/m} \in S^1$. It follows from the first assertion of Lemma 11.3 that there is a homeomorphism $J\colon S^1 \times S^1 \to \partial M$ such that $J^{-1}(\partial F) = S^1 \times \{1, \zeta, \ldots, \zeta^{m-1}\}$. Let us denote by

$p\colon S^1 \times S^1 \to S^1$ the projection to the second factor. For each $j \in \{0, \ldots, m-1\}$ the standard orientation of S^1 defines a transverse orientation of the 0–manifold $\{\zeta^j\} \subset S^1$, which in turn pulls back via the submersion $p \circ H^{-1}$ to a transverse orientation of the component $C_j = J(S^1 \times \{\zeta^j\})$ of $\partial F \subset \partial M$. For each j, we define σ_j to be $+1$ if this pulled back transverse orientation agrees with the transverse orientation of $C_j \subset \partial M$ induced by the given transverse orientation of $F \subset M$, and to be -1 otherwise.

For each interior edge e of K let us fix a parametrization α_e of $\bar{e}$. For each e and for each $s \in \{0, 1\}$ we have $\alpha_e(s) \in C_{j(e,s)}$ for a unique $j(e, s) \in \{0, \ldots, m-1\}$. Since by hypothesis the arc $\bar{e}$ has minheight at least k, there is a proper reduced homotopy $H^e\colon (I \times I, I \times \partial I, \partial I \times I) \to (M, F, \partial M)$ which starts on the $+1$ side and has length k, and such that $H^e_0 = \alpha_e$. According to the second assertion of Lemma 11.3, we may suppose the H^e to be chosen so that for each interior edge e and each $s \in \{0, 1\}$, the path $t \mapsto H^e(s, t)$ is an immersion of I in $J(\{c^e_s\} \times S^1)$ for some $c^e_s \in S^1$. In view of our description of ∂F and our definition of $j(e, s)$ and σ_j, we may assume after suitable reparametrization that the immersion $t \mapsto H^e(s, t)$ is given by

$$t \mapsto J(c^e_s, \zeta^{j(e,s)} \exp(2\pi\sigma_{j(e,s)} kt\sqrt{-1}/m))$$

for each interior edge e and each $s \in \{0, 1\}$.

We may now define the required reduced homotopy $H^+\colon (|K| \times I, |K| \times \partial I) \to (M, F)$ by setting

$$H^+(J(x, \zeta^j), t) = J(x, \zeta^j \exp(2\pi\sigma_j kt\sqrt{-1}/m))$$

for every $j \in \{0, \ldots, m-1\}$ and every $x \in S^1$, and

$$H^+(\alpha_e(s), t) = H^e(s, t)$$

for every interior edge e of K and for all $s, t \in I$. □

Lemma 11.10 *Suppose that M is an irreducible knot manifold which contains no essential annulus. Suppose that F_1 and F_2 are essential surfaces in M which intersect transversally, and that ∂F_1 and ∂F_2 intersect minimally in the sense of 1.8. Suppose that α is a component of $F_1 \cap F_2$. Then α is not properly M–homotopic to a path in ∂M.*

Proof If M is a solid torus then the components of F_1 and F_2 are disks; the boundary slopes of F_1 and F_2 must be the same, and minimality implies that $\partial F_1 \cap \partial F_2 = \emptyset$. Thus the statement is vacuously true. Now suppose that the irreducible knot manifold M is not a solid torus. Then ∂M is π_1–injective.

Suppose that some component α of $F_1 \cap F_2$ is an arc which is properly M–homotopic to a path in ∂M. Then a parametrization of α is fixed-endpoint homotopic to a path in ∂F_i for $i = 1$ and for $i = 2$. Hence α is parallel in each of the F_i to an arc $\beta_i \subset \partial F_i$ with $\partial\beta_i = \partial\alpha_i$. In particular, β_1 and β_2 are fixed-endpoint homotopic arcs in ∂M. Since ∂M is π_1–injective, β_1 and β_2 are fixed-endpoint homotopic in ∂M. But since ∂M is a torus and the 1–manifolds ∂F_1 and ∂F_2 intersect minimally, no arc in ∂F_1 can be fixed-endpoint homotopic in ∂M to an arc in ∂F_2. □

The following result was observed by Cameron Gordon, who used it in an unpublished argument giving a bound for the geometric intersection number of the boundary slopes of two essential surfaces in a hyperbolic knot manifold in terms of the intrinsic topological invariants of the surfaces (cf [1, Corollary 6.2.5]).

Lemma 11.11 *Suppose that M is an irreducible knot manifold which contains no essential annulus. Suppose that F_1 and F_2 are essential surfaces in M which intersect transversally, and that ∂F_1 and ∂F_2 intersect minimally in the sense of 1.8. Suppose that A and A' are distinct components of $F_1 \cap F_2$ which are both arcs. Then A and A' cannot be parallel both on F_1 and on F_2.*

Proof As in the proof of Lemma 11.10 we can show that the statement is vacuously true if M is a solid torus. Hence we may assume that ∂M is π_1–injective. Suppose that the arcs A and A' are parallel both on F_1 and on F_2 Then for $i = 1, 2$ there is a PL disk $R_i \subset F_i$ with frontier $R_i = A \cup A'$. Let us write the standard 1–sphere S^1 as a union of two arcs r_1 and r_2 with $r_1 \cap r_2 = \partial r_1 = \partial r_2 = \{a, a'\}$, and let $s\colon S^1 \times I \to M$ be a map which maps $r_i \times I$ homeomorphically onto R_i for $i = 1, 2$, and maps $\{a\} \times I$ and $\{a'\} \times I$ homeomorphically onto A and A' respectively. Since ∂F_1 and ∂F_2 intersect minimally, the arcs $s(r_1 \times \{0\}) \subset \partial F_1$ and $s(r_2 \times \{0\}) \subset \partial F_2$ are not fixed-endpoint homotopic. Hence $s|_{(S^1\times\{0\})}\colon (S^1 \times \{0\}) \to \partial M$ induces an injective homomorphism of fundamental groups. Since ∂M is π_1–injective, $s\colon S^1 \times I \to M$ also induces an injective homomorphism of fundamental groups. Furthermore, the proper path $t \mapsto s(a, t)$ in M is a parametrization of the arc A, and hence according to Lemma 11.10 it is not properly homotopic to a path in ∂M. This shows that s, regarded as a map from $S^1 \times I, S^1 \times \partial I)$ to $(M, \partial M)$, is non-degenerate in the sense of [8]. It then follows from the annulus theorem [8, Theorem IV.3.1] that M contains an essential annulus, in contradiction to the hypothesis. □

Lemma 11.12 *Suppose that F_1 and F_2 are connected strict essential surfaces in an irreducible knot manifold M. Suppose that F_1 and F_2 intersect transversally, and that ∂F_1 and ∂F_2 are non-empty and intersect minimally in the sense of 1.8). Assume that the interior of every component of $F_2-(F_1\cap F_2)$ is an open disk or an open annulus. For $i=1,2$, let g_i, s_i and m_i denote, respectively, the genus, boundary slope and number of boundary components of F_i, and let $\chi_i = 2-2g_i-m_i \le 0$ denote its Euler characteristic. Then there exists a positive integer $\Theta \ge |\chi_2|$ with the following property: if q is any real number greater than 1, and if we set $\tau=(7q-1)/(q-1)$, then*

$$\kappa(F_1,F_2) \le \frac{36qm_2}{m_1}\cdot\frac{\phi_\tau(\Theta)\log_2(2\Theta)}{\Theta}|\chi_1|,$$

where ϕ_τ is defined by 10.4.

Proof We set $\mathcal{A}^0 = F_1\cap F_2$. It follows from Lemma 11.10 that $\mathcal{A}_0$ is an essential arc system in F_2. We choose a reduction of $\mathcal{A}^0$ in F_2 in the sense of 10.8, and denote it by $\mathcal{A}$.

We set $\Theta=\Theta_{\mathcal{A}}$. By definition, Θ is the number of components of the arc system $\mathcal{A}$. Hence if N denotes a regular neighborhood of $\mathcal{A}$ in F_2 we have $\chi(\overline{F_2-N}) = \chi(F_2)+\Theta$. But by hypothesis, the interior of every component of $F_2-\mathcal{A} = F_2-(F_1\cap F_2)$ is an open disk or an open annulus, and hence $\chi(\overline{F_2-N})\ge 0$. It follows that

$$\Theta \ge -\chi(F_2) = |\chi_2| \tag{11.12.1}$$

Hence in order to prove the lemma it suffices to show that, if we fix any real number $q>1$, and if we set $\tau=(7q-1)/(q-1)$, then

$$\kappa(F_1,F_2) \le \frac{36qm_2}{m_1}\cdot\frac{\phi(\Theta)\log_2(2\Theta)}{\Theta}|\chi_1|. \tag{$*$}$$

The components of $\mathcal{A}$ may be regarded as properly embedded arcs in M. We denote by $\mathcal{I}$ the set of all proper homotopy classes of proper paths in M, in the sense of 11.1, which are represented by parametrizations of components of $\mathcal{A}$. We define a labeling of $\mathcal{A}$, in the sense of 10.3, by defining $\iota(e)$ to be the proper M–homotopy class of $\bar{e}$, for every interior edge e of $\Gamma_{\mathcal{A}}$. We have $\Theta=\Theta_{\mathcal{A}} = \sum_{i\in\mathcal{I}}\theta_i^\iota$. As in the statement of Proposition 10.10, we set $\theta_i=\theta_i^\iota$ for every $i\in\mathcal{I}$, and we set $\theta_\infty = \max_{i\in\mathcal{I}}\theta_i$.

When we wish to think of an element i of $\mathcal{I}$ as a proper M–homotopy class, rather than as a "label," we will denote it by $\mathcal{C}_i$.

For each interior edge of $\Gamma_{\mathcal{A}}$, the quantity minheight $\bar{e}$ is defined by 11.8. Let us denote by E^* the set of all interior edges of $\Gamma_{\mathcal{A}}$ such that minheight$(\bar{e})\le$

$(\theta^\iota(e)/(2q)) - 1$. For every $i \in \mathcal{I}$ set $E_i^* = E^* \cap E_i$ and $\theta_i^* = \#(E_i^*)$. For every interior edge e of $\Gamma_\mathcal{A}$, let $w(e)$ denote the $\mathcal{A}^0$–width of e. For each label $i \in \mathcal{I}$, set $\lambda(i) = \max_e w(e)$, where e ranges over all interior edges of $\Gamma_\mathcal{A}$ with label i.

We wish to apply Proposition 10.10. For this purpose we must verify that $\theta_i^* \leq \theta_i/q$ for every $i \in \mathcal{I}$. According to the definitions we have $\theta_i^* = \#(E_i^*)$, where E_i^* is a collection of interior edges of $\Gamma_\mathcal{A}$, whose closures all represent the proper M–homotopy class $\mathcal{C}_i$, and all have minheight at most $(\theta^\iota(e)/(2q)) - 1$.

Since $\mathcal{A}$ is a reduced arc system, the closures of the edges in E_i^* represent pairwise distinct proper F–homotopy classes. Applying Proposition 11.7 with $F = F_2$, $\mathcal{C} = \mathcal{C}_i$ and $m = (\theta^\iota(e)/(2q)) - 1$, we conclude that $\theta_i^* \leq 2m+2 = \theta_i/q$, as required for the application of 10.10.

Hence $\Gamma_\mathcal{A}$ has a subgraph K satisfying conditions (1)–(5) of Proposition 10.10. Since $\theta_\infty = \max_{i\in\mathcal{I}} \theta_i \leq \sum_{i\in\mathcal{I}} \theta_i^\iota = \Theta$, and since the function ϕ_τ defined in 10.4 is monotone increasing, it follows from condition (5) of 10.10 that

$$\frac{\operatorname{length}(K^0)}{\theta^\iota(K)} < \frac{6\phi_\tau(\Theta)\log_2(2\Theta)}{\Theta}\sum_{i\in\mathcal{I}}\lambda(i), \tag{11.12.2}$$

where K^0 is the subgraph of $\Gamma_{\mathcal{A}^0}$ associated to K.

In view of the definition of E^*, it follows from condition (2) of 10.10 that for every interior edge e of K we have $\operatorname{minheight}(\bar{e}) > (\theta^\iota(e)/(2q)) - 1$. Since by definition we have $\theta^\iota(K) = \min_e \theta_e^\iota$, where e ranges over the interior edges of K, it follows that

$$\operatorname{minheight}(\bar{e}) > (\theta^\iota(K)/(2q)) - 1$$

for every interior edge e of K. Applying Lemma 11.9, taking k to be the least integer $\geq (\theta^\iota(K)/(2q)) - 1$, we conclude that the thickness $t_{F_2}(|K|)$ is at least $(\theta^\iota(K)/q) - 1$. Since $t_{F_2}(|K|)$ is by definition a strictly positive integer, we have $2t_{F_2}(|K|) \geq t_{F_2} + 1 \geq \theta^\iota(K)/q$, and hence

$$t_{F_2}(|K|) \geq \frac{\theta^\iota(K)}{2q}. \tag{11.12.3}$$

Combining (11.12.2) and (11.12.3) we obtain

$$\frac{\operatorname{length}(K^0)}{t_{F_2}(|K|)} < \frac{12q\phi_\tau(\Theta)\log_2(2\Theta)}{\Theta}\sum_{i\in\mathcal{I}}\lambda(i). \tag{11.12.4}$$

Let K_∂^0 denote the subgraph of K^0 with $|K_\partial^0| = |K| \cap \partial F_2 = |K^0| \cap \partial F_2$. The number of vertices of K_0 is equal to $\#(|K_\partial^0| \cap \mathcal{A})$. It follows from condition (4) of 10.10 that all vertices of K^0 are of valence at least 2; hence K^0 has at most as many vertices as edges, ie,

$$\#(|K^0_\partial| \cap \mathcal{A}) \leq \text{length}(K^0). \tag{11.12.5}$$

We may write $|K^0| = |K^0_\partial| \cup \mathcal{B}$, where $\mathcal{B}$ is the submanifold of $\mathcal{A}$ made up of the closures of the interior edges of K. Let $\mathcal{B}_1$ denote a properly embedded submanifold of F_2 which is topologically parallel to $\mathcal{B}$ in F_2 and disjoint from $\mathcal{A}$. Then the 1–dimensional polyhedron $L = |K^0_\partial| \cup \mathcal{B}_1$ is isotopic in F_2 to $|K^0| = |K|$. Hence

$$t_{F_2}(L) = t_{F_2}(|K|). \tag{11.12.6}$$

The definition of L implies that

$$L \cap F_1 = L \cap \mathcal{A} = |K^0_\partial| \cap \mathcal{A},$$

so that by (11.12.5) we have

$$\#(L \cap F_1) \leq \text{length}(K^0). \tag{11.12.7}$$

Since L is isotopic in F_2 to $|K|$, it follows from conditions (1) and (3) of 10.10 that L is connected, has first Betti number equal to 2 and is π_1–injective in F_2. Hence according to 7.1 we have

$$\kappa(F_1, F_2) \leq \frac{m_2 \cdot \#(L \cap F_1)}{m_1 \cdot t_{F_2}(L)}. \tag{11.12.8}$$

It follows from (11.12.4), (11.12.6), (11.12.7) and (11.12.8) that

$$\kappa(F_1, F_2) \leq \frac{12qm_2}{m_1} \cdot \frac{\phi(\Theta)\log_2(2\Theta)}{\Theta} \sum_{i \in \mathcal{I}} \lambda(i). \tag{11.12.9}$$

We need an estimate for the factor $\sum_{i\in\mathcal{I}} \lambda(i)$ which appears in the right hand side of (11.12.9). By the definition of the $\lambda(i)$, we may choose for each $i \in \mathcal{I}$ an edge e_i such that $\iota(e_i) = i$ and $w(e_i) = \lambda_i$. The $\mathcal{A}^0$–width $w(e_i)$ is by definition the cardinality of the $\mathcal{A}^0$–parallelism class containing the component $\bar{e}_i$ of $\mathcal{A}^0$. If we denote this parallelism class by C_i, and set $C = \bigcup_{i\in\mathcal{I}} C_i$, it follows that

$$\sum_{i \in \mathcal{I}} \lambda(i) = \#(C). \tag{11.12.10}$$

Suppose that two distinct arcs in C, say A and A', are parallel in F_1. We have $A \in C_i$ and $A' \in C_j$ for some $i, j \in \mathcal{I}$. Then e_i and e_j are respectively parallel in F_2 to A and A', which by our assumption are parallel to each other in F_1; hence e_i and e_j are properly homotopic as properly embedded arcs in M, and from the definition of the labeling ι it follows that $i = \iota(e_i) = \iota(e_j) = j$. Thus

A and A' both belong to C_i, and are therefore parallel in F_2 as well as in F_1. Since by hypothesis the essential surfaces F_1 and F_2 intersect transversally, and ∂F_1 and ∂F_2 intersect minimally, this contradicts Lemma 11.11. It follows that no two distinct arcs in C can be parallel in F_1.

The cardinality of a collection of pairwise non-parallel arcs in F_1 is at most $-3\chi(F_1) = 3|\chi_1|$. In view of (11.12.10) it follows that

$$\sum_{i\in\mathcal{I}} \lambda(i) \leq 3|\chi_1|. \tag{11.12.11}$$

The inequality $(*)$ follows immediately from (11.12.9) and (11.12.11), and the proof of the lemma is complete. □

The next two lemmas will be needed in order to obtain a concrete estimate from Lemma 11.12.

Lemma 11.13 *For every real number $x > 1$ there is a real number q such that*

$$1 < q \leq \frac{1}{7}\left(6\left(\frac{2\ln x}{\ln 7}\right)^{1/2} + 31\right), \tag{1}$$

and such that, if we set $\tau = (7q-1)/(q-1)$, we have

$$\ln \phi_\tau(x) \leq 2(2(\ln 7)(\ln x))^{1/2} + 4\ln 7 + 1. \tag{2}$$

Proof We define a positive integer μ by

$$\mu = \left\lfloor \left(\frac{\ln x}{2\ln 7}\right)^{1/2} \right\rfloor + 1,$$

and we set
$$q = \frac{12\mu + 19}{7}.$$

Then (1) is clear from direct computation. On the other hand, if we set

$$\tau = \frac{7q+1}{q+1} = 7\left(1 + \frac{1}{2m+2}\right),$$

then
$$\ln \tau \leq \ln 7 + \frac{1}{2m+2}.$$

The definition of ϕ_τ (10.4) implies that

$$\ln \phi_\tau(x) = \min_m \left((2\mu+2)\ln\tau + \frac{1}{m}\ln x\right),$$

where m ranges over all positive integers. Hence

$$\begin{aligned}\ln\phi_\tau(x) &\le (2\mu+2)\ln\tau + \frac{1}{\mu}\ln x\\ &\le (2\mu+2)\ln 7 + 1 + \frac{1}{\mu}\ln x.\end{aligned}$$

Since $\quad 2\mu+2 \le 2\left(\frac{\ln x}{2\ln 7}\right)^{1/2} + 4 \quad$ and $\quad \frac{1}{\mu} \le \left(\frac{2\ln 7}{\ln x}\right)^{1/2},$

we now obtain (2) by direct computation. □

Lemma 11.14 *Let us define a function f on $(1,\infty)$ by*

$$f(x) = 2(2(\ln 7)(\ln x))^{1/2} + \ln\left(6\left(\frac{2\ln x}{\ln 7}\right)^{1/2} + 31\right) + \ln\left(\frac{\ln x}{\ln 2} + 1\right) - \ln x.$$

Then for all integers $m \ge n \ge 333$ we have $f(m) \le f(n)$.

Proof We set $\alpha = 2\sqrt{2\ln 7}$, $\beta = 6\sqrt{2/\ln 7}$, $\gamma = 31$ and $\delta = 1/\ln 2$. Then for $x > 1$ we have

$$f(x) = \alpha(\ln x)^{1/2} + \ln(\beta(\ln x)^{1/2} + \gamma) + \ln(\delta\ln x + 1) - \ln x,$$

and hence

$$xf'(x) = \frac{\alpha}{2(\ln x)^{1/2}} + \frac{\beta}{2\beta\ln x + 2\gamma(\ln x)^{1/2}} + \frac{\delta}{\delta\ln x + 1} - 1 \tag{11.14.1}$$

Since α, β, γ and δ are positive, the right hand side of (11.14.1) is obviously monotonically decreasing for $x > 1$, and by direct calculation it is seen to be negative for $x = 334$ (and positive for $x = 333$). Hence $f'(x) < 0$ for $x \ge 334$, and it follows that the conclusion of the lemma holds in the case $m \ge n \ge 334$. Since by direct computation we find that $f(334) < f(333)$, the conclusion also holds in the case $m \ge n = 333$. □

Proposition 11.15 *Suppose that F_1 and F_2 are connected strict essential surfaces in an irreducible knot manifold M. Suppose that F_1 and F_2 intersect transversally, and that ∂F_1 and ∂F_2 are non-empty and intersect minimally in the sense of 1.8). Assume that the interior of every component of $F_2 - (F_1 \cap F_2)$ is an open disk or an open annulus. For $i = 1, 2$, let g_i, s_i and m_i denote, respectively, the genus, boundary slope and number of boundary components of F_i, and let $\chi_i = 2 - 2g_i - m_i \le 0$ denote its Euler characteristic. Assume that $|\chi_2| \ge 333$. Then*

$$\kappa(F_1, F_2) \le g(\chi_2)\frac{m_2|\chi_1|}{m_1|\chi_2|}$$

where

$$g(x) = 12348\ \left(6\left(\frac{2\ln|x|}{\ln 7}\right)^{1/2}+31\right)\left(\frac{\ln|x|}{\ln 2}+1\right)\exp(1{+}2(2(\ln 7)(\ln|x|))^{1/2}).$$

Proof Let Θ be the integer given by Lemma 11.12. Thus $\Theta \geq |\chi_2|$, and if q is any real number greater than 1, and if we set $\tau = (7q-1)/(q-1)$, then

$$\kappa(F_1,F_2) \leq \frac{36qm_2}{m_1}\cdot\frac{\phi_\tau(\Theta)\log_2(2\Theta)}{\Theta}|\chi_1|. \qquad (11.15.1)$$

Applying Lemma 11.13 with $x = \Theta$, we obtain a particular value of $q > 1$ such that, if we set $\tau = (7q-1)/(q-1)$, we have

$$q \leq \frac{1}{7}\left(6\left(2\frac{\ln\Theta}{\ln 7}\right)^{1/2}+31\right), \qquad (11.15.2)$$

and

$$\ln\phi_\tau(\Theta) \leq 2(2(\ln 7)(\ln\Theta))^{1/2}+4\ln 7+1. \qquad (11.15.3)$$

If we define a function f on $(1,\infty)$ by

$$f(x) = 2(2(\ln 7)(\ln x))^{1/2}+\ln\left(6\left(2\frac{\ln x}{\ln 7}\right)^{1/2}+31\right)+\ln\left(\frac{\ln x}{\ln 2}+1\right)-\ln x, \qquad (11.15.4)$$

it follows from (11.15.2) and (11.15.3) that

$$\ln\left(\frac{q\phi_\tau(\Theta)\log_2(2\Theta)}{\Theta}\right) \leq f(\Theta)+3\ln 7+1.$$

But since $\Theta \geq |\chi_2| \geq 333$, Lemma 11.14 asserts that $f(\Theta) \leq f(|\chi_2|)$. Hence

$$\ln\left(\frac{q\phi_\tau(\Theta)\log_2(2\Theta)}{\Theta}\right) \leq f(|\chi_2|)+3\ln 7+1. \qquad (11.15.5)$$

Combining (11.15.1) and (11.15.5) we obtain

$$\kappa(F_1,F_2) \leq \frac{36m_2}{m_1}\cdot\exp(f(|\chi_2|)+3\ln 7+1)\cdot|\chi_1|. \qquad (11.15.6)$$

If we set $x = |\chi_2|$ in (11.15.4), substitute the resulting expression for $f(|\chi_2|)$ into (11.15.6) and simplify, we obtain the conclusion of Proposition 11.15. □

Theorem 11.16 *Suppose that K is a non-exceptional two-surface knot in a closed, orientable 3–manifold Σ such that $\pi_1(\Sigma)$ is cyclic. Let $\mathfrak{m}$ denote the meridian slope of K and let F_1 and F_2 be representatives of the two isotopy classes of connected strict essential surfaces in $M(K)$. For $i = 1,2$, let g_i, s_i*

and m_i denote, respectively, the genus, boundary slope (well-defined by 6.10) and number of boundary components of F_i, and let $\chi_i = 2 - 2g_i - m_i < 0$ denote its Euler characteristic. Assume that $|\chi_2| \geq 333$ and that $s_2 \neq \mathfrak{m}$. Set $q_i = \Delta(s_i, \mathfrak{m})$ (so that q_i is the denominator of s_i in the sense of 1.13), and set $\Delta = \Delta(s_1, s_2)$ (so that $\Delta \neq 0$ by 6.10). Then

$$\frac{q_1^2}{\Delta} \leq 2g(\chi_2)\frac{m_2|\chi_1|}{m_1|\chi_2|}$$

where

$$g(x) = 12348\left(6\left(\frac{2\ln|x|}{\ln 7}\right)^{1/2} + 31\right)\left(\frac{\ln|x|}{\ln 2} + 1\right)\exp(1+2(2(\ln 7)(\ln|x|))^{1/2}).$$

Proof We may assume after an isotopy that F_1 and F_2 intersect transversally, and that ∂F_1 and ∂F_2 intersect minimally in the sense of 1.8. Furthermore, F_1 and F_2 may be assumed to be chosen within their rel-boundary isotopy classes so as to minimize the number of components of $F_1 \cap F_2$. Then no component of $F_1 \cap F_2$ is a homotopically trivial simple closed curve (cf Remark 7.5). Set $A = F_1 \cap F_2$. It now follows from Theorem 7.4 that every component of $(\operatorname{int} F_i) - A$ is an open disk or an open annulus. Hence by Proposition 11.15 we have

$$\kappa(F_1, F_2) \leq g(\chi_2)\frac{m_2|\chi_1|}{m_1|\chi_2|}.$$

On the other hand, according to Theorem 7.7 we have

$$\frac{q_1^2}{\Delta} \leq 2\kappa(F_1, F_2).$$

The last two inequalities imply the inequality in the conclusion of Theorem 11.16. □

Theorem 11.16 has the following qualitative consequence.

Corollary 11.17 *There is a positive-valued function $f_1(x)$ of a positive real variable x with the following properties.*

(i) *For every $\epsilon > 0$ we have*

$$\lim_{x\to\infty} x^{1-\epsilon} f_1(x) = 0.$$

(ii) *If K is any non-exceptional two-surface knot in a closed, orientable $3-$manifold Σ such that $\pi_1(\Sigma)$ is cyclic, and if $\mathfrak{m}$, F_i, g_i, s_i, m_i, q_i and Δ are defined as in the statement of Theorem 9.5 or 11.16, then*

$$\frac{q_1^2}{\Delta} \leq \frac{m_2|\chi_1|}{m_1} f_1(|\chi_2|).$$

11.18 It is instructive to compare Corollary 11.17 with the corresponding qualitative consequence of Theorem 9.5, Corollary 9.7. Suppose that f_1 is a function with the properties stated in Corollary 11.17. After replacing f_1 by the function $x \mapsto \sup_{y\geq x} f_1(y)$, which clearly still has the same properties, we may assume that f_1 is monotone decreasing. Now suppose that K is any non-exceptional two-surface knot in a closed, orientable 3–manifold Σ such that $\pi_1(\Sigma)$ is cyclic. With the notation of 11.17 we have

$$\frac{q_1^2}{\Delta} \leq \frac{m_2|\chi_1|}{m_1} f_1(|\chi_2|).$$

On the other hand, according to Corollary 7.6 we have $|\chi_1| \leq m_1 m_2 \Delta/2$. Hence

$$\left(\frac{q_1}{\Delta}\right)^2 \leq m_2^2 \frac{f_1(|\chi_2|)}{2}.$$

Since f is monotone decreasing and since $|\chi_2| = 2g_2 + m_2 - 2 \geq g_2$ whenever $g_2 \geq 2$, it follows that

$$\left(\frac{q_1}{\Delta}\right)^2 \leq m_2^2 \frac{f_1(g_2)}{2}$$

provided that $g_2 \geq 2$. Hence the conclusions of Corollary 9.7 hold with $f_0(x) = f_1(x)/2$.

This shows that Corollary 11.17, together with the relatively easy Corollary 7.6, implies Corollary 9.7 by a purely formal argument. This suggests that Theorem 11.16 may be "stronger" than Theorem 9.5 in some vague qualitative sense. Indeed, the argument given above suggests that when the inequality given by 7.6 is far from being an equality, 11.16 will give stronger information than 9.5.

On the other hand, note that if we derive Corollary 9.7 from Theorem 9.5, we get a function $f_0(x)$ such that $xf_0(x)$ grows like $\ln x$, whereas if we derive it from Theorem 11.16 via Corollary 11.17, we get a function $f_0(x) = f_1(x)/2$ such that $xf_0(x)$ grows like $(\ln x)^{3/2}\exp(C(\ln x)^{1/2})$, and hence more rapidly than any power of $\ln x$. In view of the argument given above this suggests that Theorem 9.5 may give stronger information than Theorem 11.16 in the case where the inequality given by 7.6 is close to being an equality.

References

[1] **S Boyer**, **M Culler**, **P B Shalen**, **X Zhang**, *Characteristic subsurfaces and Dehn filling*, to appear in Trans. Amer. Math. Soc.

[2] **M Brittenham**, *Exceptional Seifert-fibered spaces and Dehn surgery on 2–bridge knots*, Topology 37 (1998) 665–672

[3] **D Cooper**, **M Culler**, **H Gillet**, **D D Long**, **P B Shalen**, *Plane curves associated to character varieties of 3–manifolds*, Invent. Math. 118 (1994) 47–84

[4] **M Culler**, **C McA Gordon**, **J Luecke**, **P B Shalen**, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987) 237–300

[5] **M Culler**, **P B Shalen**, *Varieties of group representations and splittings of 3–manifolds*, Ann. of Math. (2) 117 (1983) 109–146

[6] **M Culler**, **P B Shalen**, *Bounded, separating, incompressible surfaces in knot manifolds*, Invent. Math. 75 (1984) 537–545

[7] **M Culler**, **P B Shalen**, *Boundary slopes of knots*, Comment. Math. Helv. 74 (1999) 530–547

[8] **W H Jaco**, **P B Shalen**, *Seifert fibered spaces in 3–manifolds*, Mem. Amer. Math. Soc. 21 (1979)

[9] **B Klaff**, **P B Shalen**, *The diameter of the set of boundary slopes of a knot*, `arXiv:math.GT/0412147`

[10] **I Niven**, **H S Zuckerman**, **H L Montgomery**, *An introduction to the theory of numbers*, fifth edition, John Wiley & Sons Inc., New York (1991)

[11] **J-P Serre**, *Trees*, Springer Monographs in Mathematics, Springer–Verlag, Berlin (2003), translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation

[12] **F Waldhausen**, *Eine Klasse von* 3*–dimensionalen Mannigfaltigkeiten I, II*, Invent. Math. 3 (1967) 308–333, Invent. Math 4 (1967) 87–117

[13] **F Waldhausen**, *On irreducible* 3*–manifolds which are sufficiently large*, Ann. of Math. (2) 87 (1968) 56–88

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