INCOMPRESSIBLE SURFACES IN ONCE-PUNCTURED TORUS BUNDLES

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For the past two decades a large part of the research in the topology of 3-manifolds has been done under the hypothesis that the manifolds are sufficiently-large, that is, they contain properly embedded, incompressible surfaces. The notion of incompressible surface was introduced by W. Haken and the power of this hypothesis was exhibited in the work of F. Waldhausen. Until recently, the only known examples of orientable, irreducible 3-manifolds that are not sufficiently-large were certain 'small' Seifert fibred spaces; the only ones with infinite fundamental group are discussed by Waldhausen in [5]. However, W. Thurston discovered that most Dehn surgeries on the 'figure-eight' knot in S^3 result in orientable, irreducible 3-manifolds that are not sufficiently-large and not Seifert fibred. These new manifolds have infinite fundamental group (in fact, they are hyperbolic). This work has been extended by Hatcher and Thurston to all 2-bridge knots in S^3 [3]. The idea is quite straightforward; namely, if $\hat{M}$ is obtained from M by doing Dehn surgery along a simple closed curve k in M and $\hat{M}$ contains an orientable, incompressible surface, then the bounded manifold $M' = M - u(k)$, where $u(k)$ is an open tubular neighbourhood of k, contains a properly-embedded, orientable, incompressible and boundary-incompressible surface. The problem is, therefore, to understand the incompressible and boundary-incompressible surfaces in M'. This problem is, in its own right, extremely important to the understanding of the structure of 3-manifolds.

In this paper we classify, up to isotopy, the orientable, incompressible and boundary-incompressible surfaces in 3-manifolds that fibre over S^1 with fibre a once-punctured torus. (A once-punctured torus is a compact surface of genus 1 with one boundary component.) We call a surface in M *essential* if it is properly embedded, incompressible, boundary-incompressible, and not parallel to a surface in ∂M. In this particular situation a properly embedded, incompressible surface is essential if it is neither a boundary-parallel torus nor a boundary-parallel annulus. It follows from the classification that if M contains no essential tori, then M contains only finitely many (up to isotopy) essential surfaces. So, we can apply this knowledge to study manifolds obtained by Dehn surgeries on a section of a torus bundle over S^1. Here, we conclude that most Dehn surgeries give manifolds that have no incompressible surfaces; and therefore, these manifolds are not Haken and not reducible. We exhibit an infinite family of such manifolds that are irreducible, not Haken, and have first homology with $\mathbb{Z}_2$-rank equal to 3. All manifolds obtained by Dehn surgery on a section of a torus bundle over S^1 have Heegaard genus at most 3. Hence, this family of manifolds has Heegaard genus equal to 3. It follows that they are not Seifert fibred (and not Haken) and not obtained by Dehn surgery on a 2-bridge knot, since the Heegaard genus in both of these cases is at most 2; thus, we have given infinitely many new examples of non-Haken manifolds.

A special case of our considerations is Dehn-surgery on the 'figure-eight' knot in S^3 (the complement of the 'figure-eight' knot fibres over S^1 with genus 1 fibre). Here we give two new pieces of information. One is that the three-sheeted cyclic branched covering of S^3, branched over the 'figure-eight' knot is a Haken manifold; and another is that while neither the 16 nor -16 Dehn surgeries on the 'figure-eight' knot are Haken, both have four-sheeted cyclic coverings that are Haken. Both of these results are obtained by exhibiting essential surfaces in the three- and four-sheeted cyclic coverings of the 'figure-eight' knot space that do not project, under the covering projection, to non-singular surfaces.

In § 1, we establish some of the preliminaries and the notation in which we work. In § 2, we construct four types of essential surfaces in once-punctured torus bundles. In § 3, we prove that an essential surface in a once-punctured torus bundle is isotopic to one of the types that we constructed in § 2. In §§ 4 and 5, we study the isotopy classification of these essential surfaces. In particular, we prove that there are only finitely many isotopy classes of surfaces of a given type (and we completely analyse the situation in which there is more than one isotopy class of surfaces of the same type). Also, we prove that in each bundle there are only finitely many types of surface, except in the case that the bundle contains an essential torus. However, in this latter case we describe the situation completely. In § 6, we give a number of examples. Our methods enable us to find all essential surfaces in a given once-punctured torus bundle M, and describe the boundary curves of all essential surfaces in terms of a single coordinate system (framing) on ∂M. We present the required algorithms in § 6. In § 7 we raise some unanswered questions and make some conjectures.

Any notation or terminology which is not defined here is defined by Jaco in [**4**]. We remark that throughout this paper we will be working in the differentiable category. We will assume, without explicitly saying so, that the manifolds under consideration are orientable, that submanifolds are properly embedded, and that intersections are transverse.

The case when the once-punctured torus bundle is hyperbolic has been independently studied by W. Floyd and A. Hatcher, using the techniques of [**3**].

1. *Notation and generalities*

1.1. *A standard once-punctured torus*

Let T be the once-punctured torus constructed by identifying four sides of an octagon as shown in Fig. 1. Also shown are two simple loops a and b based at the point x, and spanning arcs α, β, c, α_+, α_-, β_+, β_-.

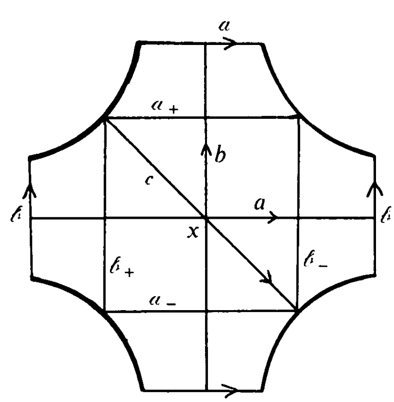


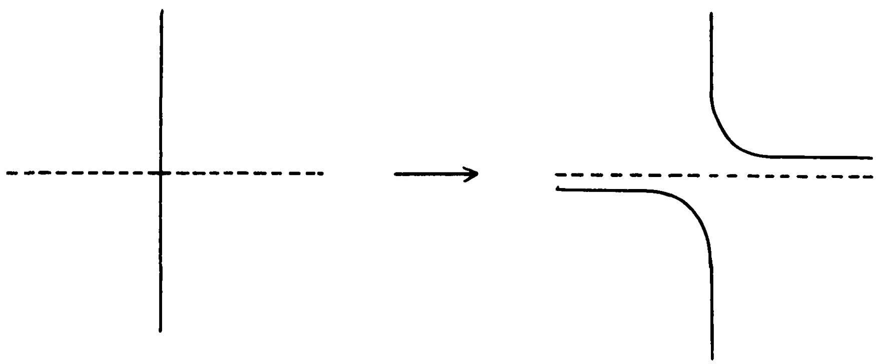
FIG. 1

We note that the homotopy classes of the loops a and b form a free basis for $\pi_1(T, *)$. Also, the loop $aba^{-1}b^{-1}$ is freely homotopic to ∂T.

1.2. *The homeotopy group of T*

The homeotopy group of a manifold M is the group $\mathscr{H}(M)$ of homeomorphisms of M modulo the subgroup of homeomorphisms that are isotopic to the identity. If M is orientable, we denote by $\mathscr{H}^+(M)$ the subgroup of isotopy classes of orientation-preserving homeomorphisms. The homeotopy group of T can be identified with the group $GL_2(\mathbb{Z})$, with $\mathscr{H}^+(T)$ corresponding to $SL_2(\mathbb{Z}) \subset GL_2(\mathbb{Z})$.

Let α and β be the left-handed Dehn twists about the curves a and b respectively.



FACTS. 1.2.1. The isotopy classes of α and β generate $\mathscr{H}^+(T)$.

1.2.2. If h is a homeomorphism of T, let H be the induced automorphism of $H_1(T)$. Then the map $h \to H$ induces an isomorphism from $\mathscr{H}(T)$ onto $\mathrm{Aut}(\mathbb{Z}+\mathbb{Z})$.

1.2.3. The homology classes of a and b form a basis for $H_1(T)$. We may use this basis to identify $\mathscr{H}(T)$ with $GL_2(\mathbb{Z})$ (of course, $\mathscr{H}^+(T)$ corresponds to $SL_2(\mathbb{Z}) \subset GL_2(\mathbb{Z})$). *Throughout this paper we will use this identification: isotopy classes of homeomorphisms of T will be denoted by* 2×2 *integer matrices with determinant* ± 1. Thus the isotopy classes of α and β correspond to the matrices

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

respectively.

1.2.4. Let P and Q be the two matrices

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

These two matrices generate $SL_2(\mathbb{Z})$ and we have the familiar presentation of $SL_2(\mathbb{Z})$ as $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$,

$$SL_2(\mathbb{Z}) = \langle P, Q\colon P^4 = Q^6 = 1,\ P^2 = Q^3 \rangle.$$

Note that

$$A = QP, \quad A^{-1} = PQ^2, \quad B = PQ, \quad B^{-1} = Q^2P.$$

Finally, observe that P is the isotopy class of the homeomorphism φ of T that is induced by rotating the octagon 90° in a counter-clockwise direction and Q is the isotopy class of the homeomorphism ψ of T that has period 6 and maps a to c, b to a, and c to $-b$.

1.2.5. An element H of $\mathscr{H}(T)$ fixes the isotopy class of an essential closed curve if and only if the trace of H is ± 2.

1.3. *Bundles over S^1 with fibre T*

Two fibre bundles with the same fibre and base are said to be *equivalent* if there is a fibre-preserving homeomorphism between them that induces the identity on the base. They are *weakly equivalent* if there is a fibre-preserving homeomorphism between them.

By the classification of fibre bundles over spheres, we know that the equivalence classes of bundles over S^1 with fibre T are in one-to-one correspondence with conjugacy classes in $\mathscr{H}(T)$. The conjugacy class that corresponds to a bundle is called its *characteristic class*. Let the conjugacy class of an element $H \in \mathscr{H}(T)$ be denoted by $[H]$. Now if M and N are fibre bundles over S^1 with fibre T and characteristic classes $[H]$ and $[G]$, respectively, then M and N are weakly equivalent if and only if $[H] = [G^{\pm 1}]$.

In general a manifold may admit fibrations over S^1, with the same fibre, which are not weakly equivalent. However, this cannot happen if the fibre is a once-punctured torus.

1.3.1. PROPOSITION (Murasugi). *Let M and N be orientable 3-manifolds that fibre over S^1 with fibre a once-punctured torus. Then M is homeomorphic to N if and only if the fibre bundle structures on M and N are weakly equivalent.*

Proof. Clearly M and N are homeomorphic if the bundles are weakly equivalent.

Conversely, suppose that M and N are homeomorphic. Let $[H]$ and $[G]$ be the characteristic classes of the respective bundles. Observe that $H_1(M)$ and $H_1(N)$ are presented with three generators and relation matrices

$$(H - I \mid 0) \quad \text{and} \quad (G - I \mid 0)$$

respectively. Now the proof breaks into two cases.

Case 1. If trace$(H) \neq 2$ then it is easily seen, from the presentation above, that the free subgroup of $H_1(M)$ has rank 1. Of course, $H_1(N) \approx H_1(M)$, so $H_1(N)$ also has free rank 1. Thus $\pi_1(M)$ and $\pi_1(N)$ each contain only one subgroup that is the kernel of an epimorphism to $\mathbb{Z}$. These subgroups must be the images of the inclusions $\pi_1(T) \to \pi_1(M)$ and $\pi_1(T) \to \pi_1(N)$. The characteristic classes of the two bundles are therefore determined by the actions, by conjugation, of $\pi_1(M)$ and $\pi_1(N)$ on $\pi_1(T)$. It follows that $[H] = [G^{\pm 1}]$.

Case 2. In the case that trace$(H) = 2 =$ trace(K) the proof is based on a fact about $\mathrm{GL}_2(\mathbb{Z})$: namely that each conjugacy class of trace 2 in $\mathrm{GL}_2(\mathbb{Z})$ contains exactly one element of the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \text{where } n \geqslant 0.$$

If H is conjugate to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, then we see from the presentation of $H_1(M)$ that n is the order of the torsion subgroup of $H_1(M) \approx H_1(N)$. Thus G is also conjugate to $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, so $[H] = [G]$.

1.4. *Constructing bundles*

Let F, X_1, X_2 be topological spaces, let γ_i: $F \times [0, 1] \to X_i$, for $i = 1, 2$, and let η: $F \to F$ be homeomorphisms. Define $X_1/_\eta X_2$ to be the quotient space obtained from $X_1 \cup X_2$ by identifying $\gamma_1(x, 1)$ to $\gamma_2(\eta(x), 0)$. Define $X_1/_\eta$ to be the quotient space obtained from X_1 by identifying $\gamma_1(x, 1)$ to $\gamma_1(\eta(x), 0)$.

If γ_i: $F \times [0, 1] \to X_i$ and η_i: $F \to F$ are homeomorphisms for $i = 1, \ldots, n$, then

$$X = X_1/_{\eta_1} X_2/_{\eta_2} \cdots /_{\eta_{n-1}} X_n/_{\eta_n}$$

is well-defined. This space clearly fibres over S^1 with fibre F and characteristic class

$$[H_n \circ H_{n-1} \circ \ldots \circ H_1]$$

where H_i denotes the isotopy class of η_i. The spaces X_i, for $i = 1, \ldots, n$, will be called *blocks* in X.

We will construct bundles over S^1, with fibre T, in this way, using homeomorphisms η_i which are powers of α or β. To construct surfaces in these bundles we will embed surfaces in each block so that, after identification of the blocks, the surfaces fit together to give a properly embedded surface in the bundle.

1.5. *Essential surfaces*

We remark that in a 3-manifold, all of whose boundary components are tori, an incompressible, boundary-compressible surface must be a boundary-parallel annulus. To see this, perform a surgery along a boundary-compressing disk and observe that the resulting surface must have a boundary component that bounds a disk on ∂M. The new surface is incompressible and hence must be a boundary-parallel disk. This implies that the original surface was a boundary-parallel annulus.

It is a corollary of this observation that any incompressible surface in a once-punctured torus bundle over S^1, which is not boundary parallel, is an essential surface.

2. *Examples*

We will describe a number of surfaces in once-punctured torus bundles and prove that they are essential. In the following section we will show that, up to isotopy, any essential surface in a once-punctured torus bundle is of the same type as one of those described here.

To describe these surfaces we will first compile a list of surfaces embedded in $T \times I$. We will then consider bundles of the form

$$T \times I/_{\eta_1} T \times I/_{\eta_2} \cdots /_{\eta_{n-1}} T \times I/_{\eta_n},$$

where each block contains one of the surfaces in our list. The surfaces and maps η_i will be such that after identification we obtain a properly embedded, connected surface in the bundle.

Sometimes the surface constructed this way will be non-orientable. In this case we will consider, instead, the orientable surface that is the boundary of a regular neighbourhood of the non-orientable surface. If $S \subset M^3$ is a properly embedded, non-orientable surface then we will use the notation $\tilde{S}$ to denote the boundary of a regular neighbourhood of S.

2.1. *Surfaces in $T \times I$*

Let

$$D_1 = a \times [0,1],$$
$$D_2 = a \times [0,1] \cup b \times [0,1],$$
$$D_3 = a \times [0,1] \cup b \times [0,1] \cup c \times [0,1].$$

Each D_i consists of pairwise-disjoint, non-parallel vertical disks in $T \times I$.

For each integer $q \neq 0$, let S_q be the surface defined as follows. Let a, a_+, and a_- be the closed curves shown in Fig. 1 and let $E = a \times [0,1]$. Let $S \subset T$ be the complement of the annular neighbourhood N of E, with a_- and a_+ as its two boundary components. Let $c_1, \ldots, c_{|q|}$ be parallel simple closed curves in N, numbered in order with c_1 being adjacent to a_-. The surface S_q is the union of $S \times \{\frac{1}{2}\}$ with a number of annuli. If $q < 0$ add annuli joining $a_- \times \{\frac{1}{2}\}$ to $c_1 \times \{0\}$, $a_+ \times \{\frac{1}{2}\}$ to $c_{-q} \times \{1\}$, and $c_{i+1} \times \{0\}$ to $c_i \times \{1\}$, for $i = 1, \ldots, |q|-1$. If $q > 0$, add annuli joining $a_- \times \{\frac{1}{2}\}$ to $c_1 \times \{1\}$, $a_+ \times \{\frac{1}{2}\}$ to $c_q \times \{0\}$, and $c_{i+1} \times \{1\}$ to $c_i \times \{0\}$, for $i = 1, \ldots, q-1$.

A schematic diagram of S_q is shown in Fig. 2.

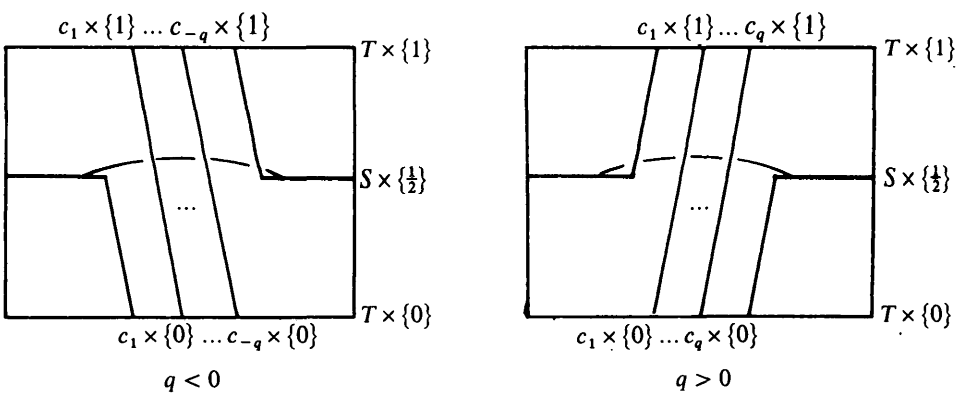


FIG. 2

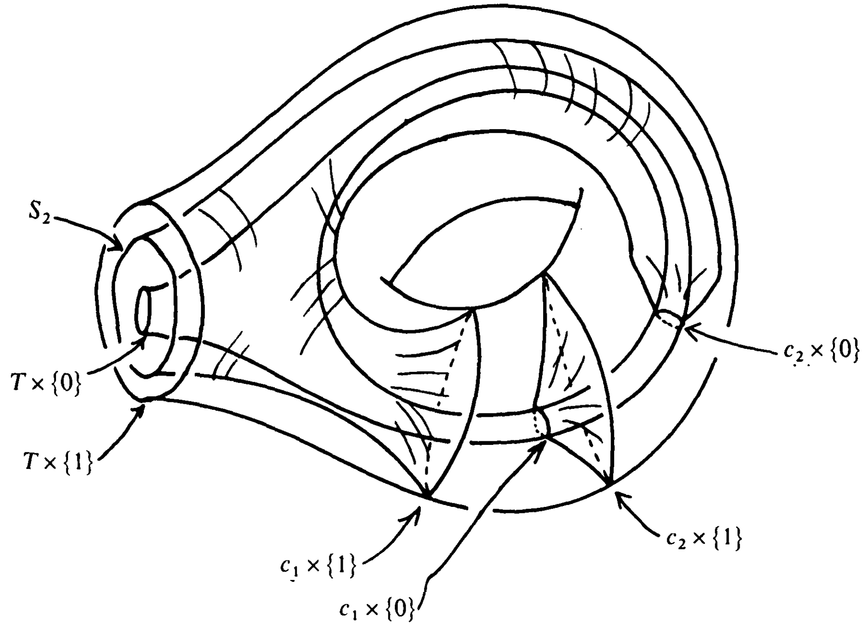
The surface S_2 is shown in Fig. 3.

It is clear that S_q can be embedded so that the projection $T \times I \to T$ restricts to a local homeomorphism $S_q \to T$.

For each integer n the surfaces $C_{a,n}$ and $C_{b,n}$ are disks in $T \times I$. The boundary of $C_{a,n}$ consists of the four arcs

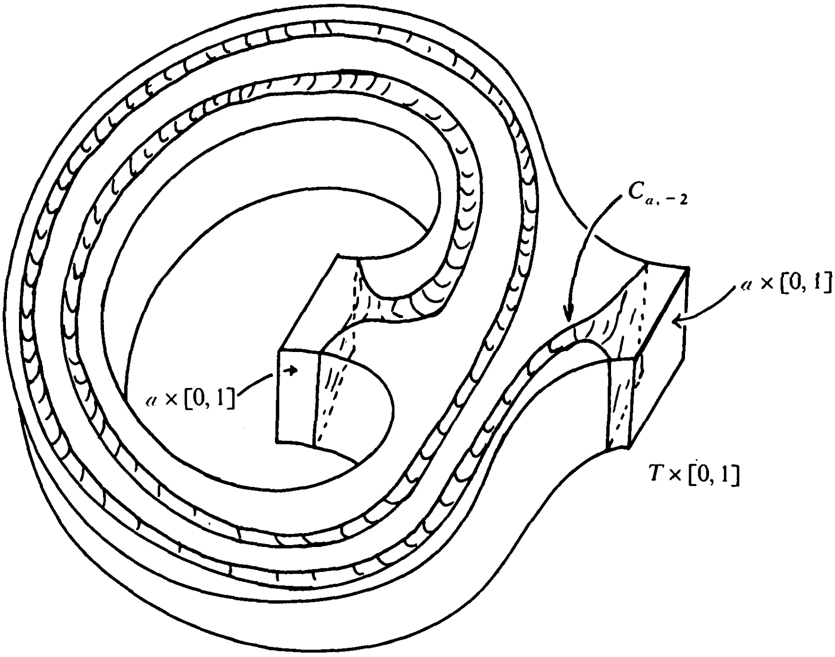
$$a_+ \times \{0\}, \quad a_- \times \{0\}, \quad \alpha^{-n}(b_+) \times \{1\}, \quad \text{and} \quad \alpha^{-n}(b_-) \times \{1\},$$

together with the four vertical arcs of the form $p \times [0,1]$ where p is an endpoint of a_+ or a_-. (Here a, a_+, a_-, b, b_+, and b_- are the curves shown in Fig. 1.) We define $C_{b,n}$ similarly, replacing a by b, b by a, and β by α. These disks look like twisted saddle surfaces—the surface $C_{a,-2}$ is shown in Fig. 4, where $T \times I$ is cut open along the disk $a \times [0,1]$.



S_2
$T \times \{0\}$
$T \times \{1\}$
$c_1 \times \{1\}$
$c_1 \times \{0\}$
$c_2 \times \{0\}$
$c_2 \times \{1\}$

FIG. 3



$C_{a,-2}$
$a \times [0, 1]$
$a \times [0, 1]$
$T \times [0, 1]$

FIG. 4

2.2. *Annuli in bundles*

The following surfaces are essential annuli contained in bundles whose characteristic class has trace 0, ± 1, or ± 2. We use the notation $\mathrm{Im}(D_i)$ to denote the images of the surfaces D_i for $1 \leqslant i \leqslant 3$ (defined in §2.1) in a quotient $T \times I/_\eta$, a bundle over S^1.

$$(\mathrm{Im}(D_1))^{\sim} \subset T \times I/_{\varphi^2},$$
$$(\mathrm{Im}(D_2))^{\sim} \subset T \times I/_{\varphi},$$
$$(\mathrm{Im}(D_3))^{\sim} \subset T \times I/_{\psi} \quad (T \times I/_{\psi} \text{ is the trefoil knot space}),$$
$$\mathrm{Im}(D_3) \subset T \times I/_{\psi^2},$$
$$\mathrm{Im}(D_1) \subset T \times I/_{\alpha^n}, \quad n \in \mathbb{Z},$$
$$(\mathrm{Im}(D_1))^{\sim} \subset T \times I/_{\varphi^2 \circ \alpha^n}, \quad n \in \mathbb{Z}.$$

2.3. *Tori in bundles*

The following surfaces are essential tori contained in bundles whose characteristic class has trace ± 2. Here we use $\mathrm{Im}(E)$ to denote the image of the surface E (defined in §2.1) in a quotient $T \times I/_\eta$.

$$\mathrm{Im}(E) \subset T \times I/_{\alpha^n}, \quad n \in \mathbb{Z},$$
$$(\mathrm{Im}(E))^{\sim} \subset T \times I/_{\varphi^2 \circ \alpha^n}, \quad n \in \mathbb{Z}.$$

2.4. *Spun surfaces in bundles*

Let p, q, and n be integers with $p > 0$ and $(p, q) = 1$. We may describe the bundle M with characteristic class $[A^n]$ as follows:

$$M = \underbrace{T \times I/_{\mathrm{id}} T \times I/_{\mathrm{id}} \cdots /_{\mathrm{id}} T \times I/_{\alpha^n}}_{p \text{ blocks}}.$$

If each block contains a copy of the surface S_q, then these surfaces will fit together in M to form a connected, properly embedded genus-1 surface with p boundary components. (Connectedness is equivalent to the condition $(p, q) = 1$.) We will call this the *spun* surface, $S(p, q, n)$. A schematic diagram of $S(5, 3, n)$ is shown in Fig. 5.

One could carry out the construction above using $\varphi^2\alpha^n$ in place of α^n. It is easily checked that the resulting (non-orientable) surface is compressible unless $p = 1$. If $p = 1$ and $q = \pm 1$ then we obtain a once-punctured Klein bottle, which is necessarily incompressible. However, the boundary of its regular neighbourhood, a twice-punctured torus, is compressible.

2.4.1. PROPOSITION. *The surface $S(p, q, n)$ is essential in the once-punctured torus bundle with characteristic class $[\alpha^n]$ for all p, q with $p > 0$ and $(p, q) = 1$.*

Proof. Let $\tilde{M}$ be the infinite cyclic cover of M, and let $\tilde{S}(p, q, n)$ be the inverse image of $S(p, q, n)$ under the covering projection. It suffices to show that $\tilde{S}(p, q, n)$ is incompressible in $\tilde{M}$, for this implies that $S(p, q, n)$ is incompressible in M. Since $S(p, q, n)$ is clearly not boundary parallel, it is therefore essential.

If $\tilde{M}$ is identified with $T \times \mathbb{R}$ in the obvious way then the projection $\tilde{M} \to T$

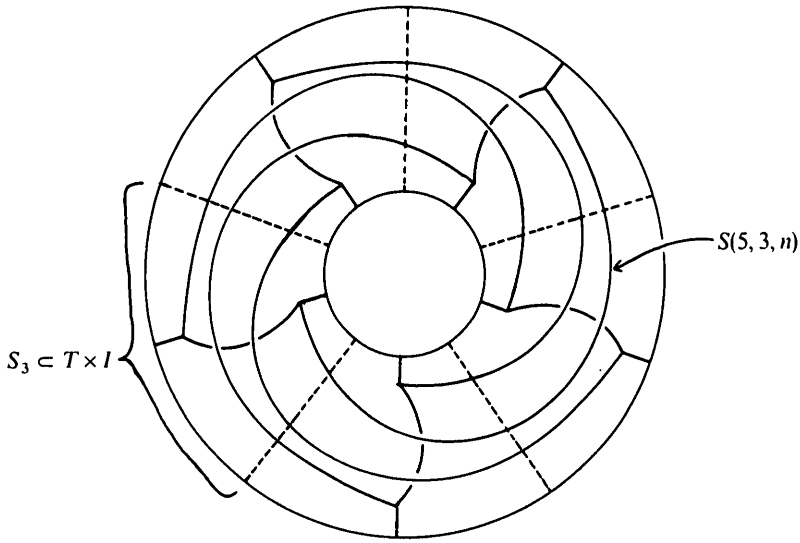


FIG. 5

restricts to a covering projection $\tilde{S}(p, q, n) \to T$. Thus the inclusion

$$\pi_1(\tilde{S}(p, q, n)) \to \pi_1(M) \approx \pi_1(T)$$

is an injection, so $\tilde{S}(p, q, n)$ is incompressible in $\tilde{M}$.

2.5. *Twisted surfaces*

Let $n(1), n(2), \dots, n(k)$ be arbitrary integers and let $J \in \{-1, 0, 1, 2\}$ have the same parity as k. Consider the bundle

$$M = T \times I/_{\alpha^{n(1)}} T \times I/_{\beta^{n(2)}} \dots /_{\gamma^{n(k)}} T \times I/_{\varphi^J}$$

where

$$\gamma = \begin{cases} \beta, & \text{if } k \text{ is even,} \\ \alpha, & \text{if } k \text{ is odd.} \end{cases}$$

Then M contains $k+1$ blocks.

Let the ith block in M, where $1 \leqslant i \leqslant k$, contain a copy of $C(a, n(i))$ if i is *odd* and a copy of $C(b, n(i))$ if i is *even*. Let the $k+1$th block contain the two vertical disks $(a_+ \cup a_-) \times I$ if i is *even* and $(b_+ \cup b_-) \times I$ if i is *odd*. After identification these surfaces fit together to give a properly embedded, connected surface R. Notice that R is orientable if k is even and R is non-orientable if k is odd. We define the twisted surface, $C(J; n(k), \dots, n(1))$, to be the orientable surface R if k is even and to be the orientable surface $\tilde{R}$ if k is odd. The surface $C(0; n(k), \dots, n(1))$ has genus $(\frac{1}{2}k - 1)$ and four boundary components; the surface $C(2; n(k), \dots, n(1))$ has genus $(\frac{1}{2}k)$ and two boundary components; the surfaces $C(1; n(k), \dots, n(1))$ and $C(-1; n(k), \dots, n(1))$ both have genus k and two boundary components.

Note that the bundle M has characteristic class

$$[P^J C^{n(k)} \dots A^{n(3)} B^{n(2)} A^{n(1)}],$$

where

$$C = \begin{cases} A & \text{if } k \text{ is odd,} \\ B & \text{if } k \text{ is even.} \end{cases}$$

2.5.1. PROPOSITION. *Let $J, n(1), \dots, n(k)$, and M be as above. The surface $C(J; n(k), \dots, n(1))$ is essential in M if and only if $|n(i)| \geqslant 2$ for $i = 1, \dots, k$.*

Proof. Let $\tilde{M}$ be the cyclic cover, corresponding to the fibre, of the bundle

$$M = T\times I/_{\alpha^{n(1)}} T\times I/_{\beta^{n(2)}} \dots T\times I/_{\varphi^{J}}$$

and let $S \subset \tilde{M}$ be a component of the inverse image of $C(J; n(k), \dots, n(1)) \subset M$ under the covering projection. Clearly it suffices to show that S is incompressible in $\tilde{M}$. We note that $\tilde{M}$ is divided into blocks, which are inverse images of the blocks in the bundle, and that each block in $\tilde{M}$ meets S in one disk, which is embedded as in Fig. 4. Let F be the union of the fibres along which the blocks in $\tilde{M}$ are joined.

To prove the proposition we will consider the family of all compressing or boundary-compressing disks for S in M. If this family is non-empty then there exists a member D for which $D \cap F$ has the minimal number of components.

We will analyse the ways that this minimal disk can meet the blocks in $\tilde{M}$, and conclude that such a disk exists if and only if $|n(i)| < 2$ for some $i = 1, \dots, k$.

Let X be a block of $\tilde{M}$ with $D \cap X \neq \varnothing$. If we cut X along the disk $X \cap S$, then we obtain two solid tori as shown in Fig. 6. In order to see how D meets X, we shall consider how D meets these tori.

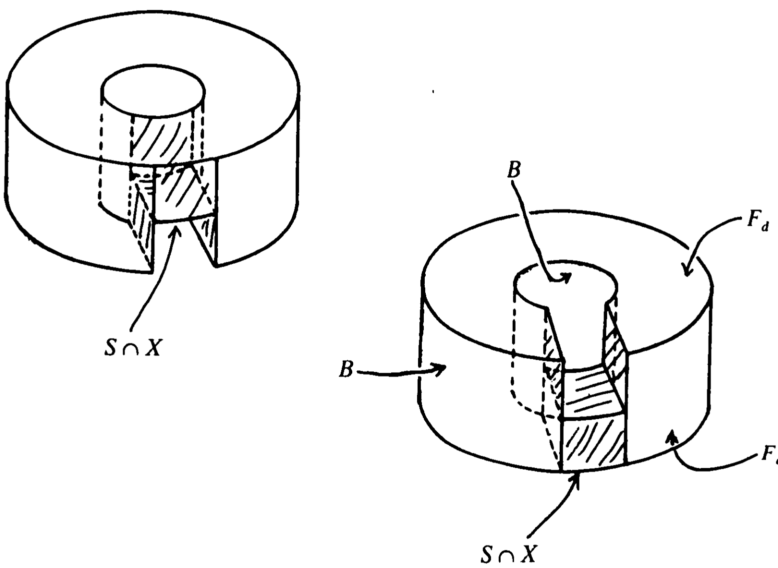


FIG. 6

Consider one of the solid tori that has non-empty intersection with D. Its boundary meets S in one disk, meets $\partial\tilde{M}$ in two disks (labelled B), and meets F in a disk, F_d, and an annulus, F_a.

These subsets of the torus are shown in Fig. 6. We will prove that $D \cap F_d = \varnothing$. In particular, it will follow that $D \cap X$ is contained in only one of these two solid tori.

First, observe that there cannot be any simple closed curve components of $F \cap D$ at all. For if there is a simple closed curve component of $F \cap D$, then there is one, say σ', that is 'innermost' on D; that is, σ' bounds a disk $D' \subset F$ and $D' \cap F = \sigma'$. Now, D' is a disk in $\tilde{M}$ and $D' \cap F = \partial D'$. However, F is incompressible, so $\partial D'$ also bounds a disk in F. Using standard techniques we may replace D by a disk which meets F fewer times but has the same boundary. Thus, we arrive at a contradiction to the minimality assumption for D.

So, suppose that σ is an arc component of $D \cap F_d$. There are four possibilities for such an arc.

(1) Both end points of σ are contained in B. If this is the situation, then D must be a boundary compression; and the arc σ is a spanning arc of D with both end points in $\partial \tilde{M}$. Among all the components of $D \cap F_d$ that are spanning arcs in D with both of their end points in $\partial \tilde{M}$, consider one, say σ', that is 'outermost' on D; that is, there is a disk $D' \subset D$ with $\partial D'$ consisting of two arcs σ' and δ', where $\delta' \subset \partial D$ and $D' \cap F_d = \sigma'$. Also, $\delta' \subset \partial \tilde{M}$. The disk $D' \subset \tilde{M}$ has $\partial D' = \sigma' \cup \delta'$ with $\sigma' = D' \cap F_d$ and $\delta' = D' \cap \partial \tilde{M}$. However, F is boundary-incompressible, and $\partial \tilde{M}$ is incompressible. Therefore, by standard techniques we may construct a boundary-compressing disk which meets S in the same arc as D, but meets F in fewer components than D. This contradicts the minimality assumption on D.

(2) Both end points of σ are in the same component of $S \cap F_d$. Here, the arc σ is a spanning arc of D with both end points in S. Among all the components of $D \cap F_d$ that have both end points in the same component of $S \cap F_d$, choose one, say σ', that is 'outermost' on F_d; that is, there is a disk $\Delta \subset F_d$, and $\partial\Delta$ consists of two arcs, σ' and δ', with $\delta' \subset S \cap F_d$ and $\Delta \cap D = \sigma'$. Now, perform a surgery of D along Δ. The result is two disks each of which meets F in fewer components than D. Moreover, at least one of these disks must be either a non-trivial compressing disk for S or a non-trivial boundary-compressing disk for S. In either case we contradict the minimality assumption for D.

(3) One end point of σ is contained in B and one is contained in S. Again, D must be a boundary compression, and the arc σ is a spanning arc of D with one end point in $\partial \tilde{M}$ and one end point in S. Since neither situation (1) nor situation (2) can occur, it is possible to choose among all the arc components of $D \cap F_d$ with one end point in B and one end point in S one, say σ', that is 'outermost' on F_d; that is, there is a disk $\Delta \subset F_d$ such that $\partial\Delta$ is the union of three consecutive arcs σ', $s' \subset S$, and $\delta' \subset B$, and $\Delta \cap D = \sigma'$. Now perform a surgery of D along Δ. The result is two disks, each of which meets F in fewer components than D. And, since $s' \subset S$ and $\delta' \subset \partial \tilde{M}$, at least one of these two disks is a non-trivial boundary compression for D. Again, this contradicts the minimality assumption for D.

(4) One end point of σ is contained in one component of $S \cap F_d$ and one is contained in another component of $S \cap F_d$. The arc σ is a spanning arc of D with both end points in S; and the arc σ is a spanning arc of F_d and is parallel in F_d into an arc component of $F_d \cap B$. Since none of the possibilities (1), (2), or (3) can occur, among all the components of $D \cap F_d$ there is one, say σ', that is 'outermost' on F_d; that is, there is a disk $\Delta \subset F_d$, $\partial\Delta$ is the union of four consecutive arcs σ', $s \subset S$, $\delta' \subset B$, and $s' \subset S$, and $\Delta \cap D = \sigma'$. Perform a surgery of D along Δ. In this situation, two disks result, each of which meets F in fewer components than D. However, there is only one of these two disks that is a candidate for a compressing or boundary-compressing disk of S; its boundary is the union of two arcs, one in S and one in $\partial \tilde{M}$. Denote this disk by D'. Now, since the end points of σ' are in different components of $F_d \cap S$, the arc in $\partial D'$ that is also in $\partial \tilde{M}$ runs between different components of ∂S; the disk D' must be a non-trivial boundary-compressing disk for S. This contradicts the minimality assumption for D.

In each possible situation we have arrived at a contradiction. We conclude that $D \cap F_d = \varnothing$.

Let X' denote the block in $\tilde{M}$ that meets the block X along the component of F containing F_a. Now, X' splits along S into two solid tori just as X did; furthermore, the same argument as above for X applies to X'. Hence D is actually contained in the union of two solid tori, one from X and one from X', that are joined along the annulus F_a. This union is again a solid torus and it meets S along an annulus as shown in Fig. 7. The annulus of S wraps around the solid torus $|n(i)|$ times for some i.

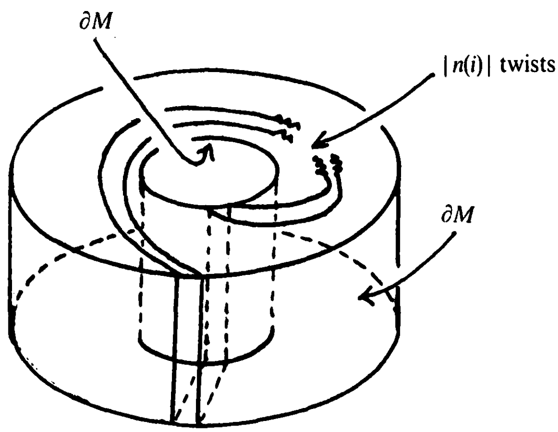
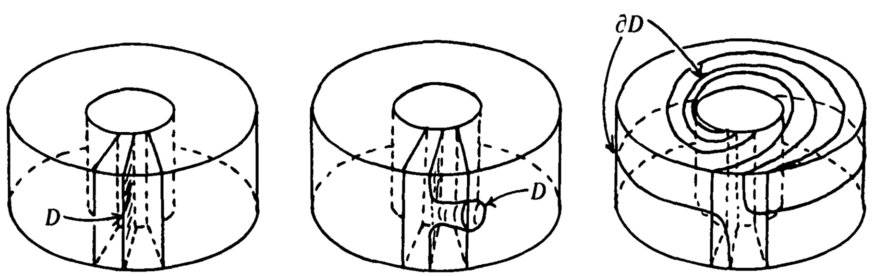


FIG. 7

If D is a compressing disk then ∂D is contained in the annulus. This implies that the annulus of S is compressible in the solid torus, so $n(i) = 0$ (Fig. 8(a)).



(a) $n(i) = 0$ (b) $n(i) = 0$ (c) $n(i) = \pm 1$

FIG. 8

If D is a boundary-compressing disk, then D meets the annulus of S in one arc and ∂M in one arc. Consideration of intersection numbers shows that this can happen exactly when $|n(i)| = 0$ or $|n(i)| = 1$ (see Figs 8(b) and 8(c)).

2.5.2. REMARK. A stronger statement than that of Proposition 2.5.1 is actually true. Let $\hat{M}$ be the manifold constructed by attaching a solid torus to M so that the boundary curves of $C(J; n(k), \ldots, n(1))$ are identified to contractible curves in the solid torus. Let $\hat{C}(J; n(k), \ldots, n(1))$ be the surface in $\hat{M}$ obtained by attaching disks to the boundary curves of $C(J; n(k), \ldots, n(1))$. It can be shown by techniques similar to those used in the proof of Proposition 2.5.1 that $\hat{C}(J; n(k), \ldots, n(1))$ is incompressible in $\hat{M}$ if and only if $|n(i)| \geqslant 2$ for $i = 1, \ldots, k$.

2.6. *Summary*

We have given four different methods of constructing once-punctured torus bundles which contain essential surfaces other than the fibre. Next we will show that any essential surface in a once-punctured torus bundle can be constructed by one of these methods.

This statement can be made more precise by means of the following definition. If S and S' are surfaces contained, respectively, in the once-punctured torus bundles M and M', then we will say that S and S' are of the *same type* if there is a bundle equivalence from M to M' that maps S to S'. In the next section we will show that any essential surface in a once-punctured torus bundle is of the same type as $\mathrm{Im}(D_i)$ or $(\mathrm{Im}(D_i))^{\sim}$ from §2.2, $\mathrm{Im}(E)$ or $(\mathrm{Im}(E))^{\sim}$ from §2.3, $S(p,q,n)$ from §2.4, or $C(J; n(k), \ldots, n(1))$ from §2.5.

3. *General position*

Throughout this section we will let M be a once-punctured torus bundle over S^1, and S an essential surface in M. We will say that S is in *general position* provided that

(1) each component of ∂S either is contained in a fibre or is transverse to every fibre,

(2) the projection of p: $M \to S^1$ restricts to a morse function on the interior of S having distinct critical values,

(3) among all surfaces isotopic to S and satisfying (1) and (2), S has the minimal number of index 0 or 2 critical points.

The usual considerations show that S can be moved by an isotopy so that it is in general position. We will assume from now on that this has been done. The level sets of $p|_S$ are, of course, the intersections of S with the fibres of M. By *level arcs* and *level curves* we mean the components of non-critical level sets.

3.1. *Level arcs*

3.1.1. LEMMA. *Each level arc of S is essential in the fibre containing it.*

Proof. Since S is boundary-incompressible it suffices to show that each level arc is essential in S. This follows from Condition (1) above.

3.2. *Upper and lower level sets*

Let x be a critical point of $p|_S$ with $p(x) = t$, and let I be the largest interval containing t so that there are no critical values other than t in I. Let X be the component of $S \cap p^{-1}(I)$ that contains x. We will call this the *critical neighbourhood* of x. For small ε, X meets the fibres $p^{-1}(t+\varepsilon)$ and $p^{-1}(t-\varepsilon)$ in 1-manifolds having at most two components. The projections (via the local product structure) of these 1-manifolds onto the fibre $p^{-1}(t)$ will be called, respectively, the *upper* and *lower level sets of* x. (They are well-defined up to isotopy.)

If x is an index 0 or 2 critical point then one of the upper and lower level sets is a contractible closed curve and the other is empty. If x is an index 1 critical point then the upper level set for x is obtained (up to isotopy) from the lower level set by taking a band sum in the fibre $p^{-1}(t)$. See Fig. 9.

Thus, given the lower level set for an index 1 critical point, the possibilities for the upper level set correspond to the isotopy classes of arcs in the fibre which are disjoint from S and have end points in the lower level set.

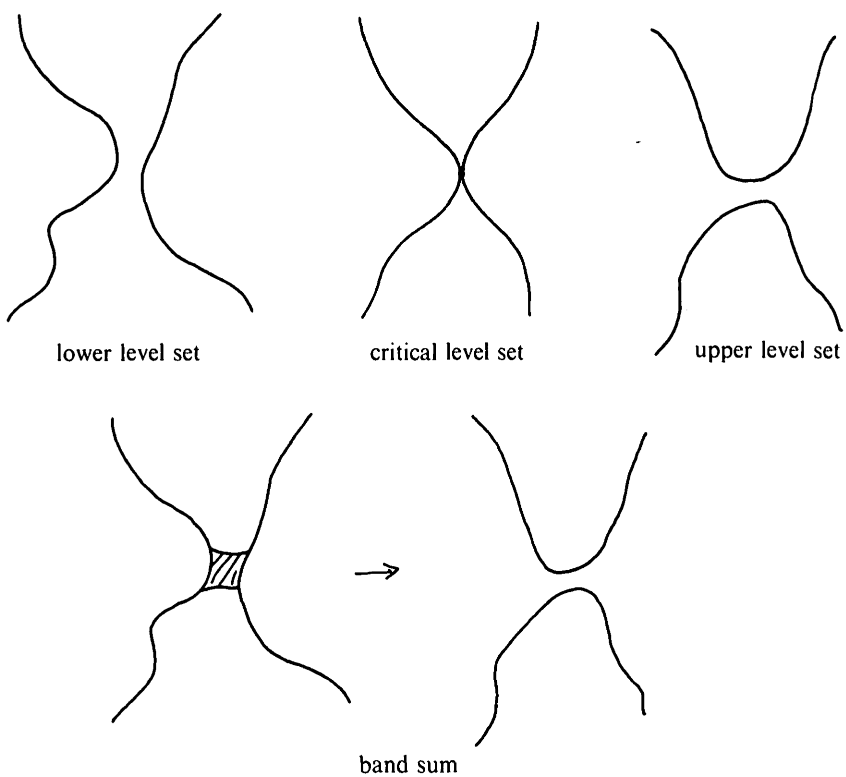


FIG. 9

3.2.1. LEMMA. *Either S meets every non-critical fibre only in arcs or S meets every non-critical fibre only in simple closed curves.*

Proof. Since S is connected it suffices to show that the upper and lower level sets of each critical point consist entirely of arcs or entirely of simple closed curves. By symmetry we need consider only the lower level sets.

Suppose that the lower level set of the critical point x consists of an arc and a simple closed curve. There are two cases depending on whether the closed curve is essential. If the curve is essential then, since the fibre split along the arc is an annulus, the isotopy class of the curve is determined. The upper and lower level sets of x must appear as in Fig. 10. In particular, the upper level set of x is a boundary-parallel arc, which contradicts Lemma 3.1.1.

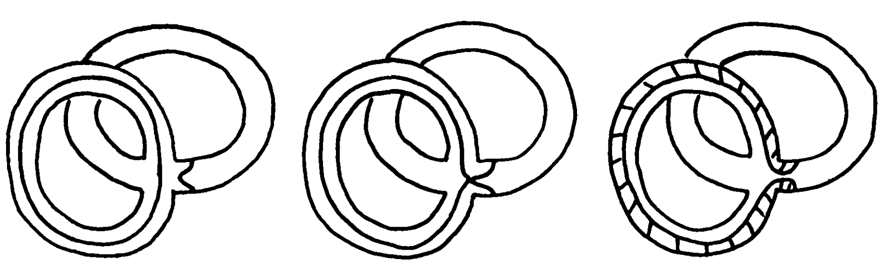


FIG. 10

If the simple closed curve component of the lower level set of x bounds a disk in the fibre $p^{-1}(t)$ then the level sets of x appear as in Fig. 11.

Since S is incompressible, the curve component of $S \cap p^{-1}(t-\varepsilon)$, for small $\varepsilon > 0$, must bound a disk on S. This disk must contain at least one index 0 critical point. However, there is an isotopy that cancels x and the index 0 critical points in the disk without introducing new critical points. This contradicts the minimality of the number of index 0 or 2 critical points on S. A careful construction of this isotopy can be made by using the observation that a level-preserving isotopy cannot introduce new critical points.

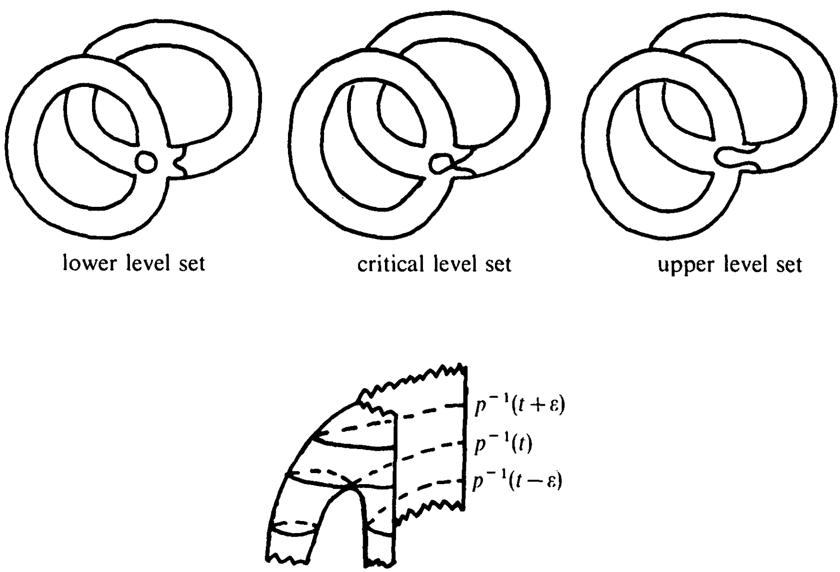


FIG. 11

As a corollary of this lemma we see that if the intersection of S with any fibre contains an arc then there are no index 0 or 2 critical points. This will happen, except when each boundary component of S is contained in a fibre.

With patience one may catalogue all of the possibilities for the upper and lower level sets of an index 1 critical point on S. Because the level sets of S consist either entirely of simple closed curves or entirely of essential arcs, there are twelve possibilities for the lower level set of x. These depend on whether it has one or two components and whether these components are contractible closed curves, boundary-parallel closed curves, essential closed curves, or essential arcs. Also, if there are two arc components, either these will be parallel or their complement will be a disk. For each of the possible lower level sets one constructs the possible upper level sets by taking band sums and eliminating those that contain boundary-parallel arcs or have both simple closed curves and arc components. Finally, one eliminates the case when the lower level set consists of one arc and the upper level set is obtained by a band sum joining opposite sides of the arc. (See Fig. 12.) This case cannot occur because the surface S is orientable.

We invite the reader to construct this catalogue. Having done so it will be easy to verify the following two facts.

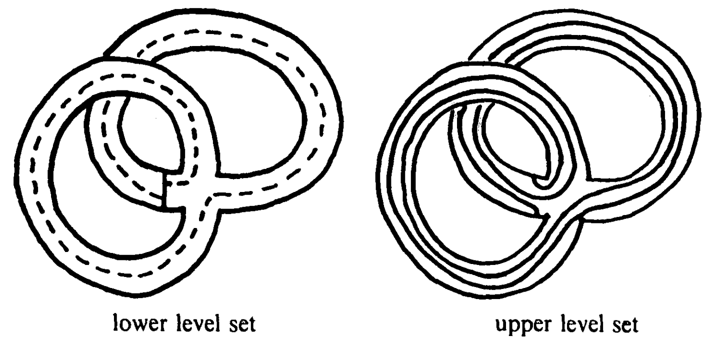


FIG. 12

3.2.2. *If both the upper and lower level sets of X contain an essential closed curve, then these curves are isotopic.*

The interesting case is shown in Fig. 13.

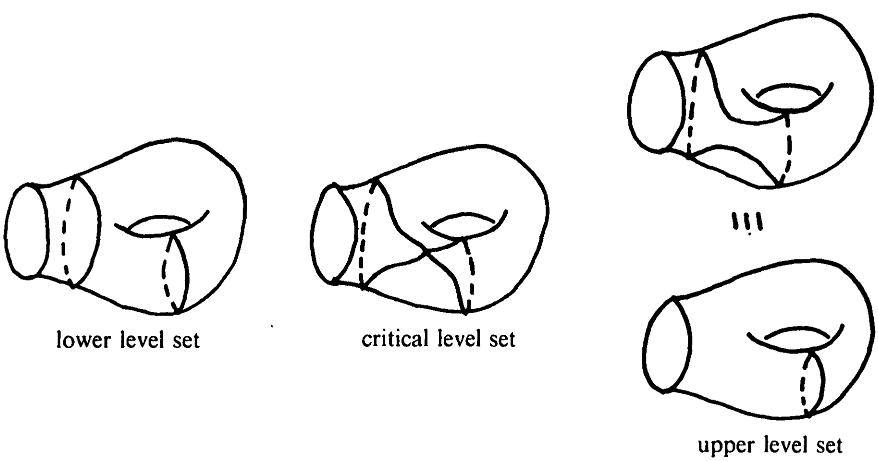


FIG. 13

3.2.3. *If the lower level set contains two arcs then*

(i) *they are parallel, and*

(ii) *the upper level set is obtained by a band sum across the annulus component of their complement* (see Fig. 14).

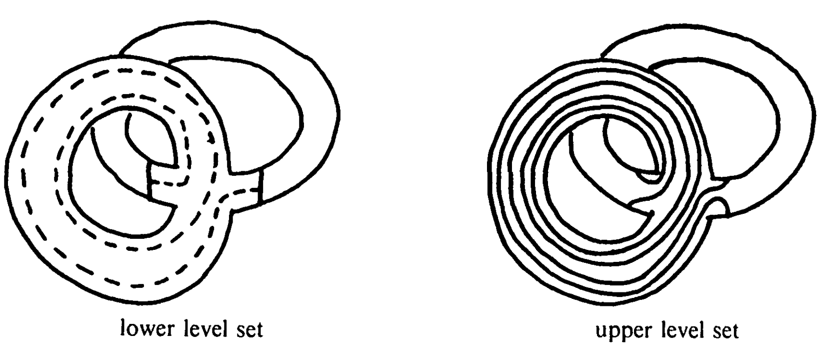


FIG. 14

3.3. *Isotopy classes of essential surfaces*

3.3.1. THEOREM. *An essential surface in a once-punctured torus bundle over S^1 is isotopic either to the fibre or to a surface having the same type as one of those given in* §§2.2, 2.3, 2.4, *or* 2.5.

Proof. We move S by an isotopy so that S is in general position. By Lemma 3.2.1 either S meets every non-critical fibre only in arcs or S meets every non-critical fibre only in simple closed curves. We consider these two cases separately.

Case 1: S meets every non-critical fibre only in simple closed curves. Suppose that, for some non-critical fibre F, none of the simple closed curve components of $S \cap F$ is essential in F (a simple closed curve is *essential* in a surface if it is not contractible and not parallel into the boundary). We can assume, possibly after an isotopy of S (we do not need to keep general position here, even though it could be maintained), not only that each component of $S \cap F$ is a simple closed curve that is not essential but that, within the isotopy class of S, the number of such curves is minimal. It follows that either $S \cap F = \emptyset$ or each component of $S \cap F$ is parallel into ∂F.

Now, split M at F to obtain a product $T \times I$ (T is a once-punctured torus). Let S' be the result of splitting S at $S \cap F$. The surface S' is incompressible in $T \times I$. Each component of $\partial S'$ is contained in $\partial T \times I$, $T \times \{0\}$, or $T \times \{1\}$ and in the latter two cases such components of $\partial S'$ are parallel into ∂T. It follows from [2, § 8, Appendix] that a component of S' is either an annulus or a once-punctured torus. Now, by considering the placement of the boundary of such an annulus or once-punctured torus, we see that the above minimality condition for the number of components of $S \cap F$ leaves precisely two possibilities. Either $S \cap F$ has one component and S' is an annulus parallel into $\partial T \times I$ or $S \cap F = \emptyset$ and S' is parallel in $T \times I$ to $T \times \{0\}$. Hence, S is either a torus and is parallel into ∂M (S is not essential) or S is parallel to a fibre.

So, suppose that *every* non-critical fibre F contains an essential simple closed curve component of $S \cap F$. In this situation observe that for F a non-critical fibre, the components of $S \cap F$ consist of one (non-empty) family of parallel, essential simple closed curves, a (possibly empty) family of simple closed curves each parallel into ∂F, and a (possibly empty) family of simple closed curves each contractible in F. We shall show that in a critical fibre the essential closed curves of the upper level set are isotopic in the fibre to those of the lower level set.

Let x be a critical point with $p(x) = t$ and let X be the critical neighbourhood of x. Suppose that, for small $\varepsilon > 0$, $S \cap p^{-1}(t-\varepsilon)$ contains an essential closed curve s that is *not* contained in X. Then the component of $S \cap p^{-1}([t-\varepsilon, t+\varepsilon])$ containing s has no critical points and hence is an annulus. Therefore S meets the fibre $p^{-1}(t+\varepsilon)$ in a curve isotopic to s. By symmetry we may therefore assume that all essential curves of $p^{-1}(t-\varepsilon) \cap S$ and $p^{-1}(t+\varepsilon) \cap S$ are contained in X. Now, it follows from observation 3.2.2 that the essential curves in the two fibres are isotopic.

By our assumption that *every* non-critical fibre F contains an essential simple closed curve component of $S \cap F$, we have established our claim that, in a critical fibre, the essential closed curves in the upper level set are isotopic to those in the lower level set. So, in particular, we know that the characteristic class of M fixes the isotopy class of an essential curve, and hence has trace ± 2.

Let F be a non-critical fibre; we can assume, possibly after an isotopy of the essential simple closed curves of S at a non-critical fibre, that $S \cap F$ does not contain any contractible simple closed curves and that relative to all the preceding conditions $S \cap F$ has a minimal number of components.

We split the manifold M at F to obtain a product $T \times I$, where T is a once-punctured torus. Let S' be the result of splitting S at $S \cap F$. Now, there is an annulus $R \subset T \times I$ with ∂R having one component in $T \times \{0\}$ and one component in $T \times \{1\}$, each an essential simple closed curve parallel to the components of $\partial S'$ corresponding to the family of essential simple closed curves in $S \cap F$. Furthermore, $R \cap S' \subset R - \partial R$. We may assume that, among all such annuli, the number of components of $R \cap S'$ is a minimum. In particular, it follows that either $R \cap S' = \varnothing$ or each component of $R \cap S'$ is a non-contractible simple closed curve.

Split $T \times I$ at R to obtain $T' \times I$, where T' is a disk with two holes. Let S'' be S' split at $R \cap S'$. Each component of S'' is incompressible in $T' \times I$; and each component of S'' is contained in $\partial T' \times I$, $T' \times \{0\}$, or $T' \times \{1\}$. Again, from [2, § 8, Appendix] we have that a component of S'' is either an annulus or a disk with two holes. We again consider the placement of the boundary of such an annulus or disk with two holes. Using the minimality conditions on the number of components of $S \cap F$ and the number of components of $S' \cap R$, we can describe completely the placement of the components of S''. We omit this analysis, but show the only possibilities for the components of S'' in Fig. 15.

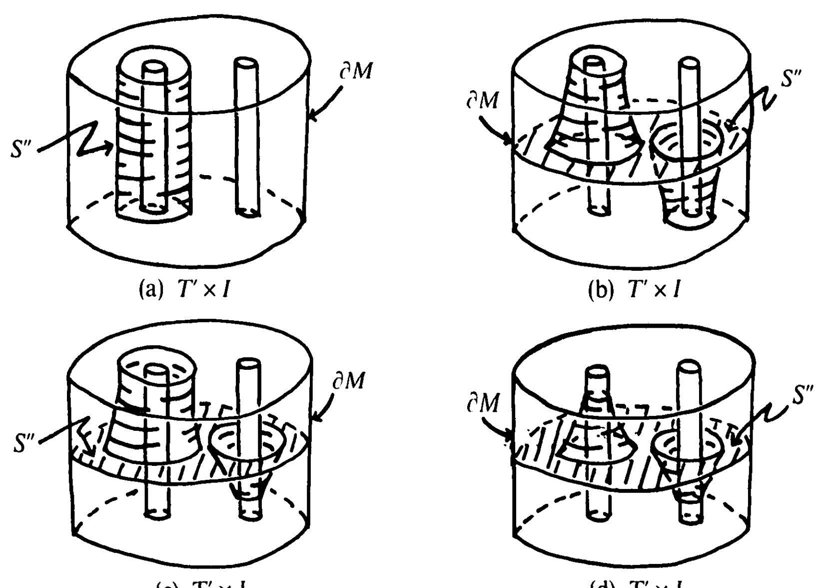


FIG. 15

We conclude that S either has the type of one of the tori, Im(E) or $(\mathrm{Im}(E))^{\sim}$, of § 2.3 or S has the type of one of the surfaces $S(p, q, n)$ of § 2.4. This completes Case 1.

Case 2: S meets every non-critical fibre only in essential arcs. Notice that a once-punctured torus T has the property that the maximum number of pairwise disjoint, non-parallel essential arcs in T is three.

If a non-critical fibre F contains three essential level arcs, no two of which are parallel, it is easy to check that there are no critical levels.

Split M at a non-critical fibre F to obtain the product $T \times I$ and let S' be S split at $F \cap S$. Then each component of S' is a vertical disk in $T \times I$ meeting each level in three distinct families of parallel, essential arcs. By connectivity (and orientability) considerations, either S has six vertical disks in $T \times I$ and S is an annulus of type $(\mathrm{Im}(D_3))^{\sim}$, coming from a bundle with characteristic class $[Q]$, or S' has three vertical disks in $T\times I$ and S is an annulus of type $\mathrm{Im}(D_3)$, coming from a bundle with characteristic class $[Q^2]$.

We can now assume that S meets each non-critical fibre in at most two parallel families of arcs.

Next we need an observation that uses the incompressibility of S. Suppose that F is a non-critical fibre that meets S in two parallel families of arcs. Let x be the critical point immediately below F and y the critical point immediately above. Let X and Y be the respective critical neighbourhoods of x and y. Recall that the upper and lower level sets of a critical point each consist of two parallel arcs, and that the upper level set is obtained by a band sum across the annulus component of the complement of the lower level set. This means that X and Y meet F in arcs that are 'outermost' in their parallel family (i.e. the parallel family is contained in the disk component of the complement of these arcs). We observe that the arcs $X\cap F$ are not in the same parallel family as $Y\cap F$. That is to say that the sequence of level sets shown in Fig. 16 cannot occur.

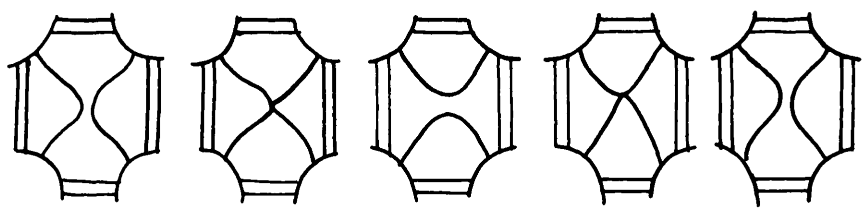


FIG. 16

To see this, notice that, since $X\cap F$ and $Y\cap F$ are outermost, if they were contained in the same family then they would be equal. Thus $X\cup Y$ would be an annulus contained in S that is contained in a 3-cell in M. This is impossible since S is incompressible. (The compression that would result is analogous to that shown in Fig. 8.)

Now we claim that either S meets some fibre in a single parallel family of arcs, or S is a surface of the same type as the annulus $(\mathrm{Im}(D_2))^{\sim}$.

For the proof of this claim suppose that S meets every non-critical fibre in two parallel families of arcs. One of these families is distinguished in each fibre by the fact that its outermost arcs are contained in the critical neighbourhood of the critical point immediately above the fibre. Let X be a critical point with $p(x)=t$. The preceding observation shows that for small $\varepsilon>0$, the distinguished family of arcs in $p^{-1}(t+\varepsilon)$ contains two fewer arcs than the distinguished family in $p^{-1}(t-\varepsilon)$. This implies that there are no critical points in S, for otherwise it would follow by induction that the distinguished family of arcs in F contains fewer arcs than itself. An easy combinatorial argument now shows that, since S is connected, there are exactly two arcs in each parallel family in $S\cap F$. Therefore S is of the same type as $(\mathrm{Im}(D_2))^{\sim}$.

Finally, suppose that S meets the fibre F in one family of parallel arcs. If S contains no critical points, then another easy combinatorial argument shows that there are at most two arcs in the family. It follows that the characteristic class of M has trace 2 and hence that S has the type of $(\mathrm{Im}(D_1))^{\sim}$ in $T\times I/_{\varphi^2}$, or of $(\mathrm{Im}(D_1))^{\sim}$ in $T\times I/_{\varphi^2\circ\alpha''}$, or of $\mathrm{Im}(D_1)$ in $T\times I/_{\alpha''}$.

If S does contain a critical point, we consider the product M' obtained by splitting M along F. Let $S'\subset M'$ be S split along $S\cap F$. The earlier observation then implies, by another combinatorial argument, that each component of S' meets each non-critical fibre in exactly two arcs. Therefore, in order for S to be connected, there must be either 2 or 4 arcs in $S \cap F$. It follows that S is of type $C(J; n(k), \ldots, n(1))$ where J is odd if $S \cap F$ has four components, and even otherwise. Note that S contains k critical points if J is even, and $2k$ if J is odd.

4. *Isotopy*

A given once-punctured torus bundle may contain several of the essential surfaces of the types that we have described. We now determine which of these are in the same isotopy class.

If M is a fibre bundle, an isotopy of M which is a bundle equivalence at each time will be called a *bundle isotopy.*

4.1. *Isotopies for surfaces of type* $C(J; n(k), \ldots, n(1))$

4.1.1. PROPOSITION. *Let M be a once-punctured torus bundle containing surfaces S and S' of types $C(J; n(k), \ldots, n(1))$ and $C(J; m(k), \ldots, m(1))$, respectively. If the k-tuple $(m(k), \ldots, m(1))$ is obtained from $(n(k), \ldots, n(1))$ by a cyclic permutation, then there is a bundle equivalence $h\colon M \to M$ so that S is isotopic to $h(S')$.*

Proof. The surface S' can be moved by a bundle isotopy so that both S and S' are divided into twisted saddles by the fibres $F_0, \ldots, F_{k-1}$ and so that for any $i \in \{0, \ldots, k-1\}$ each surface twists the same number of times in the block between F_i and $F_{i+1 \pmod k}$. Thus we can assume that $m(i) = n(i)$, for $i = 1, \ldots, k$.

The isotopy classes in F_i of the arcs $S \cap F_i$ and $S' \cap F_i$ are completely determined by the respective isotopy classes of $S \cap F_0$ and $S' \cap F_0$. While $S \cap F_0$ may not be isotopic to $S' \cap F_0$, there is a homeomorphism $\eta\colon F_0 \to F_0$ with $\eta(S' \cap F_0) = S \cap F_0$. Let $H \in \mathrm{SL}_2(\mathbb{Z})$ be the isotopy class of η. Then it follows from the definition of η that H commutes with $P^J \ldots B^{n(2)}A^{n(1)}$. Therefore η can be extended to a bundle equivalence $h\colon M \to M$. Both S and $h(S')$ are divided into twisted saddles by $F_0, \ldots, F_{k-1}$ and they meet each F_i in isotopic families of arcs. Thus S is isotopic to $h(S')$ by a bundle isotopy of M.

4.1.2. REMARK. Suppose that X and Y are commuting elements in a free product with amalgamation in which the amalgamating subgroup is central in each factor. Then X and Y are both contained in a cyclic subgroup, or X and Y are both contained in a factor, or one of X or Y is contained in the amalgamating subgroup.

In the notation of the proof of Proposition 4.1.1, this implies that either $H = P^2$ or H and $P^J \ldots B^{n(2)}A^{n(1)}$ are contained in a cyclic subgroup. Thus, under the hypotheses of Proposition 4.1.1, if the characteristic class of the bundle M is not a proper power, then S is isotopic to S'.

In general, if M has characteristic class $[G^s]$, where G is not a proper power, then there are *at most* $s-1$ bundle equivalences mapping S to a surface that is not isotopic to S. These bundle equivalences are extensions of homeomorphisms of the fibre in the isotopy classes $G, G^2, \ldots, G^{s-1}$.

4.1.3. PROPOSITION. *Let M be a once-punctured torus bundle containing surfaces S and S' of types $C(J; n(k), \ldots, n(1))$ and $C(J'; m(k'), \ldots, m(1))$, respectively. If S and S' are isotopic, then there is a bundle isotopy taking S to S'. Moreover, $k = k'$, $J = J'$, and $(m(k), \ldots, m(1))$ is a cyclic permutation of $(n(k), \ldots, n(1))$.*

Proof. Since S and S' are homeomorphic surfaces, an euler characteristic computation shows that k must equal k'. Once we have shown that S and S' are bundle isotopic it is immediate that $(m(k), \ldots, m(1))$ is a cyclic permutation of $(n(k), \ldots, n(1))$. It also follows that $J = J'$, since P^2X is not conjugate to X in $SL_2(\mathbb{Z})$. Thus we need show only that S and S' are bundle isotopic.

If $P \in \{1, -1\}$ then S and S' are boundaries of regular neighbourhoods of non-orientable surfaces. It suffices to show that these non-orientable surfaces are bundle isotopic. Thus we will assume in this case that S and S' have been replaced by the corresponding non-orientable surfaces.

Let $F_0, \ldots, F_{k-1}$ be fibres that divide S into twisted saddles and let $\mathcal{I}_t$ be an isotopy of M with $\mathcal{I}_0 = \mathrm{id}$ and $\mathcal{I}_1(S) = S'$. Standard arguments show that we can assume that $\mathcal{I}_t$ has the following properties:

(1) $\mathcal{I}_t(\partial S) = \partial S$ for all $t \in [0, 1]$;

(2) the projection $M \to S^1$ restricts to a morse function on $\mathcal{I}_t(S)$ for all $t \in [0, 1]$. (The morse function may not always have distinct critical values.)

A critical point of $\mathcal{I}_t(S)$ will be called *significant* if its upper and lower level sets consist of arcs. Since, by (1) above, $\mathcal{I}_t(S)$ meets each non-critical fibre in exactly two arcs, there can never be two significant critical points in the same fibre. Therefore there is a bundle isotopy $\mathcal{J}_t$ such that the fibres $\mathcal{J}_t(F_0), \ldots, \mathcal{J}_t(F_{k-1})$ separate the significant critical points of $\mathcal{I}_t(S)$ for all $t \in [0, 1]$. It follows that the arcs of $\mathcal{I}_t(S) \cap \mathcal{J}_t(F_i)$ are isotopic in $\mathcal{J}_t(F_i)$ to $\mathcal{J}_t(S \cap F_i)$ for all t. Therefore the fibres $\mathcal{J}_1(F_0), \ldots, \mathcal{J}_1(F_{k-1})$ divide both S' and $\mathcal{J}_1(S)$ into twisted saddles, and they meet the two surfaces in isotopic pairs of arcs. This implies that S' and $\mathcal{J}_1(S)$ are bundle isotopic, so S' and S are bundle isotopic.

4.2. *Isotopies for surfaces of type* $S(p, q, n)$

4.2.1. PROPOSITION. *Let M be the once-punctured torus bundle with characteristic class $[A^n]$, and let S and S' be surfaces of type $S(p, q, n)$ and $S(p', q', n)$ respectively. Then S is isotopic to S' if and only if $p = p'$ and $q = q'$.*

Proof. It is clear that S and S' are isotopic if $p = p'$ and $q = q'$. To prove the converse it is helpful to view M a little differently. Identify M with the bundle $T \times I/_{\alpha^n}$; and let $\hat{M}$ be the quotient of M obtained by identifying the boundary of each fibre to a point. Thus $\hat{M}$ is a torus bundle over S^1. We can also give $\hat{M}$ the structure of an S^1-bundle over a torus Z. The S^1-fibres will be circles contained in torus fibres and parallel to the image of the closed curve $a \times \{t\}$. The image in $\hat{M}$ of ∂M and the curve $b \times \{0\}$ are closed curves in $\hat{M}$ which project, respectively, to simple closed curves x and y in Z. These two curves define a framing of Z. The surfaces $\mathrm{Im}(S)$ and $\mathrm{Im}(S')$ are both saturated in the S^1-fibration of $\hat{M}$, and project to simple closed curves homologous to $p[x] + q[y]$ and $p'[x] + q'[y]$, respectively. If S and S' are isotopic in M, then their images are isotopic in $\hat{M}$, and their projections are homotopic in Z. Therefore $p = p'$ and $q = q'$.

5. *Classification*

We shall exploit the structure of $SL_2(\mathbb{Z})$ as the free product with amalgamation, $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, given in § 1.2. Recall that, with this structure, $SL_2(\mathbb{Z})$ has the presentation

$$SL_2(\mathbb{Z}) \cong | P, Q \colon P^4 = Q^6 = 1, P^2 = Q^3 |,$$

where

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

5.1. *Classification of bundles*

In Proposition 1.3.1, we proved that if M and N are once-punctured torus bundles with characteristic classes $[G]$ and $[H]$, respectively, then M is equivalent to N if and only if $[G] = [H]$. Now, by using the above structure of $SL_2(\mathbb{Z})$ and the normal form theorem for free products with amalgamation, we have that each element g of $SL_2(\mathbb{Z})$ can be written uniquely as

$$g = P^r X_1 X_2 \ldots X_s,$$

where the X_j $(1 \leqslant j \leqslant s)$ are chosen alternately from the two sets $\{P\}$ and $\{Q, Q^2\}$ and $r \in \{0, 2\}$. Furthermore, since P^2 is in the centre, the element $g \in SL_2(\mathbb{Z})$ is conjugate to an element whose normal form

$$P^r Y_1 \ldots Y_t$$

has the additional property that Y_1 and Y_t are not both contained in the same one of the two sets $\{P\}$ and $\{Q, Q^2\}$. The normal form of this latter element is obtained by cyclically reducing the normal form for g' and it is unique up to a cyclic permutation of $Y_1, \ldots, Y_t$. We will call such a normal form the *cyclically reduced normal form for g*. We will call $l(g) = t$ the *length of g*. Observe that if $l(g) = 0$, then $g = I$ or $g = P^2$; if $l(g) = 1$, then g is conjugate to one of $P^{\pm 1}, Q^{\pm 1}, Q^{\pm 2}$; otherwise, $l(g)$ is even.

We can use cyclically reduced normal forms to classify once-punctured torus bundles over S^1. Namely, except for bundles with characteristic class $[I]$, $[P^2]$, $[P^{\pm 1}]$, $[Q^{\pm 1}]$, and $[Q^{\pm 2}]$, once-punctured torus bundles over S^1 are classified by equivalence classes of words $P^r Q^{\varepsilon_1} P Q^{\varepsilon_2} P \ldots Q^{\varepsilon_s} P$, where $r = 0$ or 1, $\varepsilon_i = 1$ or 2 $(1 \leqslant i \leqslant s)$ and two words $P^r Q^{\varepsilon_1} P \ldots Q^{\varepsilon_s} P$ and $P^{r'} Q^{\delta_1} P \ldots Q^{\delta_{s'}} P$ are equivalent if $r' = r$, $s' = s$, and $(\delta_1, \ldots, \delta_s)$ is a cyclic permutation of $(\varepsilon_1, \ldots, \varepsilon_s)$. Later, in Examples 6.3, it will be convenient to use simply the notation $[\varepsilon_1, \ldots, \varepsilon_s]$ to denote the bundle with characteristic class $[Q^{\varepsilon_1} P \ldots Q^{\varepsilon_s} P]$. These correspond to characteristic classes with positive trace whereas the class $[P^2 Q^{\varepsilon_1} P \ldots Q^{\varepsilon_s} P]$ has negative trace.

5.2. *Classification of essential surfaces*

If M is a once-punctured torus bundle with characteristic class $[H]$, then for each element of $SL_2(\mathbb{Z})$ that is conjugate to H and has the form

$$P^J A^{n(k)} \ldots B^{n(2)} A^{n(1)}, \quad \text{where } J \in \{1, -1\} \text{ and } |n(i)| \geqslant 2,$$

or

$$P^J B^{n(k)} \ldots B^{n(2)} A^{n(1)}, \quad \text{where } J \in \{0, 2\} \text{ and } |n(i)| \geqslant 2,$$

we can construct an essential surface in M of type $C(J; n(k), \ldots, n(1))$.

Furthermore, by Remark 4.1.2 there are only a finite number of isotopy classes of such surfaces determined by the number J together with the k-tuple $(n(1), \ldots, n(k))$, modulo cyclic permutations. Conversely, by Theorem 3.3.1, an essential surface in the bundle, M (except for the fibre, certain annuli and tori, which appear in bundles whose characteristic class has trace of absolute value not greater than 2, and the surfaces of type $S(p, q; n)$, which appear in bundles whose characteristic class has trace

2), determines an element of $SL_2(\mathbb{Z})$ which has one of the above forms and is conjugate to H. The number J together with the k-tuple $(n(1),\dots,n(k))$, modulo cyclic permutations, are invariants of the isotopy class of the essential surface. A *special form* for an element of $SL_2(\mathbb{Z})$ is defined to be one of the two forms listed above.

The fact that there are only a finite number of types of essential twisted surfaces in a given bundle M is a corollary of the following proposition.

5.3. PROPOSITION. *The elements of a conjugacy class in* $SL_2(\mathbb{Z})$ *can be represented by only a finite number of special forms.*

Proof. The proof will follow from an estimate of the length of a word in special form. In particular, we will show that

$$l(P^J C^{n(k)} \dots B^{n(2)} A^{n(1)}) \geqslant 2 \sum_{i=1}^{k} |n(i)| - 2k,$$

where $C = A$ if k is odd and $C = B$ if k is even. Since $|n(i)| \geqslant 2$, it follows immediately from this estimate that the elements of a conjugacy class can be represented by only a finite number of special forms.

The estimate is made by analysing the (syllable) cancellation that takes place between A^n and B^m, where $n, m \in \mathbb{Z}$. The elements $A = QP$, $B = PQ$, $A^{-1} = PQ^2$, $B^{-1} = Q^2P$ are cyclically reduced, so the cancellation must take place between a power of A and a power of B. Since $|n|$, $|m| \geqslant 2$, this cancellation never involves more than half of the syllables of A^n or B^m. The situation in which the maximal number of syllables are cancelled is when a power of B (or A) appears between powers of A (or B) with the same sign; that is, if $l, m, n, > 0$, then

$$\begin{aligned} \dots A^l B^m A^n \dots &= \dots QP \dots QPPQ \dots PQQP \dots \\ &= P^2 \dots QP \dots Q^2 \dots PQ^2 P \dots \end{aligned}$$

and

$$\begin{aligned} A^{-l} B^{-m} A^{-n} &= \dots PQ^2 \dots PQ^2 Q^2 P \dots Q^2 PPQ^2 \dots PQ^2 \dots \\ &= P^2 \dots PQ^2 \dots PQP \dots Q \dots PQ^2 \dots . \end{aligned}$$

In either case four syllables are cancelled. In a word of the form $P^J A^{n(k)} \dots B^{n(2)} A^{n(1)}$, where $J \in \{1, -1\}$, the P^J may be cancelled. In this situation the maximal number of syllables are cancelled when $n(1)$ and $n(k)$ have the same sign; that is, if $m > 0$ and $n > 0$, then

$$\dots A^m P^J A^n \dots = \dots QPP^J Q \dots = P^{J+1} Q^2$$

and

$$\dots A^{-m} P^J A^{-n} = \dots Q^2 P^J PQ^2 \dots = P^{J'} \dots Q \dots,$$

where $J' = 2$ if $J = -1$, and $J' = 0$ if $J = 1$. In either case three syllables are cancelled.

So, in a word of the form $P^J B^{n(k)} \dots A^{n(1)}$, with $J \in \{0, 2\}$, which has syllable length of $2 \sum_1^k |n(i)|$, there are at most $2k$ syllables cancelled. In a word of the form $P^J A^{n(k)} \dots B^{n(2)} A^{n(1)}$, with $J \in \{1, -1\}$, which has a syllable length of $2 \sum_1^k |n(i)| + 1$, there are at most $2k+1$ syllables cancelled.

In the next section we use these results to list the essential surfaces in certain specified bundles.

6. *Computations*

Given a once-punctured torus bundle over S^1, say M, our methods enable us to find all essential surfaces in M. We are also able to put a framing (coordinate system) on ∂M and describe the boundary curves of all the essential surfaces in terms of this one framing. Using this latter information, we can draw some conclusions about the closed manifolds obtained by attaching a solid torus to ∂M; namely, we can describe all such attachments which give either reducible or Haken manifolds (see Remark 2.5.2).

6.1. *An algorithm*

Given a once-punctured torus bundle over S^1 (here the word 'given' can be interpreted simply as meaning that we are given a representative of a conjugacy class in $SL_2(\mathbb{Z})$) we shall write down the steps of a procedure for listing all essential surfaces of type $C(J; n(k), \ldots, n(1))$ in the given bundle.

Suppose $H \in SL_2(\mathbb{Z})$.

Step 1. Write H as a word in the syllables P, Q, and Q^2. (There are many ways to do this; e.g. by using row and column operations, first write H as a product of powers of A, B, and P. Then use the relations $A = QP$, $B = PQ$, $A^{-1} = PQ^2$, $B^{-1} = Q^2P$.)

Step 2. Write out the cyclically reduced normal form for H. Obtain $l(H)$. If $l(H) < 2$, then stop; there are no essential surfaces of type $C(J; n(k), \ldots, n(1))$ in the given bundle. If $l(H) \geqslant 2$, proceed.

Step 3. From the formula

$$l(P^J C^{n(k)} \ldots B^{n(2)} A^{n(1)}) \geqslant 2\left(\sum_1^k |n(i)| - 1\right) \geqslant 2k,$$

where $C = A$ if k is odd and $C = B$ if k is even, write all special forms that could possibly represent elements of $SL_2(\mathbb{Z})$ with length $l(H)$.

(For example, suppose that $l(H) = 6$. Then the possibilities for special forms that represent elements of $SL_2(\mathbb{Z})$ having length 6 are: $k = 1$ and $|n(1)| \leqslant 4$; $k = 2$ and $|n(1)| = 3$, $|n(2)| = 2$, or $|n(1)| = |n(2)| = 2$; $k = 3$ and $|n(1)| = |n(2)| = |n(3)| = 2$. Of course, for $l(H)$ large there are many possibilities; and, even in the above case, the many sign combinations must be considered. However, there are techniques that systematically eliminate many of the possibilities. Say in the above case that we were to consider $k = 2$ and $|n(1)| = |n(2)| = 2$. Then the syllable length of $P^J B^{n(2)} A^{n(1)}$, with $J = 0$ or 2, is 8. However, in arriving at the cyclically reduced normal form for $P^J B^{n(2)} A^{n(1)}$ there are precisely four cancellations when $n(1)$ and $n(2)$ have the same sign and no cancellations when $n(1)$ and $n(2)$ have opposite signs. Hence, for $k = 2$ and $|n(1)| = |n(2)| = 2$, $l(P^J B^{n(2)} A^{n(1)})$ is either 4 or 8, respectively; and so, no combination of $n(1) = \pm 2$ and $n(2) = \pm 2$ leads to a length 6 element.)

Step 4. From the special forms obtained in Step 3 list those that represent elements conjugate to H.

(This step uses a solution of the conjugacy problem in $SL_2(\mathbb{Z})$. The easiest way to do this is to write the cyclically reduced normal form for the particular special form in question and compare it to the cyclically reduced normal form for H, obtained in Step 2.)

There is an essential surface of type $C(J; n(k), \ldots, n(1))$ in the given manifold if and only if the special form $P^J C^{n(k)} \ldots B^{n(2)} A^{n(1)}$, modulo cyclic permutation of $(n(k), \ldots, n(1))$, appears in the list obtained in Step 4.

6.1.1. EXAMPLE. We carry out the algorithm in a specific case. We suppose $H = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. (This example corresponds to finding the essential surfaces of type $C(J; n(k), \ldots, n(1))$ in the knot space of the 'figure-eight' knot.)

Step 1.

$$H = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A^{-3}P = PQ^2PQ^2PQ^2P.$$

Step 2. The cyclically reduced normal form for H is QPQ^2P. Hence, $l(H) = 4$.
Step 3. From the formula

$$4 = l(P^J C^{n(k)} \ldots B^{n(2)} A^{n(1)}) \geqslant 2\left(\sum_1^k (|n(i)| - 1)\right) \geqslant 2k,$$

we have $k \leqslant 2$; and the possible special forms that represent elements of $\mathrm{SL}_2(\mathbb{Z})$ with length 4 are: $k = 1$ and $|n(1)| \leqslant 3$; $k = 2$ and $|n(1)| = |n(2)| = 2$. For $k = 1$ and $|n(1)| \leqslant 3$, the syllable length of $P^J A^{n(1)}$, where $J = 1$ or 3, is $2|n(1)| + 1$; and precisely three cancellations occur in arriving at the cyclically reduced normal form for $P^J A^{n(1)}$. So, the only possibility is for $|n(1)| = 3$. We have $PA^3, PA^{-3}, P^{-1}A^3, P^{-1}A^{-3}$. For $k = 2$ and $|n(1)| = |n(2)| = 2$, the syllable length of $P^J B^{n(2)} A^{n(1)}$, where $J = 0, 2$, is $2(|n(2)| + |n(1)|) = 8$. As we observed earlier, to obtain the cyclically reduced normal form for $P^J B^{n(2)} A^{n(1)}$ we make precisely four cancellations if $n(1)$ and $n(2)$ have the same sign and no cancellations otherwise. So, the only possibility is for $n(1)$ and $n(2)$ to have the same sign, $|n(1)| = |n(2)| = 2$. We have

$$B^2A^2, B^{-2}A^{-2}, P^2B^2A^2, P^2B^{-2}A^{-2}.$$

Step 4. The special form $PA^{-3} = PPQ^2PQ^2PQ^2$ cyclically reduces to QPQ^2P; and $P^{-1}A^3 = P^{-1}QPQPQP$ cyclically reduces to Q^2PQP. Both of these are conjugate to H. These are the only special forms listed in Step 3 that represent elements of $\mathrm{SL}_2(\mathbb{Z})$ conjugate to H.

We conclude that the only essential (orientable) surfaces in the 'figure-eight' knot space (other than the fibre) are of type $C(1; -3)$ and $C(-1; 3)$. Both are genus 1 with two holes.

6.2. *Framing*

If M is a once-punctured torus bundle over S^1, then we want to select a framing (coordinate system or pair of transverse simple closed curves) for ∂M; and in this framing describe the boundary curves of all essential surfaces in M.

First, note that there is a unique (up to isotopy) simple closed curve in ∂M that is the boundary of an orientable surface. This curve is the boundary of the fibre in a fibration of M as a once-punctured torus bundle over S^1 and is analogous to the 'longitude' in the classical knot manifold in S^3. This curve will be one of the curves of our framing for ∂M.

We fix a base point x in ∂T and let a, b be elements of $\pi_1(T, x)$, analogous to a and b of Fig. 1, oriented so that ∂T is the word $[a, b]$. The group $\pi_1(T, x)$ is freely generated by a and b. Let $\mathrm{Stab}([a, b])$ be the subgroup of the group of automorphisms of $\pi_1(T, x)$ that stabilizes $[a, b]$. Now, for $\gamma \in \mathrm{Stab}([a, b])$ there is a unique (up to isotopy fixing x) homeomorphism g: $(T, x) \to (T, x)$ such that $g_* = \gamma$. Furthermore, and here is the point, if $M = T \times I/g$, then there exists a unique simple closed curve t_γ in ∂M such that t_γ is transverse to the fibre in M and $t_\gamma a t_\gamma^{-1} = \gamma(a)$ and $t_\gamma b t_\gamma^{-1} = \gamma(b)$. The curve t_γ, along with the boundary of the fibre, gives a framing for ∂M. This framing is completely determined by the automorphism $\gamma \in \mathrm{Stab}([a, b])$. We call the elements of $\mathrm{Stab}([a, b])$ *framings for once-punctured torus bundles*, and we call a particular automorphism $\gamma \in \mathrm{Stab}([a, b])$, a *framing* for the bundle $M = T \times I/g$, where $g : (T, x) \to (T, x)$ is a homeomorphism and $g_* = \gamma$. If f maps (T, x) to (T, x) and f is isotopic to g, then $M' = T \times I/f$ is a bundle equivalent to $M = T \times I/g$. The framings $\gamma = g_*$ and $\eta = f_*$ differ by a conjugation by $[a, b]^j$; and indeed, the j explains how t_γ differs from t_η by 'twisting' around the boundary of the fibre. It is this observation that allows us to describe the boundaries of all essential surfaces in M in terms of a fixed framing.

Let $X \in \mathrm{SL}_2(\mathbb{Z})$. There is a natural map (using the basis a, b selected above) from $\mathrm{Stab}([a, b])$ onto $\mathrm{SL}_2(\mathbb{Z})$. An element $\xi \in \mathrm{Stab}([a, b])$ such that ξ is mapped to X is called a *framing for* X (X does uniquely determine a once-punctured torus bundle and we have selected a framing for it). If $Z \in \mathrm{SL}_2(\mathbb{Z})$ and Z is conjugate to X, say $X = UZU^{-1}$, then the bundle determined by Z, M_Z, is equivalent to the bundle determined by X, M_X; in fact, there is a homeomorphism h: $(T, x) \to (T, x)$ such that $h \times \mathrm{Id}$: $T \times I \to T \times I$ extends to a bundle equivalence from M_Z to M_X, where $h_* \to U$. If ξ is a framing for X (a framing for ∂M_X) and ζ is a framing for Z (a framing for ∂M_Z), then $\xi^{-1}\mu\zeta\mu^{-1}$ is a conjugation by $[a, b]^j$, where μ is any framing for U. The integer j is independent of μ. We call j the *transition index between* ξ *and* ζ.

Now, suppose that M_X is a once-punctured torus bundle over S^1, where X is a representative of the characteristic class of M. Let ξ be a framing for X. Suppose that $Z = P^J C^{n(k)} \dots B^{n(2)} A^{n(1)}$ is a special form representing the element $Z \in \mathrm{SL}_2(\mathbb{Z})$. By using the construction of § 2.2, we can select a framing, ζ, for Z and describe the boundary of the surface of type $C(J; n(k), \dots, n(1))$ in M_Z in terms of this framing. On the other hand, if Z is conjugate to X, then M_Z is bundle equivalent to M and the bundle M contains an essential surface of type $C(J; n(k), \dots, n(1))$. We wish to describe the boundary of this surface of type $C(J; n(k), \dots, n(1))$ in M in terms of the framing ξ for X. If j is the transition index between ξ and ζ, then the matrix $\begin{pmatrix} 1 & 0 \\ -j & 1 \end{pmatrix}$ defined in terms of the ordered basis $\{t_\zeta, \lambda\}$ for M_Z and $\{t_\xi, \lambda\}$ for M, where λ is the boundary of the fibre, describes the change in coordinates between the framing for ∂M_Z and the framing for ∂M.

We shall slightly abuse notation and use α, β, φ, and ψ for automorphisms in $\mathrm{Stab}([a, b])$ representing the homeomorphisms α, β, φ, and ψ, respectively, defined in § 1.2. The following framings are selected as *standard framings:*

$$\alpha: \begin{cases} a \to a \\ b \to ba^{-1} \end{cases} \quad \text{is standard for} \quad A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$\beta: \begin{cases} a \to ab \\ b \to b \end{cases} \quad \text{is standard for} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\varphi: \begin{cases} a \to aba^{-1} \\ b \to a^{-1} \end{cases} \quad \text{is standard for} \quad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$\psi: \begin{cases} a \to ab^{-1} \\ b \to bab^{-1} \end{cases} \text{ is standard for } Q = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

If $P^J \dots B^{n(2)}A^{n(1)}$ is a special form for an element X of $\mathrm{SL}_2(\mathbb{Z})$, then the *standard framing for* X corresponding to the special form

$$\begin{array}{rll} B^{n(k)} \dots B^{n(2)}A^{n(1)} & \text{is} & \beta^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}, \\ \text{for } PA^{n(k)} \dots B^{n(2)}A^{n(1)} & \text{it is} & \varphi\alpha^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}, \\ \text{for } P^2B^{n(k)} \dots B^{n(2)}A^{n(1)} & \text{it is} & \varphi^2\beta^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}, \\ \text{and for } P^{-1}A^{n(k)} \dots B^{n(2)}A^{n(1)} & \text{it is} & \varphi^{-1}\alpha^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}. \end{array}$$

We have selected these framings so that the boundary curves of the essential surface of type $C(J\,; n(k), \dots, n(1))$ all cross the framing curve transverse to the fibre at most once.

Table 1 describes the special forms, the standard framings corresponding to the special forms, the essential surface having the type of the form, and the coordinates of the boundary of the surface described in the standard framing.

TABLE 1

Form	Framing	Surface	Boundary curves
any	any	fibre	$\langle 0, 1\rangle$
P	φ	$(\mathrm{Im}(D_3))^{\sim}$, annulus	$\langle 4, 1\rangle$
P^2	φ^2	$(\mathrm{Im}(D_1))^{\sim}$ $(\mathrm{Im}(D_2))^{\sim}$, annulus	$\langle 2, 1\rangle$
P^{-1}	φ^{-1}	$(\mathrm{Im}(D_3))^{\sim}$, annulus	$\langle 4, -1\rangle$
Q	ψ	$(\mathrm{Im}(D_4))^{\sim}$, annulus	$\langle 6, -1\rangle$
Q^2	ψ^2	$\mathrm{Im}(D_4)$, annulus	$\langle 3, -1\rangle$
A^n	α^n	$\mathrm{Im}(D_1)$, annulus	$\langle 1, 0\rangle$
P^2A^n	$\varphi^2\alpha^n$	$(\mathrm{Im}(D_1))^{\sim}$, annulus	$\langle 2, 1\rangle$
$B^{n(k)} \dots B^{n(2)}A^{n(1)}$	$\beta^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}$	$C(0\,; n(k), \dots, n(1))$ genus $\frac{1}{2}k-1$ with 4-holes	$\langle 1, 0\rangle$
$P^2B^{n(k)} \dots B^{n(2)}A^{n(1)}$	$\varphi^2\beta^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}$	$C(2\,; n(k), \dots, n(1))$ genus $\frac{1}{2}k$ with 2-holes	$\langle 2, 1\rangle$
$PA^{n(k)} \dots B^{n(2)}A^{n(1)}$	$\varphi\alpha^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}$	$C(1\,; n(k), \dots, n(1))$ genus k with 2-holes	$\langle 4, 1\rangle$
$P^{-1}A^{n(k)} \dots B^{n(2)}A^{n(1)}$	$\varphi^{-1}\alpha^{n(k)} \dots \beta^{n(2)}\alpha^{n(1)}$	$C(-1\,; n(k), \dots, n(1))$ genus k with 2-holes	$\langle 4, -1\rangle$

6.2.1. EXAMPLE. Continuing with Example 6.1.1, the figure-eight knot space, we shall select a framing and describe the boundary curves of the essential surfaces of type $C(1\,; -3)$ and $C(-1\,; 3)$.

Choose the standard framing for PA^{-3}, that is $\varphi\alpha^{-3}$. In this framing the boundary of the surface of type $C(1\,; -3)$ has coordinates $\langle 4, 1\rangle$.

In the framing $\varphi^{-1}\alpha^3$, the boundary of the surface of type $C(-1\,; 3)$ has coordinates $\langle 4, -1\rangle$. We need to compute the transition index between $\varphi\alpha^{-3}$ and $\varphi^{-1}\alpha^3$.

Set $X = PA^{-3} = PPQ^2PQ^2PQ^2$ and set $Z = P^3A^3 = P^3QPQPQP$. Then for $U = Q^2PQP$, we have $X = UZU^{-1}$. Let $\beta^{-1}\alpha$ be a framing for U. We find that $\varphi\alpha^{-3}$

and $(\beta^{-1}\alpha)\varphi^{-1}\alpha^3(\beta^{-1}\alpha)^{-1}$ both take a to aba^{-1}. Hence, the transition index is zero; and the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ takes the $\langle 4, -1\rangle$ curve to the $\langle 4, -1\rangle$ curve. We conclude that for the 'figure-eight' knot space the three essential (orientable) surfaces (the fibre, the surface of type $C(1; -3)$, and the surface of type $C(3; 3)$) have boundary curves with coordinates $\langle 0, 1\rangle$, $\langle 4, 1\rangle$, and $\langle 4, -1\rangle$, respectively, in the framing $\varphi\alpha^{-3}$.

Notice that the framing $\varphi\alpha^{-3}$ is the standard framing for the 'figure-eight' knot space coming from a 'meridian' and 'longitude' pair. In this example the transition index between standard framings coming from special forms was zero. However, for any n, there is an example of a once-punctured torus bundle having a transition index larger than n between standard framings coming from special forms.

6.3. *Table of examples*

Table 2 lists all essential surfaces in once-punctured torus bundles with characteristic class of length at most 12 (and positive trace). We also select a framing for each bundle and give the coordinates of the boundary curves of the essential surfaces in terms of this framing.

Before giving the table, we will explain the notation.

The first column gives the bundle by giving a representative of its characteristic class. The representative chosen is a cyclically reduced normal form, and for classes having length not greater than 2, we use the short notation from §5.1; i.e. the characteristic class $[Q^{\varepsilon_1}P \ldots Q^{\varepsilon_s}P]$ is given by $[\varepsilon_1, \ldots, \varepsilon_s]$. So, the characteristic class listed as $[1, 2, 1, 2]$ represents the class $[QPQ^2PQPQ^2P]$.

The second column gives the trace of the class.

The third column gives a framing. It is in terms of this framing that everything is referenced. In most cases the framing was chosen as a standard framing coming from a special form representing a surface of type $C(J; n(k), \ldots, n(1))$ in the bundle. However, this is not always the case. In particular, the bundles with characteristic classes $[1, 2]$, $[1, 2, 1, 2]$, $[1, 2, 1, 2, 1, 2]$, and $[1, 2, 1, 2, 1, 2, 1, 2]$ corresponding to the 'figure-eight' knot space and its 2-sheeted, 3-sheeted, and 4-sheeted cyclic coverings, respectively, are given the framings that are 'lifts' of the framing for the 'figure-eight' knot space.

The fourth column lists the types of essential surfaces (except for the fibre), the coordinates of their boundary curves (in terms of the specified framing), and the topological type of the essential surface. For example, the bundle with characteristic class $[1, 2, 2, 2]$ has an essential surface of type $C(1; 2, 2, 3)$ with boundary curves having coordinates $\langle 4, -3\rangle$ in the framing $\varphi\alpha^{-5}$; and it is a genus 3 surface with two boundary components.

The fifth column is titled 'surgeries' and it is divided into two columns. This is explained as follows. If M is a once-punctured torus bundle with characteristic class $[H]$, then M determines a unique torus bundle over S^1, $\hat{M}$, by attaching a solid torus to ∂M, sewing a curve in the boundary of the solid torus that is the boundary of an essential disk inside the solid torus onto a curve in the boundary of M that is the boundary of the fibre. The operation of attaching a solid torus to ∂M, sewing the boundary of an essential disk in the solid torus onto a $\langle p, q\rangle$-curve in the given framing, is a $\langle p, q\rangle$-surgery on a section of $\hat{M}$. Our methods describe precisely which of these closed manifolds are not irreducible and which are Haken. We list in the column titled 'reducible' the pairs $\langle p, q\rangle$ for which, in terms of the given framing, a $\langle p, q\rangle$-surgery gives a manifold that is not irreducible. We also list the topological type of such a manifold. For example, if M is the bundle with characteristic class $[P^2]$,

TABLE 2

Characteristic class	Trace	Framing	Surfaces $\langle$Boundary curves$\rangle$	Surgeries Reducible	Surgeries Haken	Remarks
$[P]$	0	φ	$(\mathrm{Im}(D_2))^{\sim}\langle 4, 1\rangle$, Annulus	$\langle 4, 1\rangle$: $L(4, 1) \# L(2, 1)$	$\langle 0, 1\rangle$	(a) A $\langle 1, 0\rangle$-surgery gives $\mathbb{R}P^3$. There is a moebius band corresponding to the form $A^2B\langle 1, 0\rangle$.
$[P^2]$	-2	φ^2	$(\mathrm{Im}(D_1))^{\sim}\langle 2, 1\rangle$, Annulus	$\langle 2, 1\rangle$: $L(2, 1) \# L(2, 1) \# L(2, 1)$	All (except $\langle 2, 1\rangle$)	
$[Q]$	1	ψ	$(\mathrm{Im}(D_3))^{\sim}\langle 6, -1\rangle$, Annulus	$\langle 6, -1\rangle$: $L(2, 1) \# L(3, 1)$	$\langle 0, 1\rangle$	(b) A $\langle 1, 0\rangle$-surgery gives S^3. This manifold is the Trefoil knot space.
$[Q^2]$	-1	ψ^2	$\mathrm{Im}(D_3)\langle 3, -1\rangle$, Annulus	$\langle 3, -1\rangle$:	$\langle 0, 1\rangle$	(c) This manifold is the double cover of the Trefoil knot space. $\mathrm{Im}(D_4)\langle 3, 1\rangle$ covers $(\mathrm{Im}(D_4))^{\sim}\langle 6, 1\rangle$.
$[1]$	-2	χ $(\varphi\chi^{-2})$	$\mathrm{Im}(D_1)\langle 1, 0\rangle$, Annulus; $\mathrm{Im}(E)$, Torus; $S(p, q; 1)\langle 0, 1\rangle$, genus 1 with p-holes; $C(1; -2)\langle 4, 1\rangle$, genus 1 with 2-holes.	$\langle 1, 0\rangle$: $S^2 \times S^1$	All (except $\langle 1, 0\rangle$)	(d) The surface of type $C(1; -2)$ is invariant (up to isotopy) under Dehn twists about the torus, $\mathrm{Im}(E)$.
$[1, 1]$	2	χ^2	$\mathrm{Im}(D_1)\langle 1, 0\rangle$, Annulus; $\mathrm{Im}(E)$, Torus; $S(p, q; 2)\langle 0, 1\rangle$, genus 1 with p-holes; $C(2; -2, -2)\langle 2, 1\rangle$, genus 1 with 2-holes	$\langle 1, 0\rangle$: $S^2 \times S^1 \# L(2, 1)$	All (except $\langle 1, 0\rangle$)	(e) Since the characteristic class of this bundle is a power, it is possible that there is more than one surface of type $C(2; -2, -2)$; and, in fact, in this case there are two.
$[1, 2]$	3	$\varphi\chi^{-3}$	$C(1; -3)\langle 4, 1\rangle$, genus 1 with 2-holes; $C(-1; 3)\langle 4, -1\rangle$, genus 1 with 2-holes.	None	$\langle 0, 1\rangle$ $\langle 4, 1\rangle$ $\langle 4, -1\rangle$	(f) A $\langle 1, 0\rangle$-surgery gives S^3. This manifold is the figure-eight knot space.

TABLE 2 (*cont.*)

Characteristic class	Trace	Framing	Surfaces $\langle$Boundary curves$\rangle$	Surgeries: Reducible	Surgeries: Haken	Remarks
[1, 1, 1]	2	α^3	$\mathrm{Im}(D_1)\langle 1,0\rangle$, Annulus; $\mathrm{Im}(E)$, Torus; $S(p,q;3)\langle 0,1\rangle$, genus 1 with p-holes; $C(-1;-2,-2,-2)\langle 4,3\rangle$, genus 3 with 2-holes.	$\langle 1,0\rangle$: $S^2\times S^1 \# L(3,1)$	All (except $\langle 1,0\rangle$)	(g) There are three essential surfaces of type $C(3;-2,-2,-2)$. (see Remark (e)).
[1, 2, 2]	4	$\varphi\alpha^{-4}$	$C(1;-4)\langle 4,1\rangle$, genus 1 with 2-holes; $C(2;2,3)\langle 2,-1\rangle$, genus 1 with 2-holes.	None	$\langle 0,1\rangle$ $\langle 4,1\rangle$ $\langle 2,-1\rangle$	(h) A $\langle 1,0\rangle$-surgery gives $\mathbb{R}P^3$. There is a moebius band with 1-hole corresponding to the form $BA^{-2}\langle 1,0\rangle$. (See Remark (a)).
[1, 1, 1, 1]	2	α^4	$\mathrm{Im}(D_1)\langle 1,0\rangle$, Annulus; $\mathrm{Im}(E)$, Torus; $S(p,q;4)\langle 0,1\rangle$, genus 1 with p-holes; $C(0;-2,-2,-2,-2)\langle 1,1\rangle$, genus 1 with 4-holes.	$\langle 1,0\rangle$: $S^2\times S^1 \# L(4,1)$	All (except $\langle 1,0\rangle$)	(i) There are four essential surfaces of type $C(0;-2,-2,-2,-2)$. (See Remark (e)).
[1, 2, 2, 2]	5	$\varphi\alpha^{-5}$	$C(1;-5)\langle 4,1\rangle$, genus 1 with 2-holes; $C(1;2,2,3)\langle 4,-3\rangle$, genus 3 with 2-holes.	None	$\langle 0,1\rangle$ $\langle 4,1\rangle$ $\langle 4,-3\rangle$	
[1, 1, 2, 2]	6	$\beta^2\alpha^{-2}$	$C(0;2,-2)\langle 1,0\rangle$, Disk with 3-holes; $C(2;2,4)\langle 2,1\rangle$, genus 1 with 2-holes; $C(2;-2,-4)\langle 2,1\rangle$, genus 1 with 2-holes.	$\langle 1,0\rangle$: $\mathbb{R}P^3 \# \mathbb{R}P^3$	$\langle 0,1\rangle$ $\langle 2,-1\rangle$ $\langle 2,1\rangle$	(j) The homology of the bundle with characteristic class [1, 1, 2, 2] is $\mathbb{Z}+\mathbb{Z}_2+\mathbb{Z}_2$; hence a $\langle 2p,q\rangle$-surgery gives a Heegard genus 3 manifold that is *not* Haken (except $p=1$, $q=\pm 1$).
[1, 2, 1, 2]	7	$\varphi\alpha^{-3}\varphi\alpha^{-3}$	$C(2;-3,-3)\langle 2,1\rangle$, genus 1 with 2-holes; $C(2;3,3)\langle 2,1\rangle$, genus 1 with 2-holes.	None	$\langle 0,1\rangle$ $\langle 2,1\rangle$ $\langle 2,1\rangle$	(k) This manifold is the two-sheeted cover of the figure-eight knot space (see Remark (f)). Note that $C(2;-3,-3)\langle 2,1\rangle$ covers $C(1;-3)\langle 4,1\rangle$ and $C(2;3,3)\langle 2,-1\rangle$ covers $C(3;3)\langle 4,1\rangle$.
[1, 1, 1, 1, 1]	2	α^5	$\mathrm{Im}(D_1)\langle 1,0\rangle$, Annulus; $\mathrm{Im}(E)$, Torus; $S(p,q;5)\langle 0,1\rangle$, genus 1 with p-holes; $C(1;-2,-2,-2,-2,-2)\langle 4,3\rangle$, genus 5 with 2-holes.	$\langle 1,0\rangle$: $S^2\times S^1 \# L(5,1)$	All (except $\langle 1,0\rangle$)	(l) There are five essential surfaces of type $C(1;-2,-2,-2,-2,-2)$.

[1, 2, 2, 2, 2]	6	$\varphi\alpha^{-6}$	$C(1; -6)\langle 4, 1\rangle$, genus 1 with 2-holes; $C(0; 2, 2, 2, 3)\langle 1, 1\rangle$, genus with 4-holes.	None	$\langle 0, 1\rangle$ $\langle 4, 1\rangle$ $\langle 1, 1\rangle$	
[1, 1, 2, 2, 2]	8	$\beta^2\alpha^{-3}$	$C(0; 2, -3)\langle 1, 0\rangle$, Disk with 3-holes; $C(2; -2, -5)\langle 2, 1\rangle$, genus 1 with 2-holes; $C(1; 2, 2, 4)\langle 4, -3\rangle$, genus 3 with 2-holes.	$\langle 1, 0\rangle$: $L(2, 1)$ $\# L(3, 1)$	$\langle 0, 1\rangle$ $\langle 2, 1\rangle$ $\langle 4, -3\rangle$	
[1, 2, 1, 2, 2]	10	$\varphi^2\beta^{-3}\alpha^{-4}$	$C(0; -3, -4)\langle 2, 1\rangle$, genus 1 with 2 holes; $C(1; 2, 3, 3)\langle 4, -3\rangle$, genus 3 with 2-holes; $C(-1; -2, 2, 2)\langle 4, -1\rangle$, genus 3 with 2-holes.	None	$\langle 0, 1\rangle$ $\langle 2, 1\rangle$ $\langle 4, -3\rangle$ $\langle 4, -1\rangle$	
[1, 1, 1, 1, 1, 1]	2	α^6	$\mathrm{Im}(D_1)\langle 1, 0\rangle$, Annulus; $\mathrm{Im}(E)$, Torus; $S(p, q; 6)\langle 0, 1\rangle$, genus 1 with p-holes; $C(2; -2, -2, -2, -2, -2, -2)\langle 2, 3\rangle$, genus 3 with 2-holes.	$\langle 1, 0\rangle$: $S^2 \times S^1$ $\# L(6, 1)$	All (except $\langle 1, 0\rangle$)	(m) There are six essential surfaces of type $C(2; -2, -2, -2, -2, -2, -2)$.
[1, 2, 2, 2, 2, 2]	7	$\varphi\alpha^{-7}$	$C(1; -7)\langle 4, 1\rangle$, genus 1 with 2-holes; $C(-1; 3, 2, 2, 2, 2)\langle 4, -5\rangle$, genus 5 with 2-holes.	None	$\langle 4, 1\rangle$ $\langle 4, -5\rangle$	
[1, 1, 2, 2, 2, 2]	10	$\beta^2\alpha^{-4}$	$C(0; 2, -4)\langle 1, 0\rangle$, Disk with 3-holes; $C(2; -2, -6)\langle 2, 1\rangle$, genus 1 with 2-holes; $C(0; 2, 2, 2, 4)\langle 1, -1\rangle$, genus 1 with 4-holes.	$\langle 1, 0\rangle$: $L(2, 1)$ $\# L(4, 1)$	$\langle 0, 1\rangle$ $\langle 2, 1\rangle$ $\langle 1, -1\rangle$	
[1, 1, 1, 2, 2, 2]	11	$\beta^3\alpha^{-3}$	$C(0; 3, -3)\langle 1, 0\rangle$, Disk with 3-holes; $C(1; 2, 2, 5)\langle 4, -3\rangle$, genus 3 with 2-holes; $C(-1; -2, -2, -5)\langle 4, 3\rangle$, genus 3 with 2-holes.	$\langle 1, 0\rangle$: $L(3, 1)$ $\# L(3, 1)$	$\langle 0, 1\rangle$ $\langle 4, -3\rangle$ $\langle 4, 3\rangle$	
[1, 2, 1, 2, 2, 2]	13	$\varphi^2\beta^{-3}\alpha^{-5}$	$C(2; -3, -5)\langle 2, 1\rangle$, genus 1 with 2-holes; $C(-1; 2, 2, -3)\langle 4, -1\rangle$, genus 3 with 2-holes; $C(0; 3, 2, 2, 3)\langle 1, -1\rangle$, genus 1 with 4-holes.	None	$\langle 0, 1\rangle$ $\langle 1, -1\rangle$ $\langle 2, 1\rangle$ $\langle 4, -1\rangle$	

TABLE 2 (*cont.*)

Characteristic class	Trace	Framing	Surfaces ⟨Boundary curves⟩	Surgeries Reducible	Surgeries Haken	Remarks
[1, 2, 2, 1, 2, 2]	14	$\varphi^2\beta^{-4}\alpha^{-4}$ $(\varphi\alpha^{-4}\varphi\alpha^{-4})$	$C(2; -4, -4)\langle 2, 1\rangle$, genus 1 with 2-holes; $C(2; -2, 2, 2, 2)\langle 2, -1\rangle$, genus 2 with 2-holes; $C(0; 2, 3, 2, 3)\langle 1, -1\rangle$, genus 1 with 2-holes.	None	$\langle 0, 1\rangle$ $\langle 1, -1\rangle$ $\langle 2, 1\rangle$ $\langle 2, -1\rangle$	(n) This manifold covers the bundle with characteristic class [1, 2, 2]. Note that $C(2; -4, -4)\langle 2, 1\rangle$ covers $C(2; -4)\langle 4, 1\rangle$ and $C(0; 2, 3, 2, 3)\langle 1, -1\rangle$ covers $C(2; 2, 3)\langle 2, -1\rangle$. There are two essential surfaces of type $C(2; -2, 2, 2, 2)\langle 2, -1\rangle$. This shows that the $\langle 4, -1\rangle$-surgery on the bundle with characteristic class [1, 2, 2], which is *not* Haken, has a two-sheeted cover that is Haken. (We also know that these manifolds are not Seifert fibred.)
[1, 2, 1, 1, 2, 2]	15	$\varphi\alpha^{-3}\beta^{-2}\alpha^2$	$C(1; 2, 3, 4)\langle 4, -3\rangle$, genus 3 with 2-holes; $C(-1; -2, 2, 3)\langle 4, -1\rangle$, genus 3 with 2-holes; $C(1; -3, -2, 2)\langle 4, 1\rangle$, genus 3 with 2-holes; $C(-1; -4, -3, -2)\langle 4, 3\rangle$, genus 3 with 2-holes.	None	$\langle 0, 1\rangle$ $\langle 4, -3\rangle$ $\langle 4, -1\rangle$ $\langle 4, 1\rangle$ $\langle 4, 3\rangle$	(o) The transition index between $\varphi\alpha^2\beta^2\alpha^4$ and $\varphi^{-1}\alpha^{-4}\beta^{-3}\alpha^{-2}$ is -2. There are interesting symmetries in this manifold.
[1, 2, 1, 2, 1, 2]	18	$(\varphi\alpha^{-3})^3$	$C(-1; -3, -3, -3)\langle 4, 3\rangle$, genus 3 with 2-holes; $C(1; 3, 3, 3)\langle 4, -3\rangle$, genus 3 with 2-holes; $C(0; -2, -2, 2, 2)\langle 1, 0\rangle$, genus 1 with 4-holes.	None	$\langle 0, 1\rangle$ $\langle 1, 0\rangle$ $\langle 4, 3\rangle$ $\langle 4, -3\rangle$	(p) This manifold is the three-sheeted cyclic cover of the figure-eight knot space. Note that $C(3; -3, -3, -3)\langle 4, 3\rangle$ covers $C(1; -3)\langle 4, 1\rangle$ and $C(1; 3, 3, 3)\langle 4, -3\rangle$ covers $C(3; 3)\langle 4, -1\rangle$. Also, there are three essential surfaces of type $C(0; -2, -2, 2, 2)$. This shows that the three-sheeted cyclic branched cover of S^3 over the figure-eight is Haken.

[1, 2, 1, 2, 1, 2, 1, 2]	47	$(\varphi\alpha^{-3})^4$	$C(0; -3, -3, -3, -3)\langle 1, 1\rangle$, genus 1 with 4-holes; $C(0; 3, 3, 3, 3)\langle 1, -1\rangle$, genus 1 with 4-holes; $C(1; -2, 2, 2, -2, -3)\langle 4, 1\rangle$, genus 5 with 2-holes; $C(-1; 2, -2, -2, 2, 3)\langle 4, -1\rangle$, genus 5 with 2-holes.	None	$\langle 0, 1\rangle$ $\langle 1, 1\rangle$ $\langle 1, -1\rangle$ $\langle 4, 1\rangle$ $\langle 4, -1\rangle$	(q) This manifold is the four-sheeted cyclic cover of the figure-eight knot space. Here $C(0; -3, -3, -3, -3)\langle 1, 1\rangle$ covers $C(1; -3)\langle 4, 1\rangle$ and $C(0; 3, 3, 3, 3)\langle 1, -1\rangle$ covers $C(3; 3)\langle 4, 1\rangle$. There are eight essential surfaces, four each of types $C(1; -2, 2, 2, -2, -3)\langle 4, 1\rangle$ and $C(3; 2, -2, -2, 2, 3)\langle 4, 1\rangle$. This shows that both the $\langle -16, 1\rangle$-surgery and the $\langle 16, 1\rangle$-surgery on the figure-eight knot space, which are *not* Haken manifolds, have four-sheeted cyclic coverings that are Haken. These are hyperbolic manifolds [*Th*].

then a $\langle 2, 1\rangle$-surgery in the framing φ^2 gives the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3 \# \mathbb{R}P^3$, which is not irreducible. The other column is titled 'Haken'. It gives the pairs $\langle p, q\rangle$ for which, in terms of the given framing, a $\langle p, q\rangle$-surgery gives a Haken manifold. Of course, except in the case that the characteristic class has trace in absolute value not greater than 2, we only obtain Haken manifolds by doing surgery along a $\langle p, q\rangle$-curve that is also part of the boundary of an essential surface in M.

The sixth column is titled 'Remarks'. These remarks, for the most part, relate the example back to a more familiar setting (e.g., the 'figure-eight' knot space), or comment about the particular essential surfaces, or comment about a consequence of some particular $\langle p, q\rangle$-surgery.

7. *Open Problems*

There are some interesting questions that we have left unanswered.

We had hoped to classify the manifolds obtained by surgery on a section of a torus bundle. We did not do this. One of the problems is that a manifold obtained by surgery on a section of a torus bundle does not uniquely determine the bundle. Since the number of conjugacy classes of 2×2 matrices with a given trace is finite, homological considerations show that the number of bundles involved is finite (J. Birman has informed us that she can prove that if a manifold has a genus 1 open-book decomposition then, in 'most' cases, the associated bundle is unique and, in general, there are at most two distinct bundles involved). The 3-sphere has two such bundles (coming from the 'figure-eight' and the trefoil). We gave examples showing that there are two such bundles for real projective 3-space (see Remarks (a) and (h) in Table 2). However, it still seems to be unknown whether homeomorphic manifolds can be obtained by distinct surgeries on a fixed bundle.

We also proposed to answer questions of a more geometric nature. We can state exactly when a surgery on a section of a given torus bundle contains an essential torus or 2-sphere (or when it is Haken). Since Jørgensen has given a decomposition of the punctured torus bundles into ideal tetrahedra, techniques are available for trying to verify which surgeries are hyperbolic or are Seifert fibred. It should be possible to do this in the way that Thurston did it for surgeries on the 'figure-eight' knot. While we do not expect any surprises, this might be an illuminating computation. In any case, the problem remains open as to which surgeries on a section of a given torus bundle are hyperbolic or Seifert fibred (or even have finite fundamental group).

When Waldhausen first introduced examples of orientable, irreducible 3-manifolds with infinite fundamental group that were not Haken, he observed that each of the examples that he gave had a finite-sheeted covering that was Haken. This led to the conjecture that orientable, irreducible 3-manifolds with infinite fundamental group are 'almost' Haken; i.e. have a finite-sheeted covering that is Haken. This conjecture has certainly been compelling. Recently, Thurston discovered a large family of orientable, irreducible 3-manifolds with infinite fundamental group that are not Haken (and not Seifert fibred). More examples of such manifolds were added to those discovered by Thurston in the work of Hatcher and Thurston. Furthermore, in all of these examples there was no evidence for or against the above conjecture.

We have given explicitly examples of manifolds obtained by surgery on a section of a torus bundle that are not Haken and yet have cyclic coverings that are Haken. In general, many such surgeries that are not Haken will have cyclic coverings that are Haken. We believe that the best method for approaching this conjecture, in the case of those manifolds obtained by surgery on a section of a bundle over S^1, would be to acquire a better understanding, perhaps even a classification, of essential surfaces in manifolds that are bundles over S^1. In principle, it would seem possible that such a program could be carried out by methods analogous to those used here. However, the increase in complexity, when one passes from $SL_2(\mathbb{Z})$ to the mapping class group of a higher genus surface, is impressive. For such an approach to succeed, it is clear that more sophisticated techniques must be developed to deal with the combinatorics.

Perhaps, from a more general point of view, the result to prove is that if an orientable, irreducible 3-manifold with non-empty boundary has no essential annuli then there are at most a finite number of isotopy classes of simple closed curves in the boundary of the manifold that can be curves in the boundary of an incompressible and boundary-incompressible surface properly embedded in the manifold.

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