Dehn surgery, homology and hyperbolic volume

IAN AGOL
MARC CULLER
PETER B SHALEN

If a closed, orientable hyperbolic 3–manifold M has volume at most 1.22 then $H_1(M;\mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M;\mathbb{Z}_2)$ and $H_1(M;\mathbb{Z}_7)$ have dimension at most 3. The proof combines several deep results about hyperbolic 3–manifolds. The strategy is to compare the volume of a tube about a shortest closed geodesic $C \subset M$ with the volumes of tubes about short closed geodesics in a sequence of hyperbolic manifolds obtained from M by Dehn surgeries on C.

57M50; 57M27

1 Introduction

We shall prove:

Theorem 1.1 *Suppose that M is a closed, orientable hyperbolic 3–manifold with volume at most 1.22. Then $H_1(M;\mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M;\mathbb{Z}_2)$ and $H_1(M;\mathbb{Z}_7)$ have dimension at most 3. Furthermore, if M has volume at most 1.182, then $H_1(M;\mathbb{Z}_7)$ has dimension at most 2.*

The bound of 2 for the dimension of $H_1(M;\mathbb{Z}_p)$ is sharp when p is 3 or 5. Indeed, the manifolds `m003(-3,1)`, and `m007(3,1)` from the list given in [10] have respective volumes $0.94\ldots$ and $1.01\ldots$, and their integer homology groups are respectively isomorphic to $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_6$.

Apart from these two examples, the only example known to us of a closed, orientable hyperbolic 3–manifold with volume at most 1.22 is the manifold `m003(-2,3)` from the list given in [10]. These three examples suggest that the bounds for the dimension of $H_1(M;\mathbb{Z}_p)$ given by Theorem 1.1 may not be sharp for $p \neq 3, 5$.

The proof of Theorem 1.1 depends on several deep results, including a strong form of the "log 3 Theorem" of Anderson, Canary, Culler and Shalen [4; 8]; the Embedded Tube Theorem of Gabai, Meyerhoff and N Thurston [9]; the Marden Tameness Conjecture,

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recently proved by Agol [1] and by Calegari and Gabai [7]; and an even more recent result due to Agol, Dunfield, Storm and W Thurston [3]. The strategy of our proof is to compare the volume of a tube about a shortest closed geodesic $C \subset M$ with the volumes of tubes about short closed geodesics in a sequence of hyperbolic manifolds obtained from M by Dehn surgeries on C.

After establishing some basic conventions in Section 2, we carry out the strategy described above in Sections 3–6, for the case of manifolds which are "non-exceptional" in the sense that they contain shortest geodesics with tube radius greater than $(\log 3)/2$. In Section 5, for the case of non-exceptional manifolds with volume at most 1.22, we establish a bound of 3 for the dimension of $H_1(M;\mathbb{Z}_p)$ for any prime p. In Section 6, again for the case of non-exceptional manifolds with volume at most 1.22, we establish a bound of 2 for the dimension of $H_1(M;\mathbb{Z}_p)$ for any odd prime p. In Section 7 we use results from [9] to handle the case of exceptional manifolds, and complete the proof of Theorem 1.1.

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2 Definitions and conventions

2.1 If g is a loxodromic isometry of hyperbolic 3–space $\mathbb{H}^3$ we shall let A_g denote the hyperbolic geodesic which is the axis of g. The *cylinder about* A_g *of radius* r is the open set $Z_r(g) = \{x \in \mathbb{H}^3 \mid \mathrm{dist}(x, A_g) < r\}$.

2.2 Suppose that M is a complete, orientable hyperbolic 3–manifold. Let us identify M with $\mathbb{H}^3/\Gamma$, where $\Gamma \cong \pi_1(M)$ is a discrete, torsion-free subgroup of $\mathrm{Isom}_+ \mathbb{H}^3$. If C is a simple closed geodesic in M then there is a loxodromic isometry $g \in \Gamma$ with $A_g/\langle g\rangle = C$. For any $r > 0$ the image $Z_r(g)/\langle g\rangle$ of $Z_r(g)$ under the covering projection is a neighborhood of C in M. For sufficiently small $r > 0$ we have

$$\{h \in \Gamma \mid h(Z_r(g)) \cap Z_r(g) \neq \varnothing\} = \langle g\rangle.$$

Let R denote the supremum of the set of r for which this condition holds. We define $\mathrm{tube}(C) = Z_R(g)/\langle g\rangle$ to be the *maximal tube about* C. We shall refer to R as the *tube radius* of C, and denote it by $\mathrm{tuberad}(C)$.

2.3 If C is a simple closed geodesic in a closed hyperbolic 3–manifold M, it follows from [13], [2] that $M - C$ is homeomorphic to a hyperbolic manifold N of finite volume having one cusp. The manifold N, which by Mostow rigidity is unique up to isometry, will be denoted $\mathrm{drill}_C(M)$.

2.4 If C is a shortest closed geodesic in a closed hyperbolic 3–manifold M, ie, one such that $\mathrm{length}(C) \le \mathrm{length}(C')$ for every other closed geodesic C', then in particular C is simple, and the notions of 2.2 and 2.3 apply to C.

2.5 Suppose that $N = \mathbb{H}^3/\Gamma$ is a non-compact orientable complete hyperbolic manifold of finite volume. Let $\Pi \cong \mathbb{Z} \times \mathbb{Z}$ be a maximal parabolic subgroup of Γ (so that Π corresponds to a peripheral subgroup under the isomorphism of Γ with $\pi_1(N)$). Let ξ denote the fixed point of Π on the sphere at infinity and let B be an open horoball centered at ξ such that $\{g \in \Gamma \mid gB \cap B \neq \varnothing\} = \Pi$. Then $\mathcal{H} = B/\Pi$, which we identify with the image of B in N, is called a *cusp neighborhood* in N.

If $\mathcal{H}$ is a cusp neighborhood in $N = \mathbb{H}^3/\Gamma$ then the inverse image of $\mathcal{H}$ under the covering projection $\mathbb{H}^3 \to N$ is a union of disjoint open horoballs. The cusp neighborhood $\mathcal{H}$ is maximal if and only there exist two of these disjoint horoballs whose closures have non-empty intersection.

2.6 If N is a complete, orientable hyperbolic manifold of finite volume, $\widehat{N}$ will denote a compact core of N. Thus $\widehat{N}$ is a compact 3–manifold whose boundary components are all tori, and the number of these tori is equal to the number of cusps of N.

3 Drilling and packing

Lemma 3.1 *Suppose that M is a closed, orientable hyperbolic 3–manifold, and that C is a shortest geodesic in M. Set $N = \mathrm{drill}_C(M)$. If* $\mathrm{tuberad}(C) \ge (\log 3)/2$ *then* $\mathrm{vol}\, N < 3.0177\, \mathrm{vol}\, M$.

Proof The proof is based on a result due to Agol, Dunfield, Storm and W Thurston [3]. We let L denote the length of the geodesic C in the closed hyperbolic 3–manifold M, and we set $R = \mathrm{tuberad}(C)$ and $T = \mathrm{tube}(C)$. Proposition 10.1 of [3] states that

$$\mathrm{vol}\, N \le (\coth^3 2R)(\mathrm{vol}\, M + \frac{\pi}{2} L \tanh R \tanh 2R).$$

Note that
$$\mathrm{vol}\, T = \pi L \sinh^2 R = \left(\frac{\pi}{2} L \tanh R\right)(2 \sinh R \cosh R) = \left(\frac{\pi}{2} L \tanh R\right)(\sinh 2R).$$

Thus
$$\mathrm{vol}\, N \le (\coth^3 2R)\left(\mathrm{vol}\, M + \mathrm{vol}\, T \frac{\tanh 2R}{\sinh 2R}\right) = (\coth^3 2R)\left(\mathrm{vol}\, M + \frac{\mathrm{vol}\, T}{\cosh 2R}\right).$$

In the language of [16], the quantity $(\operatorname{vol} T)/(\operatorname{vol} M)$ is the density of a tube packing in $\mathbb{H}^3$. According to [16, Corollary 4.4], we have $(\operatorname{vol} T)/\operatorname{vol} M < 0.91$. Hence $\operatorname{vol} N < f(x)\operatorname{vol}(M)$, where $f(x)$ is defined for $x \geq 0$ by

$$f(x) = (\coth^3 2x)\left(1 + \frac{0.91}{\cosh 2x}\right).$$

Since $f(x)$ is decreasing for $x \geq 0$, and since a direct computation shows that $f(0.5495) = 3.01762\ldots$, we have $\operatorname{vol} N < 3.0177 \operatorname{vol} M$ whenever $R \geq 0.5495$.

It remains to consider the case in which $0.5495 > R \geq (\log 3)/2 = 0.5493\ldots$. In this case we use [16, Theorem 4.3], which asserts that the tube-packing density $(\operatorname{vol} T)/\operatorname{vol} M$ is bounded above by $(\sinh R)g(R)$, where $g(x)$ is defined for $x > 0$ by

$$g(x) = \frac{\arcsin \frac{1}{2\cosh r}}{\operatorname{arcsinh} \frac{\tanh r}{\sqrt{3}}}.$$

Since $g(x)$ is clearly a decreasing function for $x > 0$, and since $\sinh R$ is increasing for $x > 0$, we have

$$(\operatorname{vol} T)/(\operatorname{vol} M) < (\sinh 0.5495)g((\log 3)/2) = 0.90817\ldots.$$

Hence $\operatorname{vol} N < f_1(x)\operatorname{vol}(M)$, where $f_1(x)$ is defined for $x \geq 0$ by

$$f_1(x) = (\coth^3 2x)\left(1 + \frac{0.90817}{\cosh 2x}\right).$$

Again, $f_1(x)$ is decreasing for $x \geq 0$, and we see by direct computation that $f_1((\log 3)/2) = 3.017392\ldots$. Hence we have $\operatorname{vol} N < 3.0174 \operatorname{vol} M$ in this case. □

Lemma 3.2 *Suppose that M is a closed, orientable hyperbolic 3–manifold such that $\operatorname{vol} M \leq 1.22$, and that C is a shortest geodesic in M. Set $N = \operatorname{drill}_C(M)$. If $\operatorname{tuberad}(C) > (\log 3)/2$ then the maximal cusp neighborhood in N has volume less than π.*

Proof We let $d(\infty) = .853276\ldots$ denote Böröczky's lower bound [6] for the density of a horoball packing in hyperbolic space. It follows from the definition of the density of a horoball packing that the volume of a maximal cusp neighborhood in N is at most $d(\infty)\operatorname{vol} N$. Lemma 3.1 gives $\operatorname{vol} N < 3.0177 \cdot 1.22 < \pi/d(\infty)$, and the conclusion follows. □

4 Filling

As in [4], we shall say that a group is *semifree* if it is a free product of free abelian groups; and we shall say that a group Γ is k*–semifree* if every subgroup of Γ whose rank is at most k is semifree. Note that Γ is 2–semifree if and only if every rank-2 subgroup of Γ is either free or free abelian.

The following improved version of [4, Theorem 6.1] is made possible by more recent developments.

Theorem 4.1 *Let $k \geq 2$ be an integer and let Φ be a Kleinian group which is freely generated by elements $\xi_1, \dots, \xi_k$. Let z be any point of $\mathbb{H}^3$ and set $d_i = \operatorname{dist}(z, \xi_i \cdot z)$ for $i = 1, \dots, k$. Then we have*

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}.$$

In particular there is some $i \in \{1, \dots, k\}$ such that $d_i \geq \log(2k-1)$.

Proof If Γ is geometrically finite this is included in [4, Theorem 6.1]. In the general case, Γ is topologically tame according to [1] and [7], and it then follows from [15, Theorem 1.1], or from the corresponding result for the free case in [14], that Γ is an algebraic limit of geometrically finite groups; more precisely, there is a sequence of geometrically finite Kleinian groups $(\Gamma_j)_{j\geq 1}$ such that each Γ_j is freely generated by elements $\xi_{1j}, \dots, \xi_{kj}$, and $\lim_{j\to\infty} \xi_{ij} = \xi_i$ for $i = 1, \dots, k$. Given any $z \in \mathbb{H}^3$, we set $d_{ij} = \operatorname{dist}(z, \xi_{ij} \cdot z)$ for each $j \geq 1$ and for $i = 1, \dots, k$. According to [4, Theorem 6.1], we have

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_{ij}}} \leq \frac{1}{2}$$

for each $j \geq 1$. Taking limits as $j \to \infty$ we conclude that

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \frac{1}{2}. \qquad \square$$

Let us also recall the following definition from [4, Section 8]. Let Γ be a discrete torsion-free subgroup of $\operatorname{Isom}_+(\mathbb{H}^3)$. A positive number λ is termed a *strong Margulis number* for Γ, or for the orientable hyperbolic 3–manifold $N = \mathbb{H}^3/\Gamma$, if whenever ξ and η are non-commuting elements of Γ, we have

$$\frac{1}{1+e^{\operatorname{dist}(\xi\cdot z, z)}} + \frac{1}{1+e^{\operatorname{dist}(\eta\cdot z, z)}} \leq \frac{2}{1+e^{\lambda}} \, .$$

The following improved version of [4, Proposition 8.4] is an immediate consequence of Theorem 4.1.

Corollary 4.2 *Let Γ be a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$. Suppose that Γ is 2–semifree. Then $\log 3$ is a strong Margulis number for Γ.*

Lemma 4.3 *Let N be a non-compact finite-volume hyperbolic 3–manifold. Suppose that S is a boundary component of the compact core $\widehat{N}$, and $\mathcal{H}$ is the maximal cusp neighborhood in N corresponding to S. If infinitely many of the manifolds obtained by Dehn filling $\widehat{N}$ along S have 2–semifree fundamental group then $\mathcal{H}$ has volume at least π.*

Proof Suppose that (N_i) is an infinite sequence of distinct hyperbolic manifolds obtained by Dehn filling $\widehat{N}$ along S, and that $\pi_1(N_i)$ is 2–semifree for each i.

Thurston's Dehn filling theorem [5, Appendix B], implies that for each sufficiently large i, the manifold N_i admits a hyperbolic metric; that the core curve of the Dehn filling N_i of $\widehat{N}$ is isotopic to a geodesic C_i in N_i; that the length L_i of C_i tends to 0 as $i \to \infty$; and that the sequence of maximal tubes $(\mathrm{tube}(C_i))_{i\geq 1}$ converges geometrically to $\mathcal{H}$. In particular

$$\lim_{i\to\infty} \mathrm{vol}(\mathrm{tube}(C_i)) = \mathrm{vol}\,\mathcal{H}.$$

According to Corollary 4.2, log 3 is a strong Margulis number for each of the hyperbolic manifolds N_i. It therefore follows from [4, Corollary 10.5] that $\mathrm{vol}\,\mathrm{tube}(C_i) > V(L_i)$, where V is an explicitly defined function such that $\lim_{x\to 0} V(x) = \pi$. In particular, this shows that

$$\mathrm{vol}\,\mathcal{H} \geq \lim_{i\to\infty} V(L_i) \geq \pi. \qquad \square$$

5 Non-exceptional manifolds, arbitrary primes

5.1 A closed hyperbolic 3–manifold M will be termed *exceptional* if every shortest geodesic in M has tube radius at most $(\log 3)/2$.

In this section we shall prove a result, Proposition 5.3, which gives a bound of 3 for the dimension of $H_1(M;\mathbb{Z}_p)$ for any prime p when M is a non-exceptional manifold with volume at most 1.22.

Lemma 5.2 *Suppose that M is a compact, irreducible, orientable 3–manifold, such that every non-cyclic abelian subgroup of $\pi_1(M)$ is carried by a torus component of ∂M. Suppose that either*

(i) $\dim H_1(M;\mathbb{Q}) \geq 3$, *or*

(ii) *M is closed and* $\dim H_1(M;\mathbb{Z}_p) \geq 4$ *for some prime p.*

Then $\pi_1(M)$ is 2–semifree.

Proof Let X be any subgroup of $\pi_1(M)$ having rank at most 2. According to [11, Theorem VI.4.1], X is free, or free abelian, or of finite index in $\pi_1(M)$. If $\dim H_1(M;\mathbb{Q}) \geq 3$, it is clear that X has infinite index in $\pi_1(M)$. If M is closed and $H_1(M;\mathbb{Z}_p) \geq 4$ for some prime p, then Proposition 1.1 of [17] implies that every 2–generator subgroup of $\pi_1(M)$ has infinite index. Thus in either case X is either free or free abelian. This shows that $\pi_1(M)$ is 2–semifree. □

Proposition 5.3 *Suppose that M is a closed, orientable, non-exceptional hyperbolic 3–manifold such that* $\operatorname{vol} M \leq 1.22$. *Then $H_1(M;\mathbb{Z}_p)$ has dimension at most 3 for every prime p.*

Proof Since M is non-exceptional, there is a shortest geodesic C in M with $R = \operatorname{tuberad}(C) > (\log 3)/2$. We set $N = \operatorname{drill}_C(M)$. Let $\mathcal{H}$ denote the maximal cusp neighborhood in N. Since $R > (\log 3)/2$, Lemma 3.2 implies that $\operatorname{vol}\mathcal{H} < \pi$.

Now assume that $\dim H_1(M;\mathbb{Z}_p) \geq 4$ for some prime p. There is an infinite sequence (M_i) of manifolds obtained by distinct Dehn fillings of $\widehat{N}$ such that $H_1(M_i;\mathbb{Z}_p)$ has dimension at least 4 for each i. (For example, if (λ,μ) is a basis for $H_1(\partial\widehat{N},\mathbb{Z}_p)$ such that λ belongs to the kernel of the inclusion homomorphism $H_1(\partial\widehat{N},\mathbb{Z}_p) \to H_1(\widehat{N},\mathbb{Z}_p)$, we may take M_i to be obtained by the Dehn surgery corresponding to a simple closed curve in $\partial\widehat{N}$ representing the homology class $\lambda + ip\mu$.) It follows from Thurston's Dehn filling theorem [5, Appendix B] that for sufficiently large i the manifold M_i is hyperbolic. Hence by case (ii) of Lemma 5.2, the fundamental group of M_i is 2–semifree for sufficiently large i. Thus Lemma 4.3 implies that $\operatorname{vol}\mathcal{H} \geq \pi$, a contradiction. □

6 Non-exceptional manifolds, odd primes

Proposition 6.3, which is proved in this section, gives a bound of 2 for the dimension of $H_1(M;\mathbb{Z}_p)$ for any odd prime p when M is a non-exceptional manifold with volume at most 1.22.

Definition 6.1 Let N be a connected manifold, $\star \in N$ a base point, and Q a subgroup of $\pi_1(N, \star)$. We shall say that a connected based covering space $r : (N', \star') \to (N, \star)$ *carries* the subgroup Q if $Q \le r_\sharp(\pi_1(N', \star')) \le \pi_1(N, \star)$

Lemma 6.2 *Suppose that $\mathcal{H}$ is a maximal cusp neighborhood in a finite-volume hyperbolic 3–manifold N. Let $\star$ be a base point in $\mathcal{H}$, and let $P \le \pi_1(N, \star)$ denote the image of $\pi_1(\mathcal{H}, \star)$ under inclusion. Then there is an element β of $\pi_1(N, \star)$ with the following property:*

(†) *For every based covering space $r : (N', \star') \to (N, \star)$ which carries the subgroup $\langle P, \beta\rangle$ of $\pi_1(N, \star)$, there is a maximal cusp neighborhood $\mathcal{H}'$ in N' which is isometric to $\mathcal{H}$.*

Proof . We write $N = \mathbb{H}^3/\Gamma$, where Γ is a discrete, torsion-free subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. Let $q : \mathbb{H}^3 \to N$ denote the quotient map and fix a base point $\star'$ which is mapped to $\star$ by q. The components of $q^{-1}(\mathcal{H})$ are horoballs. Let B_0 denote the component of $q^{-1}(\mathcal{H})$ containing $\star'$. The stabilizer Γ_0 of B_0 is mapped onto the subgroup P of $\pi_1(N, \star)$ by the natural isomorphism $\iota : \Gamma \to \pi_1(N, \star)$.

Since $\mathcal{H}$ is a maximal cusp, there is a component $B_1 \neq B_0$ of $q^{-1}(\mathcal{H})$ such that $\overline{B_1} \cap \overline{B_0} \neq \varnothing$. We fix an element g of Γ such that $g(B_0) = B_1$, and we set $\beta = \iota(g) \in \pi_1(N, \star)$.

To show that β has property (†), we consider an arbitrary based covering space $r : (N', \star') \to (N, \star)$ which carries the subgroup $\langle P, \beta\rangle$ of $\pi_1(N, \star)$. We may identify N' with $\mathbb{H}^3/\Gamma'$, where Γ' is some subgroup of Γ containing $\langle \Gamma_0, g\rangle$.

Since $\Gamma_0 \subset \Gamma'$, the cusp neighborhood $\mathcal{H}$ lifts to a cusp neighborhood $\mathcal{H}'$ in N'. In particular $\mathcal{H}'$ is isometric to $\mathcal{H}$. The horoballs B_0 and $B_1 = g(B_0)$ are distinct components of $(q')^{-1}(\mathcal{H}')$, where $q' : \mathbb{H}^3 \to N'$ denotes the quotient map. Since $g \in \Gamma'$ and $\overline{B_1} \cap \overline{B_0} \neq \varnothing$, the cusp neighborhood $\mathcal{H}'$ is maximal. □

Proposition 6.3 *Suppose that M is a closed, orientable, non-exceptional hyperbolic 3–manifold such that* $\mathrm{vol}\, M \le 1.22$. *Then $H_1(M; \mathbb{Z}_p)$ has dimension at most 2 for every odd prime p.*

Proof Since M is non-exceptional, we may fix a shortest geodesic C in M with $R = \mathrm{tuberad}(C) > (\log 3)/2$. We set $N = \mathrm{drill}_C(M)$. Let $\mathcal{H}$ denote the maximal cusp neighborhood in N. Since $R > (\log 3)/2$, Lemma 3.2 implies that $\mathrm{vol}\, \mathcal{H} < \pi$.

As in the statement of Lemma 6.2, we fix a base point $\star \in \mathcal{H}$, and we denote by $P \le \pi_1(N, \star)$ the image of $\pi_1(\mathcal{H}, \star)$ under inclusion. We fix an element β of $\pi_1(N, \star)$ having property (†) of Lemma 6.2. We set $Q = \langle P, \beta\rangle \le \pi_1(N, \star)$.

Suppose that $\dim H_1(M;\mathbb{Z}_p) \geq 3$ for some prime p. We shall prove the proposition by showing that this assumption leads to a contradiction if p is odd.

It follows from Poincaré duality that the image of the inclusion homomorphism $\alpha : H_1(\partial\widehat{N};\mathbb{Z}_p) \to H_1(\widehat{N};\mathbb{Z}_p)$ has rank 1. Hence the image of P under the natural homomorphism $\pi_1(N,\star) \to H_1(N;\mathbb{Z}_p)$ has dimension 1. It follows that the image $\bar{Q}$ of Q under this homomorphism has dimension either 1 or 2. In the case $\dim \bar{Q} = 1$ we shall obtain a contradiction for any prime p. In the case $\dim \bar{Q} = 2$ we shall obtain a contradiction for any odd prime p.

First consider the case $\dim \bar{Q} = 1$. We have assumed $\dim H_1(M;\mathbb{Z}_p) \geq 3$. Thus there is a $\mathbb{Z}_p \times \mathbb{Z}_p$–regular based covering space $(N',\star')$ of $(N,\star)$ which carries Q. By property (†), there is a maximal cusp neighborhood $\mathcal{H}'$ in N' which is isometric to $\mathcal{H}$. In particular $\operatorname{vol}\mathcal{H}' < \pi$.

Since in particular $(N',\star')$ carries P, the boundary of the compact core $\widehat{N}$ lifts to $\widehat{N}'$. As N' is a p^2–fold regular covering, it follows that $\widehat{N}'$ has $p^2 \geq 4$ boundary components.

It follows from Thurston's Dehn filling theorem [5, Appendix B] that there are infinitely many hyperbolic manifolds obtained by Dehn filling one boundary component of $\widehat{N}'$. If Z is any hyperbolic manifold obtained by such a filling, then Z has at least three boundary components, and it follows from case (i) of Lemma 5.2 that $\pi_1(Z)$ is 2–semifree. It therefore follows from Lemma 4.3 that each maximal cusp neighborhood in N' has volume at least π. Since we have seen that $\operatorname{vol}\mathcal{H}' < \pi$, this gives the desired contradiction in the case $\dim \bar{Q} = 1$.

It remains to consider the case in which $\dim \bar{Q} = 2$ and the prime p is odd. Since we have assumed that $\dim H_1(M;\mathbb{Z}_p) \geq 3$, there is a p–fold cyclic based covering space $(N',\star')$ of $(N,\star)$ which carries Q. Since N' carries P, the boundary of the compact core $\widehat{N}$ lifts to $\widehat{N}'$, and as N' is a p–fold regular covering, it follows that $\widehat{N}'$ has p boundary components.

We claim that the inclusion homomorphism $\alpha' : H_1(\partial\widehat{N}',\mathbb{Z}_p) \to H_1(\widehat{N}',\mathbb{Z}_p)$ is not surjective. To establish this, we consider the commutative diagram

$$\begin{array}{ccc} H_1(\partial\widehat{N}';\mathbb{Z}_p) & \xrightarrow{\alpha'} & H_1(N';\mathbb{Z}_p) \\ \downarrow & & \downarrow r_* \\ H_1(\partial\widehat{N};\mathbb{Z}_p) & \xrightarrow[\alpha]{} & H_1(N;\mathbb{Z}_p) \end{array}$$

where $r : N' \to N$ is the covering projection. Since $(N',\star')$ carries Q we have $\bar{Q} \subset \operatorname{Im} r_*$. Hence surjectivity of α' would imply $\bar{Q} \subset \operatorname{Im}\alpha$. This is impossible: we observed above that $\mathrm{Im}\,\alpha$ has rank 1, and we are in the case $\dim \bar{Q} = 2$. Thus α' cannot be surjective.

Since $\widehat{N}'$ has p boundary components, it follows from Poincaré duality that $\dim \mathrm{Im}\,\alpha' = p \geq 3$. Since α' is not surjective and p is an odd prime, it follows that $\dim H_1(N'; \mathbb{Z}_p) \geq p + 1 \geq 4$.

Since $(N', \star')$ carries Q, some subgroup Q' of $\pi_1(N', \star')$ is mapped isomorphically to Q by $r_\sharp$. In particular Q' has rank at most 3. Since $\dim H_1(N'; \mathbb{Z}_p) \geq 4$, there is a p–fold cyclic based covering space $(N'', \star'')$ of $(N', \star')$ which carries Q'. Hence $(N'', \star'')$ is a p^2–fold (possibly irregular) based covering space of $(N, \star)$ which carries Q. By property (†), there is a maximal cusp neighborhood $\mathcal{H}''$ in N'' which is isometric to $\mathcal{H}$. In particular $\mathrm{vol}\,\mathcal{H}'' < \pi$.

Since $P \leq Q$, there is a component T of $\partial \widehat{N}'$ such that Q' contains a conjugate of the image of $\pi_1(T)$ under the inclusion homomorphism $\pi_1(T) \to \pi_1(N')$. Hence T lifts to the p–fold cyclic covering space N'' of N'. It follows that the covering projection $r' : N'' \to N'$ maps $p \geq 3$ components of $(r')^{-1}(\partial \widehat{N}')$ to T. As $\widehat{N}'$ has at least three boundary components, $\widehat{N}''$ must have at least five boundary components.

Hence if Z is any hyperbolic manifold obtained by Dehn filling one boundary component of $\widehat{N}''$, we have $\dim H_1(Z; \mathbb{Q}) \geq 4 > 3$, and it follows from case (i) of Lemma 5.2 that $\pi_1(Z)$ is 2–semifree. It therefore follows from Lemma 4.3 and Thurston's Dehn filling theorem that each maximal cusp neighborhood in N'' has volume at least π. Since we have seen that $\mathrm{vol}\,\mathcal{H}'' < \pi$, we have the desired contradiction in this case as well. □

7 Exceptional manifolds

Our treatment of exceptional manifolds begins with Proposition 7.1 below, the proof of which will largely consist of citing material from [9]. In order to state it we must first introduce some notation.

For $k = 0, \ldots, 6$ we define constants τ_k as follows:

$$\begin{aligned}
\tau_0 &= 0.4779 \\
\tau_1 &= 1.0756 \\
\tau_2 &= 1.0527 \\
\tau_3 &= 1.2599 \\
\tau_4 &= 1.2521 \\
\tau_5 &= 1.0239 \\
\tau_6 &= 1.0239
\end{aligned}$$

For $k = 0, \dots, 6$ let $\mathcal{E}_k$ be the 2–generator group with presentation

$$\mathcal{E}_k = \langle x, y \colon r_{1,k}, r_{2,k} \rangle,$$

where the relators $r_{1,k} = r_{1,k}(x,y)$ and $r_{2,k} = r_{2,k}(x,y)$ are the words listed below (in which we have set $X = x^{-1}$ and $Y = y^{-1}$):

$$\begin{aligned}
r_{1,0} &= xyXyyXyxyy,\\
r_{2,0} &= XyxyxYxyxy,\\[4pt]
r_{1,1} &= XXyXYXYxYXYXyXXyy,\\
r_{2,1} &= XXyyXyxyxYxyxyXyy,\\[4pt]
r_{1,2} &= XyxyxYxxYxyxyXyy,\\
r_{2,2} &= XXyXXyyXyxyxyXyy,\\[4pt]
r_{1,3} &= XXyxyXXyyXYXyXYxYXYxxYXYxYXyXYXyy,\\
r_{2,3} &= XXyxyXyxYxyxYYxyxYxyXyxyXXyyXYXyy,\\[4pt]
r_{1,4} &= XXyxyXyxYxyxYxyXyxyXXyyXYXyXYXyy,\\
r_{2,4} &= XXyxyXyxyXXyyXYXyXYxYXYxYXyXYXyy,\\[4pt]
r_{1,5} &= XyXYXyXyxyxYxyxy,\\
r_{2,5} &= XyxyxYxYXYxYxyxy,\\[4pt]
r_{1,6} &= XYXyXYxYXyXYXyxy,\\
r_{2,6} &= XYXyxyXyxYxyXyxy.
\end{aligned}$$

The group $\mathcal{E}_0$ is the fundamental group of an arithmetic hyperbolic 3–manifold which is known as Vol3. This manifold, which was studied in [12], is described as `m007(3,1)` in the list given in [10], and can also be described as the manifold obtained by a $(-1,2)$ Dehn filling of the once-punctured torus bundle with monodromy $-R^2L$.

Proposition 7.1 *Suppose that M is an exceptional closed, orientable hyperbolic 3–manifold which is not isometric to* Vol3. *Then there exists an integer k with $1 \le k \le 6$ such that the following conditions hold:*

1. *M has a finite-sheeted cover $\widetilde{M}$ such that $\pi_1(\widetilde{M})$ is isomorphic to a quotient of $\mathcal{E}_k$; and*
2. *there is a shortest closed geodesic C in M such that* $\mathrm{vol}(\mathrm{tube}(C)) \ge \tau_k$.

Proof This is in large part an application of results from [9], and we begin by reviewing some material from that paper.

We begin by considering an arbitrary simple closed geodesic C in a closed, orientable hyperbolic 3–manifold $M = \mathbb{H}^3/\Gamma$. As we pointed out in 2.2, there is a loxodromic isometry $f \in \Gamma$ with $A_f/\langle f\rangle = C$. If we set $R = \text{tuberad}(C)$ and $Z = Z_R(f)$, it follows from the definitions that $\text{tube}(C) = Z/\langle f\rangle$, that $h(Z) \cap Z = \varnothing$ for every $h \in \Gamma - \langle f\rangle$, and that there is an element $w \in \Gamma - \langle f\rangle$ such that $w(\bar{Z}) \cap \bar{Z} \neq \varnothing$.

Let us define an ordered pair (f, w) of elements of Γ to be a *GMT pair* for the simple geodesic C if we have (i) $A_f/\langle f\rangle = C$, (ii) $w \notin \langle f\rangle$, and (iii) $w(\bar{Z}) \cap \bar{Z} \neq \varnothing$. Note that since $\langle f\rangle$ must be a maximal cyclic subgroup of Γ, condition (ii) implies that the group $\langle f, w\rangle$ is non-elementary.

Set $\mathcal{Q} = \{(L, D, R) \in \mathbb{C}^3 : \text{Re}\, L, \text{Re}\, D > 0\}$. For any point $P = (L, D, R) \in \mathcal{Q}$ we will denote by (f_P, w_P) the pair $(f, w) \in \text{Isom}_+(\mathbb{H}^3) \times \text{Isom}_+(\mathbb{H}^3)$, where $f, w \in PGL_2(\mathbb{C}) = \text{Isom}_+(\mathbb{H}^3)$ are defined by

$$f = \begin{bmatrix} e^{L/2} & 0 \\ 0 & e^{-L/2} \end{bmatrix}$$

and
$$w = \begin{bmatrix} e^{R/2} & 0 \\ 0 & e^{-R/2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{D/2} & 0 \\ 0 & e^{-D/2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

With this definition, f_P has (real) translation length $\text{Re}\, L$, and the (minimum) distance between A_f and $w(A_f)$ is $(\text{Re}\, D)/2$.

In [9, Section 1], it is shown that if (f, w) is a GMT pair for a shortest geodesic C in a closed, orientable hyperbolic 3–manifold and $\text{tuberad}(C) \le (\log 3)/2$, then (f, w) is conjugate by some element of $\text{Isom}^+(\mathbb{H}^3)$ to a pair of the form (f_P, w_P) where $P \in \mathcal{Q}$ is a point such that $\exp(P) \doteq (e^L, e^D, e^R)$ lies in the union $X_0 \cup \cdots \cup X_6$ of seven disjoint open subsets of $\mathbb{C}^3$ that are explicitly defined in [9, Proposition 1.28].

For every k with $0 \le k \le 6$ and every point $P = (L, D, R)$ such that $\exp(P) \in X_k$, it follows from [9, Definition 1.27 and Proposition 1.28] that

(I) the isometries $r_{1,k}(f_P, w_P)$ and $r_{2,k}(f_P, w_P)$ have translation length less than $\text{Re}\, L$;

and it follows from [9, Table 1.1] that

(II) $\pi\, \text{Re}(L) \sinh^2(\text{Re}(D)/2) > \tau_k$.

According to [9, Proposition 3.1], if C is a shortest geodesic in a closed, orientable hyperbolic 3–manifold, and if some GMT pair for C has the form (f_P, w_P) for some P with $\exp(P) \in X_0$, then M is isometric to Vol3.

Now suppose that M is an exceptional closed, orientable hyperbolic 3–manifold. Let us choose a shortest closed geodesic C in M. By the definition of an exceptional manifold, C has tube radius $\le (\log 3)/2$. Hence the facts recalled above imply that C has a GMT pair of the form (f_P, w_P) for some P such that $\exp(P) \in X_k$ for some k

with $0 \leq k \leq 6$; and furthermore, that if M is not isometric to Vol3, then $1 \leq k \leq 6$. We shall show that conclusions (1) and (2) hold with this choice of k.

For $i = 1, 2$ it follows from property (I) above that the element $r_{i,k}(f, \omega)$ has real translation length less than the real translation length $\operatorname{Re} L$ of f. Since C is a shortest geodesic in M, it follows that the conjugacy class of $r_{i,k}(f, \omega)$ is not represented by a closed geodesic in M. As M is closed it follows that $r_{i,k}(f, \omega)$ is the identity for $i = 1, 2$. Hence the subgroup of Γ generated by f and ω is isomorphic to a quotient of $\mathcal{E}_k$. Since we observed above that $\langle f, \omega \rangle$ is non-elementary, there is a non-abelian subgroup Y of $\pi_1(M)$ which is isomorphic to a quotient of $\mathcal{E}_k$. In particular Y has rank 2, and it cannot be a free group of rank 2 since the relators $r_{1,k}$ and $r_{2,k}$ are non-trivial. Hence by [11, Theorem VI.4.1] we must have $|\pi_1(M) : Y| < \infty$. This proves (1).

Finally, we recall that

$$\operatorname{vol}\operatorname{tube}(C) = \pi(\operatorname{length}(C))\sinh^2(\operatorname{tuberad}(C)) = \pi(\operatorname{Re} L)\sinh^2((\operatorname{Re} D)/2).$$

Hence (2) follows from (II). □

We shall also need the following slight refinement of [17, Proposition 1.1].

Proposition 7.2 *Let p be a prime and let M be a closed 3–manifold. If p is odd assume that M is orientable. Let X be a finitely generated subgroup of $\pi_1(M)$, and set $n = \dim H_1(X; \mathbb{Z}_p)$. If $\dim H_1(M; \mathbb{Z}_p) \geq \max(3, n+2)$, then X has infinite index in $\pi_1(M)$. In fact, X is contained in infinitely many distinct finite-index subgroups of $\pi_1(M)$.*

Proof In this proof, as in [17, Section 1], for any group G we shall denote by G_1 the subgroup of G generated by all commutators and p-th powers, where p is the prime given in the hypothesis. Since $\dim H_1(X; \mathbb{Z}_p) = n$ we may write $X = EX_1$ for some rank-n subgroup E of X.

We first assume that $n \geq 1$. Set $\Gamma = \pi_1(M)$. Let $\mathcal{S}$ denote the set of all finite-index subgroups Δ of Γ such that $\Delta \geq X$ and $\dim H_1(\Delta; \mathbb{Z}_p) \geq n + 2$. The hypothesis gives $\Gamma \in \mathcal{S}$, so that $\mathcal{S} \neq \varnothing$. Hence it suffices to show that every subgroup $\Delta \in \mathcal{S}$ has a proper subgroup D such that $D \in \mathcal{S}$.

Any group $\Delta \in \mathcal{S}$ may be identified with $\pi_1(\widetilde{M})$ for some finite-sheeted covering space $\widetilde{M}$ of M. In particular, $\widetilde{M}$ is a closed 3–manifold, and is orientable if p is odd. Since $\Delta \in \mathcal{S}$ we have $X \leq \Delta = \pi_1(\widetilde{M})$ and $\dim H_1(\widetilde{M}; \mathbb{Z}_p) = \dim H_1(\Delta; \mathbb{Z}_p) \geq n+2$. Now set $D = E\Delta_1 \leq \Delta$. Applying [17, Lemma 1.5], with $\widetilde{M}$ in place of M, we deduce that

D is a proper, finite-index subgroup of Δ, and that $\dim H_1(D;\mathbb{Z}_p) \geq 2n+1 \geq n+2$. On the other hand, since $\Delta \in \mathcal{S}$, we have $X \leq \Delta$, and hence $X = EX_1 \leq E\Delta_1 = D$. It now follows that $D \in \mathcal{S}$, and the proof is complete in the case $n \geq 1$.

If $n = 0$ then, since $\dim H_1(M;\mathbb{Z}_p) \geq 3$, there exists a finitely generated subgroup $X' \geq X$ such that $H_1(X';\mathbb{Z}_p)$ has dimension 1. The case of the Lemma which we have already proved shows that X' has infinite index. Thus X has infinite index as well. □

Corollary 7.3 *Let p be a prime and let M be a closed, orientable 3–manifold. Let X be a finite-index subgroup of $\pi_1(M)$, and set $n = \dim H_1(X;\mathbb{Z}_p)$. Then* $\dim H_1(M;\mathbb{Z}_p) \leq \max(2, n+1)$.

Lemma 7.4 *Suppose that M is an exceptional hyperbolic 3–manifold with volume at most 1.22. Then $H_1(M;\mathbb{Z}_p)$ has dimension at most 2 for every prime $p \neq 2, 7$, and $H_1(M;\mathbb{Z}_2)$ and $H_1(M;\mathbb{Z}_7)$ have dimension at most 3. Furthermore, if M has volume at most 1.182, then $H_1(M;\mathbb{Z}_7)$ has dimension at most 2.*

Proof If M is isometric to Vol3 then $\pi_1(M)$ is generated by two elements, and the conclusions follow. For the rest of the proof we assume that M is not isometric to Vol3, and we fix an integer k with $1 \leq k \leq 6$ such that conditions (1) and (2) of Proposition 7.1 hold.

By condition (2) of Proposition 7.1, we may fix a shortest closed geodesic C in M such that $\mathrm{vol}(T) \geq \tau_k$, where $T = \mathrm{tube}(C)$. It follows from a result of Przeworski's [16, Corollary 4.4] on the density of cylinder packings that $\mathrm{vol}\, T < 0.91 \,\mathrm{vol}\, M$, and so $\mathrm{vol}\, M > \tau_k/0.91$. If $k = 3$ we have $\tau_k/0.91 > 1.22$, and we get a contradiction to the hypothesis. Hence $k \in \{1, 2, 4, 5, 6\}$.

Furthermore, we have $\tau_1/0.91 > 1.182$. Hence if $\mathrm{vol}\, M \leq 1.182$ then $k \in \{2, 4, 5, 6\}$.

By condition (1) of Proposition 7.1, $\pi_1(M)$ has a finite-index subgroup X which is isomorphic to a quotient of $\mathcal{E}_k$. From the defining presentations of the groups $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$ $\mathcal{E}_5$ and $\mathcal{E}_6$, we find that $H_1(\mathcal{E}_1;\mathbb{Z})$ is isomorphic to $\mathbb{Z}_7 \oplus \mathbb{Z}_7$, that $H_1(\mathcal{E}_2;\mathbb{Z})$ and $H_1(\mathcal{E}_4;\mathbb{Z})$ are isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$, while $H_1(\mathcal{E}_5;\mathbb{Z})$ and $H_1(\mathcal{E}_6;\mathbb{Z})$ are isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. (One can check that the two groups $\mathcal{E}_5$ and $\mathcal{E}_6$ are isomorphic to each other.) In particular, since $k \in \{1, 2, 4, 5, 6\}$ we have $\dim H_1(\mathcal{E}_k;\mathbb{Z}_p) \leq 1$ for any prime $p \neq 2, 7$, and $\dim H_1(\mathcal{E}_k;\mathbb{Z}_p) \leq 2$ for $p = 2$ or 7. As X is isomorphic to a quotient of $\mathcal{E}_k$, it follows that $\dim H_1(X;\mathbb{Z}_p) \leq 1$ for any prime $p \neq 2, 7$, and $\dim H_1(X;\mathbb{Z}_p) \leq 2$ for $p = 2$ or 7. Hence by Corollary 7.3, we have $\dim H_1(M;\mathbb{Z}_p) \leq 2$ for $p \neq 2, 7$, and $\dim H_1(M;\mathbb{Z}_p) \leq 3$ for $p = 2, 7$.

It remains to prove that if vol $M \leq 1.182$ then $\dim H_1(M;\mathbb{Z}_7) \leq 2$. We have observed that in this case $k \in \{2,4,5,6\}$. By the list of isomorphism types of the $H_1(\mathcal{E}_k;\mathbb{Z})$ given above, it follows that $\dim H_1(\mathcal{E}_k;\mathbb{Z}_7) = 0 < 1$. Hence in this case the argument given above for $p \neq 2,7$ goes through in exactly the same way to show that $\dim H_1(M;\mathbb{Z}_7) \leq 2$. □

Proof of Theorem 1.1 For the case in which M is non-exceptional, the theorem is an immediate consequence of Propositions 5.3 and 6.3. For the case in which M is exceptional, the assertions of the theorem are equivalent to those of Lemma 7.4. □

References

[1] **I Agol**, *Tameness of hyperbolic 3-manifolds* `arXiv:math.GT/0405568`

[2] **I Agol**, *Volume change under drilling*, Geom. Topol. 6 (2002) 905–916 MR1943385

[3] **I Agol**, **N M Dunfield**, **P A Storm**, **W P Thurston**, *Lower bounds on volumes of hyperbolic Haken 3-manifolds* `arXiv:math.DG/0506338`

[4] **J W Anderson**, **R D Canary**, **M Culler**, **P B Shalen**, *Free Kleinian groups and volumes of hyperbolic 3-manifolds*, J. Differential Geom. 43 (1996) 738–782 MR1412683

[5] **M Boileau**, **J Porti**, *Geometrization of 3-orbifolds of cyclic type*, Appendix A by M Heusener and J Porti, Astérisque 272 (2001) MR1844891

[6] **K Böröczky**, *Packing of spheres in spaces of constant curvature*, Acta Math. Acad. Sci. Hungar. 32 (1978) 243–261 MR512399

[7] **D Calegari**, **D Gabai**, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc. 19 (2006) 385–446 MR2188131

[8] **M Culler**, **P B Shalen**, *Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds*, J. Amer. Math. Soc. 5 (1992) 231–288 MR1135928

[9] **D Gabai**, **G R Meyerhoff**, **N Thurston**, *Homotopy hyperbolic 3-manifolds are hyperbolic*, Ann. of Math. (2) 157 (2003) 335–431 MR1973051

[10] **C Hodgson**, **J Weeks**, *Downloadable text file* Available at `http://www.geometrygames.org/SnapPea/`

[11] **W H Jaco**, **P B Shalen**, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. 21 (1979) viii+192 MR539411

[12] **K N Jones**, **A W Reid**, *Vol3 and other exceptional hyperbolic 3-manifolds*, Proc. Amer. Math. Soc. 129 (2001) 2175–2185 MR1825931

[13] **S Kojima**, *Deformations of hyperbolic 3-cone-manifolds*, J. Differential Geom. 49 (1998) 469–516 MR1669649

[14] **H Namazi**, **J Souto**, *Nonrealizability in handlebodies and ending laminations* in preparation

[15] **K Ohshika**, *Realizing end invariants by limits of minimally parabolic, geometrically finite groups* arXiv:math.GT/0504546

[16] **A Przeworski**, *A universal upper bound on density of tube packings in hyperbolic space*, J. Differential Geom. 72 (2006) 113–127 MR2215457

[17] **P B Shalen**, **P Wagreich**, *Growth rates, Z_p-homology, and volumes of hyperbolic 3-manifolds*, Trans. Amer. Math. Soc. 331 (1992) 895–917 MR1156298

Department of Mathematics, Statistics, and Computer Science (M/C 249)
University of Illinois at Chicago, 851 S Morgan St, Chicago, IL 60607-7045, USA

agol@math.uic.edu, culler@math.uic.edu, shalen@math.uic.edu

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