Pergamon

Topology Vol. 37, No. 4, pp. 805–849, 1998

0040-9383/98 $19.00 + 0.00

PII: S0040-9383(97)00055-4

# THE FIRST BETTI NUMBER OF THE SMALLEST CLOSED HYPERBOLIC 3-MANIFOLD

MARC CULLER, SA'AR HERSONSKY and PETER B. SHALEN

(*Received* 4 *July* 1996; *accepted* 25 *June* 1997)

## 0. INTRODUCTION

It follows from the work of Gromov, Jørgensen and Thurston (see [3]) that the real numbers which arise as volumes of hyperbolic 3-manifolds form a well-ordered set. It is not known at present which closed 3-manifold has the minimal volume (or whether such a manifold is unique). The techniques developed in the series of papers [6–9, 1], bear on this question since they give volume estimates which depend on topological properties of the manifold. If a certain topological hypothesis can be shown to imply a volume bound that exceeds the volume of a known manifold, one obtains topological information about any minimal volume manifold. The first estimates to have interesting qualitative consequences of this sort appeared in [1]. In the present paper we prove the following result.

THEOREM A. *If $M$ is a closed orientable hyperbolic 3-manifold of minimal volume then the first Betti number of $M$ is at most* 2.

In fact, we will prove a stronger result than Theorem A. Recall that a torsion free Kleinian group $\Gamma$ is said to be *topologically tame* if the corresponding covering space of $M$ is homeomorphic to the interior of a compact 3-manifold with (possibly empty) boundary. It is a conjecture of Marden's that every finitely generated Kleinian group without torsion is topologically tame. As our main theorem we will prove:

THEOREM B. *Let $M = \mathbb{H}^3/\Gamma$ be a closed orientable 3-manifold of minimal volume. Either $\Gamma = \pi_1(M)$ has a 2-generator subgroup of finite index or there is a 2-generator subgroup of $\Gamma$ which is not topologically tame.*

Theorem A follows from Theorem B by virtue of [6, Proposition 10.2], which implies that if the first Betti number of $M$ is at least 3 then every 2-generator subgroup of $\Gamma = \pi_1(M)$ is of infinite index and topologically tame.

According to [15], the arithmetic 3-manifold obtained by $(-5/1, -5/2)$ Dehn surgery on the Whitehead link has volume 0.94270 … . Thus Theorem B follows from the following result, which will be proved in the body of the paper.

THEOREM 1.1. *Let $M = \mathbb{H}^3/\Gamma$ be a closed orientable hyperbolic 3-manifold such that every 2-generator subgroup of $\Gamma = \pi_1(M)$ is topologically tame and of infinite index. Then the volume of $M$ exceeds* 0.94689.

Theorem 1.1 is a refinement of a result proved in [6] which under the same hypotheses gives a lower bound of 0.92 for the volume of $M$. In order to explain how we refine the arguments of [6] in this paper, we must first review them. The basic setting may be described in terms of the "displacement cylinder" $Z_\lambda(X) \subset \mathbb{H}^3$ that is associated to a cyclic group $X$ of loxodromic isometries of $\mathbb{H}^3$ and a positive number $\lambda$. By definition, $Z_\lambda(X)$ consists of all points $z \in \mathbb{H}^3$ such that $d(z, \xi \cdot z) < \lambda$ for some element $\xi \neq 1$ of $X$, where $d$ denotes hyperbolic distance. The main theorem of [6], the "log 3 theorem," which was later generalized in [1], asserts that if $\xi$ and $\eta$ are non-commuting orientation-preserving isometries of $\mathbb{H}^3$ that generate a purely loxodromic discrete group which is topologically tame but not co-compact, then for any point $z \in \mathbb{H}^3$ we have

$$\max(d(z, \xi \cdot z), d(z, \eta \cdot z)) \geqslant \log 3.$$

From this it is easy to deduce that if $M = \mathbb{H}^3/\Gamma$ is a compact hyperbolic 3-manifold and if all 2-generator subgroups of $\Gamma$ are topologically tame and of infinite index then the sets of the form $Z_{\log 3}(X)$ , where $X$ ranges over the maximal cyclic subgroups of $\Gamma$, are pairwise disjoint; in particular these sets cannot cover $\mathbb{H}^3$. If $z$ is any point of $\mathbb{H}^3 - \bigcup Z_{\log 3}(X)$ then the ball of radius $\frac{1}{2}(\log 3)$ about $z$ embeds in $M$. From the existence of a ball of radius $\frac{1}{2}(\log 3)$ in $M$, the volume estimate can be deduced via sphere-packing estimates.

The starting point for the proof of the log 3 theorem is a topological argument which shows that the group $F = \langle \xi, \eta \rangle$ is free on the generators $\xi$ and $\eta$. The free group has a "Banach–Tarski" decomposition

$$F = J_\xi \amalg J_\eta \amalg J_{\xi^{-1}} \amalg J_{\eta^{-1}} \amalg \{1\}$$

where $J_\xi$ consists of all reduced words beginning with the letter $\xi$, and the other terms are defined similarly. This decomposition leads to a decomposition

$$\mu = \nu_\xi + \nu_\eta + \nu_{\xi^{-1}} + \nu_{\eta^{-1}}$$

of a Patterson–Sullivan measure on the limit set $\Lambda$ of $F$. (The definition of the Patterson–Sullivan measure depends on identifying $\mathbb{H}^3$ conformally with a ball in $\mathbb{R}^3$ and hence on the choice of a center point; we take the center to be the point $z$ that appears in the statement of the log 3 theorem.)

Let $L$ denote the common perpendicular to the axes of $\xi$ and $\eta$. An elementary argument shows that the quantities $d(z, \xi \cdot z)$ and $d(z, \eta \cdot z)$ cannot increase when $z$ is replaced by its projection to $L$. Thus one can assume without loss of generality that $z \in L$. This implies that the total masses $|\nu_\xi|$ and $|\nu_{\xi^{-1}}|$ are equal. Since $|\nu_\xi| + |\nu_{\xi^{-1}}| + |\nu_\eta| + |\nu_{\eta^{-1}}| = |\mu| = 1$, one can assume by symmetry that $|\nu_\xi| \leqslant 1/4$. The group-theoretical identity $\xi^{-1} J_\xi = F - J_{\xi^{-1}}$ then implies that

$$\int \lambda_{\xi^{-1}}^{\delta} \, d\nu\xi = 1 - |\nu_{\xi^{-1}}| = 1 - |\nu\xi| \geqslant \frac{3}{4}$$

where $\lambda_{\xi^{-1}}$ is the conformal expansion factor of $\xi^{-1}$ and $\delta$ is the critical exponent of the Poincaré series of $F$. The function $\lambda_{\xi^{-1}} : S_\infty \to \mathbb{R}^+$ turns out to be a monotonically decreasing function of the spherical distance from the "pole" $P_{\xi^{-1}}$ of $\xi^{-1}$, which is defined to be the endpoint of the ray in $\mathbb{H}^3$ which begins at $z$ and passes through the point $\xi \cdot z$. To prove the log 3 theorem one first proves a variant of the statement, in which the assumption that $F$ is topologically tame is replaced by the assumption that $\mu$ is the ordinary area measure on the sphere at infinity $S_\infty$ (so that in particular $\Lambda = S_\infty$). In this case we have $\delta = 2$. Furthermore, using the monotonic behavior of $\lambda_{\xi^{-1}}$ and the fact that $\nu_\xi$ is bounded above by the area measure $A$, it is not hard to show that the expression $\int \lambda^2_{\xi^{-1}}\, dv_\xi$ is bounded above by $\int_{C_{\xi^{-1}}} \lambda^2_{\xi^{-1}}\, dA$, where $C_{\xi^{-1}}$ is a spherical cap of area $|v_\xi|$, centered at $P_{\xi^{-1}}$. Thus

$$\int_{C_{\xi^{-1}}} \lambda^2_{\xi^{-1}}\, dA \geqslant \frac{3}{4}.$$

If the integral above is modified by replacing the cap $C_{\xi^{-1}}$ by a larger cap of area $\frac{1}{4}$ then it can be evaluated in closed form; the result allows one to deduce from the inequality above that the displacement of the point $z$ under $\xi^{-1}$ is at least log 3. This gives the conclusion of the log 3 theorem under the assumption that $\mu$ is the area measure.

To complete the proof of the log 3 theorem one must replace this assumption by the assumption that $F$ is topologically tame. If $F$ is topologically tame but not geometrically finite, it is a result of Canary's that $F$ has a property, called analytic tameness, which implies that the area measure is in fact the unique Sullivan–Patterson measure. The case where $F = \langle \xi, \eta \rangle$ is geometrically finite requires additional work. The pairs $(\xi, \eta)$ such that $\langle \xi, \eta \rangle$ is discrete, free of rank 2, purely loxodromic and geometrically finite form an open set $V \subset PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$. The function $(\xi, \eta) \mapsto \max(d(z, \xi \cdot z), d(z, \eta \cdot z))$ is easily seen to have no local minimum on $V$; hence if the conclusion of the log 3 theorem fails for some pair in $V$, it also fails for some pair lying in the boundary $B$ of $V$ in $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$. It is shown in [6], and generalized in [1], that $B$, which consists of pairs $(\xi, \eta)$ such that $\langle \xi, \eta \rangle$ is discrete and free of rank 2, has a dense subset consisting of pairs $(\xi, \eta)$ such that $\langle \xi, \eta \rangle$ is also purely loxodromic and analytically tame; by continuity it follows that the conclusion of the log 3 theorem holds when $(\xi, \eta) \in B$, and hence when $(\xi, \eta) \in V$.

Many of these steps need to be refined in order to prove Theorem 1.1. First of all, the lower bound for vol $M$ given by Theorem 1.1 is not derived in all cases from a lower bound on the volume of a ball in $M$. Instead, we show, for certain constants $\varepsilon_0$ and $\lambda_0$, that $M$ contains either a ball of radius $\frac{1}{2}(\log 3 + \varepsilon_0)$ or a closed geodesic of length $> \lambda_0$. In the first case, we use a sphere-packing argument to obtain the lower bound for vol $M$. In the case where $M$ contains a closed geodesic $C$ of length $< \lambda_0$, we use results from [1] to obtain a lower bound on the volume of a certain tube about the geodesic $C$.

This volume estimate given in [1] depends on the hypothesis that 2-generator subgroups of $\Gamma$ are topologically tame and of infinite index. The 2-generator groups that come up here are of the form $\langle \gamma, \delta \rangle$ where $\gamma$ is a representative of the conjugacy class corresponding to $C$ and $\delta$ is an arbitrary element which does not commute with $\gamma$. The estimate is based on a stronger version of the log 3 theorem which asserts that if $\xi$ and $\eta$ generate a non-cocompact topologically tame group then

$$\frac{1}{1 + e^{d(z, \xi \cdot z)}} + \frac{1}{1 + e^{d(z, \eta \cdot z)}} \leqslant \frac{1}{2}$$

for any $z \in \mathbb{H}^3$. Combining the above inequality with a little hyperbolic trigonometry one obtains a lower bound for the distance between the axis $A_\gamma$ and $\delta \cdot A_\gamma$, and therefore for the radius, and the volume, of a tube about $C$.

Assume, then, that $M$ contains neither a "big" ball (of radius $> \frac{1}{2}(\log 3 + \varepsilon_0)$) nor a "short" geodesic (of length $< \lambda_0$). The assumption that $M$ contains no big ball implies, by the argument that was reviewed above, that the displacement cylinders $Z_\lambda(X)$, where $\lambda = \log 3 + \varepsilon_0$ and $X$ ranges over the maximal cyclic subgroups of $\Gamma$, form a covering of $\mathbb{H}^3$. By a largely topological argument which is given in Section 2, we can then conclude that there are four distinct maximal cyclic subgroups $X_i$ $(i = 0, 1, 2, 3)$ of $\Gamma$ such that $\bigcap_{0 \leqslant i \leqslant 3} Z_\lambda(X_i) \neq \emptyset$. Let $z$ be a point of this intersection, and for $i = 0, 1, 2, 3$, let $\xi_i$ be an element of $X_i$ such that $d(z, \xi_i \cdot z) < \lambda$. Then for any two distinct elements $i, j$ of $\{0, 1, 2, 3\}$, the elements $\xi_i$ and $\xi_j$ fail to commute. Since by the hypotheses of Theorem 1.1 $\langle \xi_i, \xi_j \rangle$ is topologically tame but not co-compact we may apply the log 3 theorem to $\xi_i$ and $\xi_j$. If for some $i$ and $j$ we knew that $d(z, \xi_i \cdot z), d(z, \xi_j \cdot z) < \log 3$, we would have a contradiction to the log 3 theorem. What we actually know, for any distinct $i$ and $j$, is that $d(z, \xi_i \cdot z), d(z, \xi_j \cdot z) < \log 3 + \varepsilon_0$. This allows us to apply a refined version of the log 3 theorem, which is proved in this paper as Theorem 4.1 (and is quite distinct from the version proved in [6] ); Theorem 4.1 gives restrictions on the angles $\angle(\xi_i^{\pm 1} \cdot z, z, \xi_j^{\pm 1} \cdot z)$ that must hold if $d(z, \xi_i \cdot z), d(z, \xi_j \cdot z) < \lambda$, where $\lambda$ is somewhat greater than log 3. Actually Theorem 4.1 also requires as hypotheses certain lower bounds for the translation lengths of the elements $\xi_i^{\pm 1}\xi_j^{\pm 1}$; in the application, these are satisfied according to our assumption that $M$ contains no short geodesics. The latter assumption also gives lower bounds for the angles $\angle(\xi_i \cdot z, z, \xi_i^{-1} \cdot z)$ for $i = 0, 1, 2, 3$.

The rays starting from $z$ and passing through the eight points $\xi_i^{\pm 1} \cdot z$ define eight points on $S_\infty$, and the restrictions on the angles between these rays may be read as restrictions on the spherical distances between these points. We prove an elementary theorem about the 2-sphere, Theorem 5.1, which shows that these conditions are incompatible; this completes the proof of Theorem 1.1.

The proof of Theorem 1.1, like that of the log 3 theorem, involves a reduction to the case where the point $z$ lies on the common perpendicular to the axes of the two given hyperbolic isometries. However, this reduction is quite complicated in the case of Theorem 1.1. If $\xi$ and $\eta$ are hyperbolic isometries satisfying the hypotheses of the log 3 theorem, and if $z$ is on the common perpendicular $L$ to the axes of $\xi$ and $\eta$ such that $d(z, \xi \cdot z)$ and $d(z, \eta \cdot z)$ are less than $\lambda$, Theorem 3.1 gives a lower bound for the angle $\angle(\xi \cdot z, z, \eta \cdot z)$. This relatively simple statement does not generalize in the obvious way to the case where $z \notin L$, because replacing $z$ by its perpendicular projection to $L$ may well increase the angle in question. This is why the counterpart of Theorem 3.1 in the general case is the much more technical Theorem 4.1, which involves a complicated expression in the two angles $\angle(\xi \cdot z, z, \eta \cdot z)$ and $\angle(\xi^{-1} \cdot z, z, \eta^{-1} \cdot z)$ and must be deduced from Theorem 3.1 by an elaborate calculation involving hyperbolic trigonometry and hard differential calculus.

Still, the heart of the matter is Theorem 3.1, and its proof is a refinement of the case $z \in L$ of the proof of the log 3 theorem. As in the latter proof, one first considers the case in which the area measure is a Patterson measure. Reasoning by contradiction, we assume that $\angle(\xi \cdot z, z, \eta \cdot z)$ is small; we must obtain a contradiction by showing that $\max(d(z, \xi \cdot z), d(z, \eta \cdot z)) \geqslant \lambda$, which is an improvement over the conclusion of the log 3 theorem. We will use the notation introduced above in the sketch of the proof of the log 3 theorem. Recall that $|\nu_\xi| + |\nu_\eta| = \frac{1}{2}$; an examination of the sketch of the proof given above shows that a straightforward improvement is possible except in the case where $|\nu_\xi|$ and $|\nu_\eta|$ are both close to $\frac{1}{4}$. For the purpose of this outline of the argument, we therefore focus on the case where $|\nu_\xi| = |\nu_\eta| = \frac{1}{4}$ . In this case, we can make an improvement over the inequality

$$\int \lambda_{\xi^{-1}}^2 \, d\nu_\xi \leqslant \int_{C_{\xi^{-1}}} \lambda_{\xi^{-1}}^2 \, dA \tag{$*$}$$

which was used in the proof of the log 3 theorem. What makes the improvement possible is that the pole $P_{\eta^{-1}}$ of $\eta^{-1}$ is close to $P_{\xi^{-1}}$ on $S_\infty$, the spherical distance being equal to $\angle(\xi \cdot z, z, \eta \cdot z)$ ; this gives a lower bound, say $2I$, for the area of $C_{\xi^{-1}} \cap C_{\eta^{-1}}$, where $C_{\eta^{-1}}$ of course denotes the spherical cap of area $|\nu_\eta|$ centered at $P_\eta$. Using that $\nu_\xi + \nu_\eta$ is bounded above by the area measure, we conclude, after interchanging $\xi$ and $\eta$ if necessary, that

$$\nu_\xi(C_{\xi^{-1}}) \leqslant A(C_{\xi^{-1}}) - \tfrac{1}{2}A(C_{\xi^{-1}} \cap C_{\eta^{-1}}) \leqslant |\nu_\xi| - I.$$

This permits us to replace the inequality (*) by one of the form

$$\int_{C_0 \cup R} \lambda_{\xi^{-1}}^2 \, dA \geqslant \int \lambda_{\xi^{-1}}^2 \, d\nu_\xi$$

where $C_0$ and $R$ are, respectively, a disk centered at $P_{\xi^{-1}}$ and an annulus disjoint from $C_{\xi^{-1}}$, and $A(C_0 \cup R) = |\nu_\xi|$. This leads to the desired improvement over the inequality $\max(d(z, \xi \cdot z), d(z, \eta \cdot z)) \geqslant \log 3$.

Replacing the hypothesis that the area measure is a Patterson measure by the hypothesis of tameness is achieved by essentially the same technique as in the proof of the log 3 theorem, but more work is required in the geometrically finite case in the step where a pair $(\xi, \eta) \in V$ is replaced by a pair in $V$, in order to preserve the condition $z \in L$.

The paper is organized according to the following plan. In Section 1 we deduce the main theorem from Proposition 2.5, Proposition 5.1 and a special case of Theorem 4.1 which is somewhat less technical and is stated as Corollary 4.9. Sections 2, 4, and 5 are devoted to these results. In Section 3 we prove Theorem 3.1, the key case of Theorem 4.1 in which the point $z$ lies on the common perpendicular to the two axes. To maintain the flow of the argument we have relegated to appendices three technical results that are needed along the way but have self-contained proofs. The appendices are indexed by letters, so e.g. Lemma A1 is a reference to the first lemma in Appendix A.

Throughout the paper there are times when we need approximate values of certain real constants. We will give these values as decimal numbers followed by ellipses, so 1.234... denotes a real number in the interval $[1.234, 1.235)$.

We will be making computations in spherical geometry which will involve the following conventions and notation. We shall think of $S^2$ as the unit sphere in $\mathbb{R}^3$. Any point $P \in S^2$ may be written in the form $(\cos\theta\cos\lambda, \sin\theta\cos\lambda, \sin\lambda)$ with $-\pi/2 \leqslant \lambda \leqslant \pi/2$ and $0 \leqslant \theta < 2\pi$. The *latitude* $\lambda = \lambda(P)$ is uniquely determined by $P$; we have $\lambda(P) = \pi/2 - \phi(P)$, where $\phi(P)$ is the *polar angle*. The *longitude* $\theta = \theta(P)$ is uniquely determined unless $P$ is one of the poles $N = (0, 0, 1)$ or $S = (0, 0, -1)$.

We shall denote by $\ell\colon S^2 \setminus \{N, S\} \to S^1$ the projection map defined by

$$\ell(P) = (\cos\theta(P), \sin\theta(P)).$$

For any two points $P, Q \in S^2 \setminus \{N, S\}$ we shall let $\Theta(P, Q)$ denote the circular distance between $\ell(P)$ and $\ell(Q)$. Thus $\Theta(P, Q)$ is the absolute value of the unique element of the interval $(-\pi, \pi]$ which is congruent to $\theta(P) - \theta(Q)$ modulo $2\pi$. Note that $\ell\colon S^2 \setminus \{N, S\} \to S^1$ and $\Theta : (S^2 \setminus \{N, S\}) \times (S^2 \setminus \{N, S\}) \to [0, \pi]$ are continuous although $\theta : S^2 \setminus \{N, S\} \to [0, 2\pi)$ is not.

A *meridian* in $S^2$ is the closure of a fiber of the map $\ell$. Two points $P, P' \in S^2$ lie on a common meridian if and only if either (i) $P$ or $P'$ is a pole or (ii) $P, P' \in S^2 - \{N, S\}$ and $\ell(P) = \ell(P')$.

We will use the notation $\mathrm{dist}_s(P, Q)$ for the spherical distance between two points $P$ and $Q$ of $S^2$. To minimize confusion when we are working on the sphere at infinity of hyperbolic space, we will write $\mathrm{dist}_h(x, y)$ for the distance between two points $x$ and $y$ of the hyperbolic 3-space $\mathbb{H}^3$. We will use the notation $\mathrm{Isom}_+(\mathbb{H}^3)$ to denote the group of orientation preserving isometries of $\mathbb{H}^3$.

If $A$, $B$ and $C$ are points in $\mathbb{H}^3$ with $A \neq B \neq C$, we denote the angle between the lines $BA$ and $BC$ by $\angle(A, B, C)$. If $\Gamma$ is a group, we shall write $H \leqslant \Gamma$ to indicate that $H$ is a subgroup of $\Gamma$. We shall use $\langle x_1, \dots, x_n \rangle$ to denote the subgroup generated by elements $x_1, \dots, x_n$ of $\Gamma$.

## 1. PROOF OF THE MAIN THEOREM

In this section we give a derivation of the main estimate of the paper. The argument depends on results which are proved in later sections of the paper. We shall state these results as they are needed in the course of the argument.

THEOREM 1.1. *Let $M = \mathbb{H}^3/\Gamma$ be a closed orientable hyperbolic 3-manifold such that every 2-generator subgroup of $\Gamma = \pi_1(M)$ is topologically tame and of infinite index. Then the volume of $M$ exceeds 0.94689.*

*Proof.* The group $\Gamma < \mathrm{Isom}_+(\mathbb{H}^3)$ is discrete, torsion-free, co-compact and, consequently, purely loxodromic.

The proof uses four carefully chosen constants. We set

$$\beta_0 = 0.51\pi, \quad \varepsilon_0 = 0.0065, \quad \lambda_0 = 1.00485, \quad \delta_0 = 0.71497\pi.$$

It follows from [1, Corollary 7.3, Proposition 8.1 and Lemma 10.3] that if every 2-generator subgroup of $\pi_1(M) = \Gamma$ has infinite index and is topologically tame, and if $M$ contains a non-trivial closed geodesic whose length is less than some given positive number $\lambda$, then the volume of $M$ is at least

$$V(\lambda) = \frac{\pi\lambda}{e^\lambda - 1}\left(\frac{e^{2\lambda} + 2e^\lambda + 5}{2(\cosh\frac{\lambda}{2})(e^\lambda + 3)}\right) - \frac{\pi\lambda}{2}.$$

We have $V(\lambda_0) = 0.94689\ldots$. Thus the conclusion of Theorem 1.1 is certainly true if $M$ contains a non-trivial closed geodesic of length $\leqslant \lambda_0$.

We shall now assume that every non-trivial closed geodesic in $M$ has length $> \lambda_0$. In particular, every non-trivial element of $\Gamma$ has translation length $> \lambda_0$. Recall that by hypothesis every 2-generator subgroup of $\Gamma$ is topologically tame and of infinite index.

We shall show that under these conditions $M$ contains a hyperbolic ball of radius $\frac{1}{2}(\log 3 + \varepsilon_0)$. As was observed by Meyerhoff [13], this implies the conclusion of Theorem 1.1 by an estimate due to Böröczky [4] for the density of a hyperbolic sphere-packing. Given an embedded ball in $M$, one obtains a sphere-packing by considering all of the lifts of the boundary sphere to hyperbolic space. Böröczky's result gives an explicit estimate for the volume of the Dirichlet domain for the sphere-packing which, in this case, is also a Dirichlet domain for $M$. Applying this estimate, exactly as was done in [6, Corollary 10.4], one obtains that if $M$ contains a ball of radius $\frac{1}{2}(\log 3 + \varepsilon_0)$ then the volume of $M$ is at least $0.94689\ldots$.

Let us now assume that $M$ contains no ball of radius $\frac{1}{2}(\log 3 + \varepsilon_0)$. We will show that this assumption leads to a contradiction.

We begin by applying the following general result, which is proved in Section 2.

PROPOSITION 2.5. *Let $\Gamma$ be a co-compact, torsion-free, discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$. Let $M$ denote the closed hyperbolic 3-manifold $\mathbb{H}^3/\Gamma$. Let $\Delta$ be a positive real number.*

*Then either*

(i) *M contains a hyperbolic ball of radius* $\frac{1}{2}\Delta$, *or*

(ii) *there exist a point* $z \in \mathbb{H}^3$ *and pairwise non-commuting elements* $\xi_0, \xi_1, \xi_2, \xi_3$ *of* $\Gamma$ *such that* $\mathrm{dist}_h(z, \xi_i \cdot z) < \Delta$ *for* $i = 0, 1, 2, 3$.

We set $\Delta = \log 3 + \varepsilon_0$ in Proposition 2.5. Since we have assumed that conclusion (i) does not hold, we find a point $z$ of $\mathbb{H}^3$ which is moved a distance less than $\frac{1}{2}(\log 3 + \varepsilon_0)$ by pairwise non-commuting elements $\xi_0, \xi_1, \xi_2, \xi_3$ of $\Gamma = \pi_1(M)$. Note that, since the $\xi_i$ are non-trivial elements of $\Gamma$, they are loxodromic and have translation length $> \lambda_0$.

Let us identify $H^3$ conformally with the open unit ball in $\mathbb{R}^3$ in such a way that $z$ is the center of the ball. Then the unit sphere $S^2$ is identified with the sphere at infinity, and every ray in $\mathbb{H}^3$ emanating from $z$ has a well-defined endpoint in $S^2$. For every $(i, u) \in \{0, 1, 2, 3\} \times \{-1, 1\}$, let $P_{(i,u)} \in S^2$ denote the endpoint of the ray that emanates from $z$ and passes through the point $\xi_i^u(z)$ (which is distinct from $z$ since $\xi_i$ is loxodromic). For any indices $(i, u)$ and $(j, v)$ in $\{0, 1, 2, 3\} \times \{1, -1\}$ we have $\mathrm{dist}_s(P_{(i,u)}, P_{(j,v)}) = \angle(\xi_i^u \cdot z, z, \xi_j^v \cdot z)$. In Sections 3 and 4 we derive explicit lower bounds for the spherical distances between pairs of the points $P_{(i,u)}$. To state these results we must introduce certain auxiliary functions.

First we define a function $\phi: \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\phi(l, t, s, \alpha) = \frac{2\cosh^2 s - 2\sinh^2 s \cos\alpha - \cosh \ell - \cos t}{\cosh l - \cos t}.$$

By direct calculation we find that

$$\phi(\lambda_0, \pi, \log 3 + \varepsilon_0, 0) = 0.56936\ldots < 1 < 3.40499\ldots = \phi(\lambda_0, \pi, \log 3 + \varepsilon_0, \pi).$$

Since $\phi$ is monotonically increasing with respect to the fourth variable, there is a unique $\alpha_{-\infty} \in [0, \pi]$ such that $\phi(\lambda_0, \pi, \log 3 + \varepsilon_0, \alpha_{-\infty}) = 1$, and for any $\alpha \in (\alpha_{-\infty}, \pi]$ we have $\phi(\lambda_0, \pi, \log 3 + \varepsilon_0, \alpha) > 1$. Solving numerically, one finds that $\alpha_{-\infty} = 0.80060$. We set

$$\sigma(\alpha) = \cosh^{-1} \phi(\lambda_0, \pi, \log 3 + \varepsilon_0, \alpha)$$

for $\alpha \in (\alpha_{-\infty}, \pi]$.

We also need a constant $K$ whose value is a slight perturbation of $\sigma(\beta_0)$. By direct calculation we find that

$$\phi(\lambda_0, \pi, \log 3 - \varepsilon_0, \beta_0) = 1.98495\ldots > 1$$

and

$$\cosh^{-1} \phi(\lambda_0, \pi, \log 3 - \varepsilon_0, \beta_0) = 1.30822\ldots.$$

We set

$$K = 1.30822$$

so that $\cosh^{-1} \phi(\lambda_0, \pi, \log 3 - \varepsilon_0, \beta_0) > K$.

The following result is included in the main result of Section 4. It is obtained by assigning particular values to the parameters appearing in the latter result.

COROLLARY 4.9. *Let* $\xi$ *and* $\eta$ *be isometries which generate a subgroup of* $\mathrm{Isom}_+(\mathbb{H}^3)$ *which is discrete, free of rank 2, purely loxodromic and topologically tame. Suppose that* $\xi^{-1}\eta$ *has translation length* $\geq \lambda_0$. *Let* $z$ *be a point of* $\mathbb{H}^3$ *such that*

$$\max\{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\} < \log 3 + \varepsilon_0.$$

*Set* $\alpha_1 = \angle(\xi\cdot z, z, \eta\cdot z)$ *and* $\alpha_{-1} = \angle(\xi^{-1}\cdot z, z, \eta^{-1}\cdot z)$. *Then* $\alpha_1$ *and* $\alpha_{-1}$ *lie in the interval* $[\alpha_{-\infty}, \pi]$ *and*

$$\sigma(\alpha_1) + \sigma(\alpha_{-1}) > 2K.$$

To first order, the conclusion of Corollary 4.9 consists of a lower bound on the sum of the angles $\alpha_1$ and $\alpha_{-1}$. Corollary 4.9 is a refinement of Theorem 9.1 of [6], which gives a lower bound of log 3 for the quantity $\max\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\}$ without any restriction on $\alpha_1$ and $\alpha_{-1}$.

Suppose that $(i, u)$ and $(j, v)$ are indices in $\{0, 1, 2, 3\}\times\{1, -1\}$ with $i \neq j$. We wish to apply Corollary 4.9 with $\xi = \xi_i^u$, $\eta = \xi_j^v$, $(\beta, \varepsilon) = (\beta_0, \varepsilon_0)$ , $\lambda = \lambda_0$ and $\theta = \pi$. Since the first Betti number of $M$ is at least 3, and since $\xi_i$ and $\xi_j$ do not commute, it follows from [1, Corollary 7.2; 5, Proposition 3.2] that $\xi$ and $\eta$ generate a free group $F$ of rank 2 which is topologically tame. Since $\Gamma$ is co-compact, $F$ is purely loxodromic. Since $\xi^{-1}\eta$ is a nontrivial element of $\Gamma$, it has translation length $> \lambda_0$. It follows from the defining properties of $\xi_0, \ldots, \xi_3$ that $\mathrm{dist}_h(z, \xi_i^u\cdot z)$ and $\mathrm{dist}_h(z, \xi_j^v\cdot z)$ are less than $\log 3 + \varepsilon_0$. Thus all the hypotheses of Corollary 4.9 hold. Setting $\alpha_1 = \angle(\xi\cdot z, z, \eta\cdot z) = \mathrm{dist}_s(P_{(i,u)}, P_{(j,v)})$ and $\alpha_{-1} = \angle(\xi^{-1}\cdot z, z, \eta^{-1}\cdot z) = \mathrm{dist}_s(P_{(i,-u)}, P_{(j,-v)})$, we conclude that

$$\sigma(\alpha_1) + \sigma(\alpha_{-1}) > 2K.$$

This shows that for any two indices $(i, u)$ and $(j, v)$ in $\{0, 1, 2, 3\}\times\{1, -1\}$ such that $i \neq j$, we have $\mathrm{dist}_s(P_{(i,u)}, P_{(j,v)}) > \alpha_{-\infty}$, $\mathrm{dist}_s(P_{(i,-u)}, P_{(j,-v)}) > \alpha_{-\infty}$ and

$$\sigma(\mathrm{dist}_s(P_{(i,u)}, P_{(j,u)})) + \sigma(\mathrm{dist}_s(P_{(i,-u)}, P_{(j,-v)})) > 2K.$$

The next step is to apply the following proposition about configurations of eight points on a sphere which is proved in Section 5.

PROPOSITION 5.1. *Suppose that we are given an indexed family*

$$(P_{(i,u)})_{(i,u)\in\{0,1,2,3\}\times\{-1,1\}}$$

*of points in* $S^2$. *Assume that for any two indices* $(i, u)$ *and* $(j, v)$ *in* $\{0, 1, 2, 3\}\times\{-1, 1\}$ *with* $i \neq j$, *we have* $\mathrm{dist}_s(P_{(i,u)}, P_{(j,v)}) > \alpha_{-\infty}$. *Then either*

(i) *there is an element* $i$ *of* $\{0, 1, 2, 3\}$ *such that* $\mathrm{dist}_s(P_{(i,1)}, P_{(i,-1)})) \leqslant \delta_0$, *or*
(ii) *there exist indices* $(i, u)$ *and* $(j, v)$ *in* $\{0, 1, 2, 3\}\times\{-1, 1\}$, *with* $i \neq j$, *such that*

$$\sigma(\mathrm{dist}_s(P_{(i,u)}, P_{(j,v)})) + \sigma(\mathrm{dist}_s(P_{(i,-u)}, P_{(j,v)})) \leqslant 2K.$$

Observe that the family $(P_{(i,u)})$ satisfies the hypothesis of Proposition 5.1 but, by Corollary 4.9, does not satisfy alternative (ii) of the conclusion.

On the other hand, at the end of Section 4 we prove a result which shows that the family $(P_{i,u})$ does not satisfy alternative (i) either. The statement requires one more auxiliary function. For $\Delta > \lambda > 0$ we define

$$\omega(\lambda, \Delta) = \cos^{-1}\left(1 - \frac{2(\cosh\lambda - 1)}{\cosh\Delta - 1}\right).$$

PROPOSITION 4.10. *Let* $0 < \lambda < \Delta$ *be real numbers. Let* $\xi$ *be a loxodromic isometry of* $\mathbb{H}^3$ *with translation length* $\geqslant \lambda$, *and let* $z$ *be a point of* $\mathbb{H}^3$ *such that* $\mathrm{dist}_h(z, \xi\cdot z) < \Delta$. *Then we have* $\angle(\xi^{-1}\cdot z, z, \xi\cdot z) > \omega(\lambda, \Delta)$.

Since each $\xi_i$ has translation length $>\lambda_0$ and since $\mathrm{dist}_h(z, \xi_i \cdot z) < \Delta$, it follows from Proposition 4.10 that $\angle(\xi_i^{-1} \cdot z, z, \xi_i \cdot z) > \omega(\lambda_0, \Delta)$. By direct computation we have

$$\omega(\lambda_0, \log 3 - \varepsilon_0) = (0.71497\ldots)\pi \geqslant \delta_0.$$

This means that $\mathrm{dist}_s(P_{(i,1)}, P_{(i,-1)}) > \delta_0$ for $i = 0, 1, 2, 3$. Thus alternative (i) of Proposition 5.1 also fails to hold for the family $(P_{(i,u)})$. This gives a contradiction to Proposition 5.1, and the proof of Theorem 1.1 is therefore complete. □

## 2. INTERSECTIONS OF CYLINDERS IN $\mathbb{H}^3$

In this section we prove Proposition 2.5, which was the starting point for the proof of Theorem 1.1. The proof is based on an analysis of coverings of hyperbolic space by cylinders, which is formulated in Proposition 2.1.

We let $\hat{\mathbb{H}}^3$ denote the union of $\mathbb{H}^3$ with the sphere at infinity, equipped with the standard topology that makes it homeomorphic to a closed 3-ball. In this section we will use the *Beltrami–Klein* model for $\hat{\mathbb{H}}^3$, i.e. we will identify $\hat{\mathbb{H}}^3$ with the closed unit ball in $\mathbb{R}^3$ in such a way that the lines in $\mathbb{H}^3$ are open Euclidean line segments in $\mathbb{R}^3$. If we use this model, the Euclidean metric on the unit ball becomes a metric on $\hat{\mathbb{H}}^3$ which will be denoted $\mathrm{dist}_e$ to distinguish it from the hyperbolic metric $\mathrm{dist}_h$ on $\mathbb{H}^3$. A subset of $\mathbb{H}^3$ is (strictly) convex in the hyperbolic sense if and only if it is identified with a set in the Beltrami–Klein model which is (strictly) convex in the Euclidean sense.

For any closed set $X \subset \mathbb{H}^3$, we let $\hat{X}$ denote the closure of $X$ in $\hat{\mathbb{H}}^3$, and we set $\dot{X} = \hat{X} - X$. In particular, $\dot{\mathbb{H}}^3$ denotes the sphere at infinity. For any line $l \subset \mathbb{H}^3$, the set $\hat{l}$ is identified with a closed line segment in the Beltrami–Klein model, and $\dot{l}$ is identified with the set of endpoints of this segment.

Recall that two lines $l$ and $l'$ in $\mathbb{H}^3$ are said to be *parallel* if $\dot{l} \cap \dot{l}' \neq \emptyset$. In this case, either $\dot{l} \cap \dot{l}'$ consists of a single point, or $l = l'$.

If $A$ is a line in $\mathbb{H}^3$ and $r$ is a positive real number, we shall denote by $Z_r(A)$ the set of all points in $\mathbb{H}^3$ whose hyperbolic distance from $A$ is $\leqslant r$. By a *cylinder* in $\mathbb{H}^3$ we shall mean a set of the form $Z = Z_r(A)$, where $A \subset \mathbb{H}^3$ is a line and $r$ is a positive number. The line $A$ is uniquely determined by the set $Z$ because $\dot{A} = \dot{\mathbb{H}}^3 \cap \hat{Z}$. It follows easily that $r$ is also uniquely determined by $Z$. We shall refer to $A$ and $r$ respectively as the *core* and the *radius* of the cylinder $Z$.

If $Z$ is a cylinder with core $A$ and radius $r$, then, by [6, Proposition 1.2], $Z$ is a strictly convex subset of $\mathbb{H}^3$. It follows that $\hat{Z}$ is a strictly convex, compact subset of $\hat{\mathbb{H}}^3$ and is therefore identified in the Beltrami–Klein model with a strictly convex, compact subset of the unit ball in $\mathbb{R}^3$. Since this set clearly has non-empty interior, it follows that $\hat{Z}$ is a topological 3-ball, and that the boundary of this ball, $\partial\hat{Z}$, is identified with its frontier in $\mathbb{R}^3$ in the Beltrami–Klein model. On the other hand, it is clear that $\dot{Z}$ is equal to $\dot{A}$, a two-point subset of $\dot{\mathbb{H}}^3$. Hence $\partial Z = \partial\hat{Z} - \dot{Z}$ is a topological annulus, and coincides with the frontier of $Z$ in $\mathbb{H}^3$.

The first main result of this section is the following.

PROPOSITION 2.1. *Let $\mathscr{Z}$ be a locally finite collection of cylinders whose interiors cover $\mathbb{H}^3$. Suppose that the cores of any two distinct cylinders in $\mathscr{Z}$ are non-parallel (and in particular distinct). Let $\mathscr{R} \subset (0, \infty)$ denote the set of all radii of cylinders in $\mathscr{Z}$. Suppose that $\mathscr{R}$ has a greatest element $r_0$, and let $Z_0$ be any cylinder of radius $r_0$ in $\mathscr{Z}$. Then there are cylinders $Z_1, Z_2, Z_3 \in \mathscr{Z}$, distinct from one another and from $Z_0$, such that $\partial Z_0 \cap \mathrm{Int}\, Z_1 \cap \mathrm{Int}\, Z_2 \cap \mathrm{Int}\, Z_3 \neq \emptyset$. In particular we have $\mathrm{Int}\, Z_0 \cap \mathrm{Int}\, Z_1 \cap \mathrm{Int}\, Z_2 \cap \mathrm{Int}\, Z_3 \neq \emptyset$.*

Note that if the open set $\operatorname{Int} Z_1 \cap \operatorname{Int} Z_2 \cap \operatorname{Int} Z_3$ meets the frontier $\partial Z_0$ of $\operatorname{Int} Z_0$ in $\mathbb{H}^3$, then $\operatorname{Int} Z_1 \cap \operatorname{Int} Z_2 \cap \operatorname{Int} Z_3$ must also meet the set $\operatorname{Int} Z_0$ itself. Thus the last assertion of the above proposition does follow from the first.

The following two lemmas are needed for the proof of Proposition 2.1. An open subset $U$ of a connected, non-simply-connected 2-manifold $\Sigma$ will be termed *inessential* if (i) $U$ has compact closure in $\Sigma$ and (ii) if $U$ is non-empty then, for every component $V$ of $U$, the inclusion homomorphism $\pi_1(V) \to \pi_1(\Sigma)$ is trivial.

LEMMA 2.2. *Let $\Sigma$ be a connected, non-compact, non-simply-connected 2-manifold without boundary. Let $\mathcal{U}$ be a locally finite collection of inessential open subsets of $\Sigma$ that covers $\Sigma$. Then there are three distinct sets $U_1, U_2, U_3 \in \mathcal{U}$ such that $U_1 \cap U_2 \cap U_3 \neq \emptyset$.*

*Proof.* We may choose a compact set $X_U \subset U$ for each $U \in \mathcal{U}$, in such a way that the interiors of the sets $X_U$ cover $\Sigma$. Let us fix a piecewise linear structure on $\Sigma$. After possibly enlarging the $X_U$ we may assume that each $X_U$ is a polyhedral subset of $\Sigma$. After replacing the $X_U$ by their regular neighborhoods we may assume that they are compact polyhedral 2-manifolds with boundary.

For each $U$ there is a polyhedral disk $D_U \subset \Sigma$ such that $\partial D_U \subset X_U \subset D_U$. Indeed, the hypothesis that each $U \in \mathcal{U}$ is inessential implies that each component $C$ of $\partial X_U$ bounds a disk $\Delta_C \subset \Sigma$. We must have either $X_U \subset \Delta_C$ or $\Delta_C \cap X_U = C$. But if $\Delta_C \cap X_U = C$ for every component $C$ of $\partial X_U$, then the set $X_U \cup \bigcup_C \Delta_C$, where $C$ ranges over all components of $\partial X_U$, is a closed 2-manifold; this is impossible since $\Sigma$ is connected and non- compact. Hence for some component $C$ of $\partial X_U$ we must have $X_U \subset \Delta_C$. The disk $D_U = \Delta_C$ then has the asserted properties.

We claim that there exists a set $W \in \mathcal{U}$ with the property that there is no $U \in \mathcal{U}$ for which $D_W \subset \operatorname{Int} D_U$. Assume that this is false. Then there is a sequence $(U_i)_{i \geqslant 0}$ of sets in $\mathcal{U}$ such that if we set $D_i = D_{U_i}$ for $i = 0, 1, \ldots$, we have $D_i \subset \operatorname{Int} D_{i+1}$ for every $i \geqslant 1$. Set $D = \bigcup_{i \geqslant 0} D_i$. Then $D$ is an open, simply-connected subset of $\Sigma$. Since $\Sigma$ is connected and non-simply-connected, $D$ must have a non-empty frontier. Let $P$ be a point of the frontier of $D$, and let $W$ be a connected neighborhood of $P$ in $\Sigma$ whose closure is compact. The set $W$ meets $D$, and therefore meets $D_j$ for some $j \geqslant 0$; hence $W$ meets $D_i$ for every $i \geqslant j$. On the other hand, there is no $i$ for which $W \subset D_i$; for this would imply $W \subset \operatorname{Int} D_{i+1} \subset \operatorname{Int} D$, and $P$ would not be on the frontier of $D$. Since $W$ is connected it follows that $W$ meets the boundary of $D_i$ for each $i \geqslant j$. But $\partial D_i = \partial D_{U_i} \subset X_{U_i} \subset U_i$; thus $W \cap U_i \neq \emptyset$ for each $i \geqslant j$. Since it is clear from the choice of the sequence $(U_i)_{i \geqslant 0}$ that the $U_i$ are all distinct, this contradicts the local finiteness of $\mathcal{U}$. This proves the claim. Now if $W \in \mathcal{U}$ is the set given by the claim just proved, let us set $S_U = \partial D_W \cap X_U$ for each $U \in \mathcal{U}$. Since $\Sigma$ is covered by the sets $\operatorname{Int} X_U$ for $U \in \mathcal{U}$, the simple closed curve $\partial D_W$ is covered by the sets $S_U$ for $U \in \mathcal{U}$. Since $X_W \subset D_W$, we have $S_W = \emptyset$. Thus $\partial D_W$ is covered by the sets $S_U$ for $W \neq U \in \mathcal{U}$. Our choice of $W$ guarantees that each $S_U$ is a proper subset of $\partial D_W$. Since the $S_U$ are open in $\partial D_W$ and $\partial D_W$ is connected, there are two sets $U, U' \in \mathcal{U}$, distinct from $W$ and from each other, such that $S_U \cap S_{U'} \neq \emptyset$. We have

$$\emptyset \neq S_U \cap S_{U'} = \partial D_W \cap \operatorname{Int} X_U \cap \operatorname{Int} X_{U'} \subset X_W \cap X_U \cap X_{U'} \subset W \cap U \cap U'$$

and the lemma is proved. □

LEMMA 2.3. *Let $Z_0$ and $Z$ be two cylinders in $\mathbb{H}^3$ with non-parallel cores, and let $r_0$ and $r$ denote their respective radii. Suppose that $r_0 \geqslant r$. Then* $\operatorname{Int} Z \cap \partial Z_0$ *is inessential in* $\partial Z_0$.

*Proof.* Let $A$ and $A_0$ denote the cores of $Z$ and $Z_0$, respectively. Since $A$ and $A_0$ are non-parallel, we have $\dot{Z} \cap \dot{Z}_0 = \dot{A} \cap \dot{A}_0 = \emptyset$. Hence $Z \cap Z_0 \subset \mathbb{H}^3$ is compact. In particular $\operatorname{Int} Z \cap \partial Z_0$ has compact closure in $\partial Z_0$.

It remains to show that for every component $V$ of $\operatorname{Int} Z \cap \partial Z_0$, the inclusion homomorphism $\pi_1(V) \to \pi_1(\Sigma)$ is trivial. We first prove this in the "generic" case in which $A \cap A_0 = \emptyset$. In this case, since $A$ and $A_0$ are not parallel, we have $\hat{A} \cap \hat{A}_0 = \emptyset$.

To prove the assertion in this case, we begin by defining a map $f\colon \hat{A}_0 \to \partial \hat{Z}_0$ as follows. Given any point $z \in \hat{A}_0$, we have $z \notin \hat{A}$ since $\hat{A} \cap \hat{A}_0 = \emptyset$. Hence there is a unique line $L_z$ such that $\hat{L}_z$ contains $z$ and $L_z$ meets $A$ perpendicularly. Let $P_z$ denote the point of intersection of $L_z$ with $A$. We have $P_z \neq z$ since $z \notin A$. In the Beltrami–Klein model, $\hat{L}_z$ is a non-degenerate closed line segment. Let us write $\hat{L}_z = X_z \cup Y_z$, where $X_z$ and $Y_z$ are closed line segments such that $P_z \in Y_z$ and $X_z \cap Y_z = \{z\}$. Note that $Y_z$ is always non-degenerate, but that $X_z$ will be degenerate if $z \in \dot{A}_0$. Since $z \in \hat{A}_0 \subset \hat{Z}_0$, and since (according to the remarks at the beginning of this section) $\hat{Z}_0$ is a strictly convex subset of the unit ball, there is a unique point of intersection of $X_z$ with $\partial \hat{Z}_0$. We define $f(z)$ to be this point of intersection. Note that we may characterize $f(z)$ as the unique point of $\hat{L}_z \cap \partial \hat{Z}_0$ such that $\operatorname{dist}_e(f(z), P_z) \geqslant \operatorname{dist}_e(f(z), z)$. Note also that $f$ restricts to the identity on $\dot{A}_0$ and that $f(A_0) \subset \partial Z_0$.

We claim that $f\colon \hat{A}_0 \to \partial \hat{Z}_0$ is continuous. To prove this, it suffices to show that if $(z_i)$ is a sequence of points of $A_0$ converging to a point $z \in \dot{A}_0$, then $(f(z_i))$ converges to $z$. If this is false, then after passing to a subsequence we may assume that $(f(z_i))$ converges to some point $w \neq f(z)$ in $\partial \hat{Z}_0$.

Now since $(z_i)$ converges to $z$, it is clear that the sequence $(P_i) = (P_{z_i})$ converges to $P_z$ in the metric $d_e$, and hence that the sequence of line segments $(\hat{L}_i) = (\hat{L}_{z_i})$ converges to $L_z$ in the Hausdorff metric defined by $d_e$. Since $f(z_i) \in L_i$ for each $i$, it follows that $w \in L_z$. Similarly,

$$\operatorname{dist}_e(w, P_z) = \lim_{i \to \infty} \operatorname{dist}_e(f(z_i), P_i) \geqslant \lim_{i \to \infty} \operatorname{dist}_e(f(z_i), z_i) = \operatorname{dist}_e(w, z).$$

From the above characterization of $f(z)$ we conclude that $w = f(z)$. This contradiction establishes the continuity of $f$.

Now consider the subset $S = \partial Z_0 - f(A_0)$ of $\partial Z_0$. Since $f$ restricts to the identity on $\dot{A}_0$ and $f(A_0) \subset Z_0$, we have $S = \partial \hat{Z}_0 - f(\hat{A}_0)$. The set $f(\hat{A}_0)$ is compact and connected since $f$ is continuous and $\hat{A}_0$ is homeomorphic to a line segment. Since, according to the remarks at the beginning of this section, $\partial \hat{Z}_0$ is a topological 2-sphere, it follows that each component of $S$ is simply connected. We shall complete the proof that $\pi_1(V) \to \pi_1(\Sigma)$ is trivial for every component $V$ of $\operatorname{Int} Z \cap \partial Z_0$ by showing that $\operatorname{Int} Z \cap \partial Z_0 \subset S$.

For this purpose we consider an arbitrary point $w \in \operatorname{Int} Z \cap \partial Z_0$. Thus

$$\operatorname{dist}_h(w, A) < r \leqslant r_0 = \operatorname{dist}_h(w, A_0). \tag{2.3.1}$$

Assume that $w \notin S$, so that $w = f(z)$ for some $z \in A_0$. According to the definition of the map $f$, the point $w$ lies on the line $L_z$, which meets $A$ perpendicularly at $P_z$, and $z$ lies on the segment of $L_z$ with endpoints $w$ and $P_z$. Hence

$$\operatorname{dist}_h(w, A) = \operatorname{dist}_h(w, P_z) \geqslant \operatorname{dist}_h(w, z) \geqslant \operatorname{dist}_h(w, A_0).$$

This contradicts (2.3.1), and the proof is thus complete in the case $A \cap A_0 = \emptyset$.

Finally, we consider the case in which $A \cap A_0 \neq \emptyset$. Assume that $\operatorname{Int} Z \cap \partial Z_0$ has a component $V$ for which $\pi_1(V) \to \pi_1(\Sigma)$ is non-trivial. Then there is a continuous map $g\colon S^1 \to \operatorname{Int} Z \cap \partial Z_0$ which is homotopically non-trivial in $\Sigma$. Now let us choose a sequence of lines $(A^{(i)})$ in $\mathbb{H}^3$ such that the sequence $(\hat{A}^{(i)})$ converges to $(\hat{A})$ in the Hausdorff metric defined by $\text{dist}_e$, and such that $\hat{A}^{(i)} \cap \hat{A}_0 = \emptyset$ for every $i$. Set $Z^{(i)} = Z_r(A^{(i)})$ for every $i$. By the case of the assertion already proved, $\pi_1(V) \to \pi_1(\Sigma)$ is trivial for every $i$ and every component $V$ of $\text{Int}Z^{(i)} \cap \partial Z_0$. Hence there is no $i$ for which $g(S^1) \subset Z^{(i)}$. This means that for each $i$ there is a point $w^{(i)}$ in the compact set $g(S^1)$ such that $\text{dist}_h(w^{(i)}, A^{(i)}) \geqslant r$. After passing to a subsequence we may assume that $(w^{(i)})$ converges to a point $w \in g(S^1)$. In particular we have $w \in \text{Int}\, A$, so that $\text{dist}_h(w, P) < r$ for some point $P \in A$. According to our choice of the $A^{(i)}$, there is a sequence of points $(P^{(i)})$ in $\hat{\mathbb{H}}^3$ such that $P^{(i)} \in \hat{A}^{(i)}$ for every $i$ and $(P^{(i)})$ converges to $P$ in the metric $\text{dist}_e$. We must have $P^{(i)} \in A^{(i)}$ for all large enough $i$; thus after again passing to a subsequence we may assume that $P^{(i)} \in A^{(i)}$ for all $i$. It then follows that $(P^{(i)})$ converges to $P$ in the metric $\text{dist}_h$. Hence for large enough $i$ we have $\text{dist}_h(w^{(i)}, P^{(i)}) < r$. Since $P^{(i)} \in A^{(i)}$ and $\text{dist}_h(w^{(i)}, A^{(i)}) \geqslant r$, we have a contradiction. □

2.4. *Proof of* 2.1. Let $\mathscr{U}$ denote the collection of all subsets of $\partial Z_0$ having the form $\partial Z_0 \cap \text{Int}\, Z$ where $Z$ is an element of $\mathscr{Z}$ distinct from $Z_0$. Since the interiors of the cylinders in $\mathscr{Z}$ cover $\mathbb{H}^3$, and since $\partial Z_0$ is disjoint from $\text{Int}\, Z_0$, the collection $\mathscr{U}$ covers the topological open annulus $\partial Z_0$. Since $\mathscr{Z}$ is locally finite, so is $\mathscr{U}$. By Lemma 2.3, each set in $\mathscr{U}$ is inessential in $\partial Z_0$. Hence by Lemma 2.2, applied with $\Sigma = \partial Z_0$, there are distinct sets $U_1, U_2, U_3 \in \mathscr{U}$ with $U_1 \cap U_2 \cap U_3 \neq \emptyset$. Writing $U_i = \partial Z_0 \cap \text{Int}\, Z_i$ with $Z_i \in \mathscr{Z}$ for $i = 1, 2, 3$, we obtain the conclusion of the proposition. □

We are now ready to prove the result that was quoted in the proof of Theorem 1.1.

PROPOSITION 2.5. *Let* $\Gamma$ *be a co-compact, torsion-free, discrete subgroup of* $\text{Isom}_+(\mathbb{H}^3)$. *Let* $M$ *denote the closed hyperbolic* 3*-manifold* $\mathbb{H}^3/\Gamma$. *Let* $\Delta$ *be a positive real number. Then either* (i) $M$ *contains a hyperbolic ball of radius* $\Delta/2$ *or* (ii) *there exist a point* $z \in \mathbb{H}^3$ *and pairwise non-commuting elements* $\xi_0, \xi_1, \xi_2, \xi_3$ *of* $\Gamma$ *such that* $\text{dist}_h(z, \xi_i \cdot z) < \Delta$ *for* $i = 0, 1, 2, 3$.

*Proof.* As in [7], for every maximal cyclic subgroup of $\Gamma$ we denote by $Z(X) = Z_\Delta(X)$ the set of all points $z \in \mathbb{H}^3$ such that $\text{dist}_h(z, \xi \cdot z) < \Delta$ for some non-trivial element $\xi$ of $X$. The set $Z(X)$ is the interior of a cylinder $\bar{Z}(X)$ if the maximal subgroup $X$ is generated by an element of translation length $< \Delta$, and otherwise $Z(X)$ is empty. Let $\mathscr{Z}$ denote the collection consisting of all the cylinders $\bar{Z}(X)$, where $X$ ranges over the maximal cyclic subgroups of $\Gamma$ that are generated by elements of translation length $< \Delta$. According to [7, Proposition 3.2], either $M$ contains a hyperbolic ball of radius $\frac{1}{2}\Delta$ or $\mathscr{Z}$ covers $\mathbb{H}^3$. In the latter case we shall show that conclusion (ii) of the proposition holds.

The discreteness of $\Gamma$ implies that for every point $z \in \mathbb{H}^3$ there are only finitely many elements $\gamma \in \Gamma$ such that $\text{dist}(z, \gamma \cdot z) \leqslant \Delta$. Hence the collection $\mathscr{Z}$ is locally finite. Since $\Gamma$ is discrete and co-compact, every element of $\Gamma$ lies in a unique maximal cyclic subgroup, which is its centralizer. The stabilizer in $\Gamma$ of any point of $\dot{\mathbb{H}}^3$ is either a maximal cyclic subgroup or the trivial group. Thus if $Z(X)$ is a cylinder in $\mathscr{Z}$ then $X$ is the unique maximal cyclic subgroup fixing either point of $\dot{Z}(X)$. It follows that any two distinct cylinders in $\mathscr{Z}$ have non-parallel cores.

On the other hand, since $\Gamma$ is co-compact, it contains only finitely many conjugacy classes of elements with translation length $< \Delta$. It follows that the set $R \subset (0, \infty)$, consisting of all radii of cylinders in $\mathscr{Z}$, is finite and hence has a greatest element $r_0$. Let $Z_0$ be any cylinder of radius $r_0$ in $\mathscr{Z}$. Then it follows from Proposition 2.1 that there are cylinders

$Z_1, Z_2, Z_3 \in \mathscr{Z}$, distinct from one another and from $R$, such that $\operatorname{Int} Z_0 \cap \operatorname{Int} Z_1 \cap \operatorname{Int} Z_2 \cap \operatorname{Int} Z_3 \neq \emptyset$. We may write $Z_i = Z(X_i)$ for $i = 0, 1, 2, 3$, where $X_i$ is a maximal cyclic subgroup of $\Gamma$. For $i = 0, 1, 2, 3$ there is a non-trivial element $\xi_i$ of $X_i$ such that $\operatorname{dist}_h(\xi_i \cdot z, z) < \Delta$. Since the $\xi_i$ lie in distinct maximal cyclic subgroups of $\Gamma$, no two of them can commute. □

### 3. ANGLES AND DISPLACEMENTS, I

Corollary 4.9, which was used in the proof of Theorem 1.1, is a special case of the main result of Section 4, Theorem 4.1, which in turn may be regarded as a refined version of Theorem 9.1 of [6]. If two hyperbolic isometries $\xi$ and $\eta$ generate a free Kleinian group of rank 2 which is topologically tame and has no parabolic elements, then Theorem 9.1 of [6] asserts that an arbitrary point $z$ of hyperbolic space is displaced at least a distance log 3 by either $\xi$ or $\eta$. Theorem 4.1 gives an improved lower bound for this displacement involving the angles $\angle(\xi \cdot z, z, \eta \cdot z)$ and $\angle(\xi^{-1} \cdot z, z, \eta^{-1} \cdot z)$. To first order, the refined result says that the lower bound of log 3 can be increased slightly under the assumption that the sum of these two angles is small.

Theorem 4.1 is proved in two stages. In this section we consider the case when $z$ lies on the common perpendicular to the axes of $\xi$ and $\eta$. Here we are able to take advantage of the existence of an involution that fixes $z$ and conjugates $\xi$ and $\eta$ to their inverses. In particular, this symmetry implies that the two angles mentioned above are equal. The general case, where $z$ need not lie on the common perpendicular, is handled in the next section.

The proof of Theorem 9.1 of [6] makes use of a Patterson–Sullivan construction to produce a certain measure-theoretic decomposition of the limit set of the group $\Gamma = \langle \xi, \eta \rangle$ into four measures corresponding to the two generators and their inverses. As in [6, Lemma 5.3], the existence of an involution that fixes $z$ and conjugates the generators to their inverses will be used to conclude that the measure associated to a generator has the same total mass as that associated to its inverse. Actually, as in [6, Proposition 5.2] these observations are used only in the special case in which every $\Gamma$-invariant positive super-harmonic function on $\mathbb{H}^3$ is constant; the general case of Theorem 4.1 is reduced to this case. In this case the argument given in [6, Lemma 5.5 and Proposition 5.2] involves comparing the four measures with characteristic measures of four spherical caps whose areas sum to 1. The angle $\alpha = \angle(\xi \cdot z, z, \eta \cdot z) = \angle(\xi^{-1} \cdot z, z, \eta^{-1} \cdot z)$ is the spherical distance between the centers of two of these caps. In the crucial special case where the caps have areas close to $\frac{1}{4}$, the assumption that $\alpha$ is small implies that there is a substantial overlap between these two caps. This leads to a refinement of the estimate established in [6, Proposition 5.2].

We will need some more notation regarding spherical geometry. If $P$ is a point of $S^2$ and $r$ is a number in the interval $(0, \pi)$, we shall denote by $C(P, r)$ the spherical "cap" consisting of all points of $S^2$ whose spherical distance from $P$ is at most $r$. An easy computation shows that the area of $C(P, r)$ is $2\pi(1 - \cos r)$. For any three real numbers $r_1, r_2, \alpha$ in the interval $(0, \pi)$ we shall denote by $\iota(\alpha, r_1, r_2)$ the area of the intersection of two spherical caps $C(P_1, r_1)$ and $C(P_2, r_2)$, where $P_1$ and $P_2$ are two points of $S^2$ such that $\operatorname{dist}_s(P_1, P_2) = \alpha$. A closed form expression for the function $\iota$ is derived in Appendix B.

We define functions $E: (0, \infty) \to (0, \frac{1}{4})$ and $r: (0, \infty) \to (0, \pi)$ by

$$E(\varepsilon) = \frac{1}{1 + 3e^{\varepsilon}} \quad \text{and} \quad r(\varepsilon) = \cos^{-1}(1 - 2E(\varepsilon)).$$

We define functions $I$ and $f$ on $(0, \pi) \times (0, \infty) \subset \mathbb{R}^2$ by

$$I(\alpha, \varepsilon) = \frac{1}{8\pi} \imath(\alpha, r(\varepsilon), r(\varepsilon)) \tag{3.0.1}$$

and

$$f(\alpha, \varepsilon) = \tfrac{1}{2} - E(\varepsilon) - I(\alpha, \varepsilon).$$

Note that it follows from the definition of $\imath$ that it is non-negative and is monotonically decreasing as a function of the first variable. Hence $I$ is also non-negative and monotonically decreasing as a function of the first variable.

We denote by $\mathscr{G}$ the subset of $\mathbb{R}^3$ consisting of all points $(x, u, t)$ such that $x > 1$ and $0 \leqslant t \leqslant u$. We define a real-valued function $g$ on $\mathscr{G}$ by

$$g(x, u, t) = \frac{tx}{1 + t(x-1)} + \frac{u - t}{(1 + u(x-1))(1 + (2u - t)(x - 1))} + u.$$

We define a constant $\varepsilon_\infty = 0.05$. We shall denote by $\mathscr{D}$ the open subset of $(0, \pi) \times (0, \varepsilon_\infty) \subset \mathbb{R}^2$ consisting of all points $(\beta, \varepsilon)$ satisfying the following conditions:

(1) $f(\beta, \varepsilon) > 0$;

(2) $g(9e^{2\varepsilon}, \frac{1}{2} - E(\varepsilon), f(\beta, \varepsilon)) < 1$; and

(3) $I(\beta, \varepsilon) < \frac{1}{3} E(\varepsilon)$.

Note that if (1) holds then $(9e^{2\varepsilon}, \frac{1}{2} - E(\varepsilon), f(\beta, \varepsilon)) \in \mathscr{G}$, so that condition (2) makes sense. The main result of this section is the following theorem, the proof of which will occupy the rest of the section.

THEOREM 3.1. *Let $(\beta, \varepsilon)$ be any point in $\mathscr{D}$. Let $\xi$ and $\eta$ be two loxodromic isometries of $\mathbb{H}^3$ such that the group $\Gamma$ generated by $\xi$ and $\eta$ is discrete, topologically tame, purely loxodromic and free on the generators $\xi$ and $\eta$. Let $z$ be a point on the common perpendicular to the axes of $\xi$ and $\eta$. Suppose that $\angle(\xi \cdot z, z, \eta \cdot z) < \beta$. Then we have*

$$\max\{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\} \geqslant \log 3 + \varepsilon.$$

We follow the notation of Section 5 of [6]. We denote by $\lambda_{\gamma,z} : S_\infty \to \mathbb{R}$ the conformal expansion factor of a hyperbolic isometry $\gamma$ relative to a point $z \in \mathbb{H}^3$. It is shown in paragraph 2.4 of [6] that if we identify $\hat{\mathbb{H}}^3 = \mathbb{H}^3 \cup S_\infty$ conformally with the closed unit ball in $\mathbb{R}^3$ in such a way that $z$ is the origin and $\gamma^{-1} \cdot z$ is on the positive vertical axis, then the conformal expansion factor of $\gamma$ is given by the formula

$$\lambda_{\lambda, z}(\zeta) = (c - s\cos\phi)^{-1} \tag{3.1.1}$$

where $c = \cosh \mathrm{dist}_h(z, \gamma \cdot z)$, $s = \sinh \mathrm{dist}_h(z, \gamma \cdot z)$, and $\phi = \phi(\zeta)$ is the polar angle of $\zeta$. In this paper we define the *pole* of $\gamma$ (relative to $z$) to be the endpoint $P \in S^\infty$ of the ray emanating from $z$ and passing through $\gamma^{-1} \cdot z$. In the coordinate system just described, $P$ is the north pole of $S^2$ and the polar angle of a point of $S^2$ is its spherical distance from $P$. Thus $\lambda_{\gamma,z}$ is a positive-valued, decreasing function of the spherical distance of a point from the pole of $\gamma$. In particular, the pole is the unique maximum point of $\lambda_{\gamma,z}$.

For any point $z \in \mathbb{H}^3$ we denote by $A_z$ the area measure on the sphere at infinity $S_\infty$ determined by the round metric centered at $z$, normalized so as to have total mass 1. Thus in the coordinate system described above, $A$ is obtained from the ordinary area measure by dividing by $4\pi$.

The following lemma is a key step in the proof of Theorem 3.1 which replaces one of the basic estimates in [6]. The original estimate contained in [6, Lemma 5.5], depends on the observation that if $\nu$ is a Borel measure on $S_\infty$ which is bounded above by $A_z$, then

$$\int_{S_\infty} (\lambda_{\gamma,z})^2 \, d\nu \leqslant \int_{C_0} (\lambda_{\gamma,z}) \, dA_z$$

where $C_0$ is a spherical cap centered at the pole of $\gamma$ with area $u = \nu(S_\infty)$ . Here we observe that the cap $C_0$ can be replaced by the union of a smaller cap of area $t = \nu(C_0)$ and an annulus of area $u - t$. Later we will use our condition on angles to get bounds on $u$ and $t$. The conclusion that $g(e^{2\Delta}, u, t) \geqslant 1$ can, as we shall see, be regarded as giving a lower bound for the displacement $\Delta = \mathrm{dist}_h(z, \gamma \cdot z)$.

LEMMA 3.2. *Let $\gamma$ be a loxodromic isometry of $\mathbb{H}^3$ and let $z$ be any point of $\mathbb{H}^3$. Set $\Delta = \mathrm{dist}_h(z, \gamma \cdot z)$. Let $\nu$ be a Borel measure on $S_\infty$ such that*

(i) $\nu \leqslant A_z$ *and*
(ii) $\int_{S_\infty} (\lambda_{\gamma,z})^2 \, d\nu = 1 - \nu(S_\infty)$.

*Set $u = \nu(S_\infty)$. Let $C_0$ be the spherical cap with center at the pole of $\gamma$ and with area $u$. Set $t = \nu(C_0)$. Then we have $g(e^{2\Delta}, u, t) \geqslant 1$.*

*Proof.* Let $z \in \mathbb{H}^3$ be given. As in the discussion preceding the statement of the lemma, we identify $\hat{\mathbb{H}}^3 = \mathbb{H}^3 \cup S_\infty$ conformally with the closed unit ball in $\mathbb{R}^3$ in such a way that $z$ is the origin and $\gamma^{-1} \cdot z$ is on the positive vertical axis. We set $A = A_z$, $\phi_1 = \arccos(1 - 2t)$, and $C_1 = C(P, \phi_1)$. Since $A$ is $1/4\pi$ times the area measure on $S_\infty$, and since $C_1$ has area $2\pi(1 - \cos t)$, we have

$$A(C_1) = \tfrac{1}{2}(1 - \cos \phi_1) = t.$$

Now we set $\phi_0 = \cos^{-1}(1 - 2u)$ and $C_0 = C(P, \phi_0)$. We also set $\phi_R = \arccos(1 - 4u + 2t)$, and we let $R \subset S_\infty$ denote the annulus $C(P, \phi_R) \backslash \mathrm{Int}\, C(P, \phi_0)$. Then we have

$$A(R) = \tfrac{1}{2}(1 - \cos \phi_R) - \tfrac{1}{2}(1 - \cos \phi_0) = u - t.$$

Let us denote by $\nu_1, \nu_2$ the restrictions of the measure $\nu$ to $C_0$ and to $S_\infty \backslash C_0$, respectively. Then we have $\nu = \nu_1 + \nu_2$. Hence hypothesis (ii) may be rewritten in the form

$$\int_{C_0} (\lambda_{\gamma,z})^2 \, d\nu_1 + \int_{S_\infty - C_0} (\lambda_{\gamma,z})^2 \, d\nu_2 = 1 - u. \tag{3.2.1}$$

Since $\nu \leqslant A$ we have $\nu_1 \leqslant A$ and $\nu_2 \leqslant A$. Since, by the discussion preceding the statement of the lemma, $\lambda_{\gamma,z}(\zeta)$ is a positive-valued monotonically decreasing function of the polar angle $\phi$, we apply Lemma 5.4 of [6] with $C = C_1$, $\mu_0 = \nu_1$, $\mu = A$, $X = C_0$ and $f = \lambda_{\gamma,z}$ to obtain the following inequality:

$$\int_{C_0} (\lambda_{\gamma,z})^2 \, d\nu_1 \leqslant \int_{C_1} (\lambda_{\gamma,z})^2 \, dA. \tag{3.2.2}$$

Applying the same lemma from [6] with $C = R$, $\mu_0 = \nu_2$, $\mu = A$, $X = S_\infty \backslash C_0$ and $f = \lambda_{\gamma,z}$ we obtain that

$$\int_{S_\infty \backslash C_0} (\lambda_{\gamma,z})^2 \, d\nu_2 \leqslant \int_R (\lambda_{\gamma,z})^2 \, dA. \tag{3.2.3}$$

Now we evaluate the right-hand sides of (3.2.2) and (3.2.3) using the formula (3.1.1) for $\lambda_{\gamma,z}$. We find that

$$\begin{aligned}\int_{C_1} (\lambda_{\gamma,z})^2\, dA &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\phi_1} \frac{\sin\phi}{(c - s\cos\phi)^2}\, d\phi\, d\theta \\ &= \frac{1}{2s}\left(\frac{1}{c-s} - \frac{1}{c - s\cos\phi_1}\right) \\ &= \frac{te^{2\Delta}}{1 + t(e^{2\Delta-1})} \end{aligned} \tag{3.2.4}$$

where in the last step we used that $c = \cosh\Delta$, $s = \sinh\Delta$ and $\cos\phi_1 = 1 - 2t$. Likewise, setting $x = e^{2\Delta}$, we have

$$\begin{aligned}\int_{R} (\lambda_{\gamma,z})^2\, dA &= \int_0^{2\phi}\int_0^{\phi_R} \frac{\sin\phi}{(c - s\cos\phi)^2}\, d\phi\, d\theta \\ &= \frac{1}{2s}\left(\frac{1}{c - s\cos\phi_0} - \frac{1}{c - s\cos\phi_R}\right) \\ &= \frac{u-t}{(1 + u(x-1))(1 + (2u - t)(x-1))}. \end{aligned} \tag{3.2.5}$$

Now adding (3.2.2) and (3.2.3), substituting for the right-hand sides the expressions given by (3.2.4) and (3.2.5), and rewriting the left-hand side of the resulting inequality as $1 - u$ by virtue of (3.2.1), we obtain

$$1 - u \leqslant \frac{tx}{1 + t(x-1)} + \frac{u-t}{(1 + u(x-1))(1 + (2u-t)(x-1))}$$

which is equivalent to the conclusion of the lemma. □

The proof of Theorem 3.1, like that of Theorem 9.1 of [6], reduces to the case of a hyperbolic manifold which admits no non-constant positive super-harmonic functions. This case is contained in the next Proposition. We need two lemmas for the proof.

LEMMA 3.3. *Let $\varepsilon$ be a given positive number. Let $\xi$ and $\eta$ be two loxodromic isometries of $\mathbb{H}^3$. Suppose that the group $\Gamma$ generated by $\xi$ and $\eta$ is discrete and free on the generators $\xi$ and $\eta$, and that every $\Gamma$-invariant, positive, superharmonic function on $\mathbb{H}^3$ is constant. Let $z$ be a point on the common perpendicular to the axes of $\xi$ and $\eta$ and set $\alpha = \angle(\xi\cdot z, z, \eta\cdot z)$. If we have*

$$\max\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\} < \log 3 + \varepsilon.$$

*Then there exist real numbers $\Delta$, $t$ and $u$ with $\Delta \in \{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\}$, $u \in [E(\varepsilon), \frac{1}{2} - E(\varepsilon)]$ and $t \in [0, u - I(\alpha, \varepsilon)]$, such that*

$$g(e^{2\Delta}, t, u) \geqslant 1.$$

*Note that the inequalities $\Delta \geqslant 0$ and $0 \leqslant t \leqslant u$ imply that $(e^{2\Delta}, t, u) \in \mathcal{G}$, so that the last conclusion of the lemma makes sense if the others hold.*

*Proof.* Let us identify $\hat{\mathbb{H}}^3 = \mathbb{H}^3 \cup S_\infty$ conformally with the closed unit ball in $\mathbb{R}^3$ in such a way that $z$ is the origin. Let $P_\xi$ and $P_\eta$ denote the poles of $\xi$ and $\eta$ relative to $z$. By definition $P_\xi$ and $P_\eta$ are the endpoints of the rays emanating from $z$ and passing through $\xi^{-1}\cdot z$ and $\eta^{-1}\cdot z$. Hence we have $\mathrm{dist}_s(P_\xi, P_\eta) = \angle(\xi^{-1}\cdot z, z, \eta^{-1}\cdot z)$ . If we define $\tau \in \mathrm{Isom}_+(\mathbb{H}^3)$ to be the rotation about $L$ through an angle $\pi$, we have $\tau\xi\tau = \xi^{-1}$ and $\tau\eta\tau = \eta^{-1}$. Since $\tau$ fixes $z$ it follows that

$$\angle(\xi^{-1}\cdot z, z, \eta^{-1}\cdot z) = \angle(\tau\xi\tau\cdot z, z, \tau\eta\tau\cdot z) = \angle(\xi\cdot z, z, \eta\cdot z) = \alpha.$$

Hence

$$\mathrm{dist}_s(P_\xi, P_\eta) = \alpha. \tag{3.3.1}$$

We set $\Psi = \{\xi, \xi^{-1}, \eta, \eta^{-1}\} \subset \Gamma$. We also set $A = A_z$.

According to [6; Lemma 5.3] and the hypotheses of the lemma there exist four Borel measures $(\nu_\xi, \nu_{\xi^{-1}}, \nu_\eta, \nu_{\eta^{-1}})$ on $S_\infty$ such that

$$A = \nu_\xi + \nu_{\xi^{-1}} + \nu_\eta + \nu_{\eta^{-1}} \tag{3.3.2}$$

for each $\psi \in \Psi$ we have

$$\int_{S_\infty} (\lambda_{\psi,z})^2 \, d\nu_{\psi^{-1}} = 1 - \nu_\psi(S_\infty) \tag{3.3.3}$$

$$\nu_\xi(S_\infty) = \nu_{\xi^{-1}}(S_\infty) \quad \text{and} \quad \nu_\eta(S_\infty) = \nu_{\eta^{-1}}(S_\infty). \tag{3.3.4}$$

Next we apply [6, Lemma 5.5] (after correcting a typographical error by interchanging the numerator and denominator of the fraction in the conclusion). Setting $a = \nu_\psi(S_\infty)$ and $b = 1 - a$, we obtain that

$$\log \frac{1 - \nu_\psi(S_\infty)}{\nu_\psi(S_\infty)} \leqslant \mathrm{dist}_h(z, \psi(z)) < \log 3 + \varepsilon$$

for each $\psi \in \Psi$. We therefore obtain for each $\psi \in \Psi$ the inequality

$$\nu_\psi(S_\infty) > \frac{1}{1 + 3e^\varepsilon} = E(\varepsilon). \tag{3.3.5}$$

Conditions (3.3.2) and (3.3.4) above imply that

$$\nu_{\xi^{-1}}(S_\infty) + \nu_{\eta^{-1}}(S_\infty) = \tfrac{1}{2}. \tag{3.3.6}$$

From (3.3.5) and (3.3.6) it follows that

$$\nu_{\xi^{-1}}(S_\infty), \nu_{\eta^{-1}}(S_\infty) < \tfrac{1}{2} - E(\varepsilon). \tag{3.3.7}$$

Let us choose two spherical caps $C_\xi$ and $C_\eta$, centered at $P_\xi$ and $P_\eta$, such that $A(C_\xi) = \nu_{\xi^{-1}}(S_\infty)$ and $A(C_\eta) = \nu_{\eta^{-1}}(S_\infty)$. Since a spherical cap $C(P, r)$ has area $2\pi(1 - \cos r)$, and since $A$ is $1/4\pi$ times the area measure, we have $C_\xi = C(P_\xi, r_\xi)$ and $C_\eta = C(P_\xi, r_\eta)$, where $r_\xi = \cos^{-1}(1 - 2\nu_{\xi^{-1}}(S_\infty))$ and $r_\eta = \cos^{-1}(1 - 2\nu_{\eta^{-1}}(S_\infty))$. Since $r(\varepsilon) = \cos^{-1}(1 - 2E(\varepsilon))$, it follows from (3.3.5) that

$$r_\xi, r_\eta \geqslant r(\varepsilon).$$

If we set $G = C_\xi \cap C_\eta$, then by (3.3.1) and (3.0.1) and the definition of the function $\iota$, we have

$$A(G) = \frac{1}{4\pi}\iota(\alpha, r_1, r_2) \geqslant \frac{1}{4\pi}\iota(\alpha, r(\varepsilon), r(\varepsilon)) = 2I(\alpha, \varepsilon). \tag{3.3.8}$$

By property (3.3.2) of the $\nu_\psi$ we have

$$\nu_{\xi^{-1}}(G) + \nu_{\eta^{-1}}(G) \leqslant A(G).$$

By symmetry we may assume that

$$\nu_{\xi^{-1}}(G) \leqslant \tfrac{1}{2} A(G). \tag{3.3.9}$$

Set $t = \nu_{\xi^{-1}}(C_\xi)$ and $u = \nu_{\xi^{-1}}(S_\infty)$. By (3.3.5) and (3.3.7) we have

$$E(\varepsilon) < u < \tfrac{1}{2} - E(\varepsilon). \tag{3.3.10}$$

Using (3.3.2) and (3.3.9) and the definition of $C_\xi$, we find

$$\begin{aligned} t = \nu_{\xi^{-1}}((C_\xi \backslash G) \cup G) &= \nu_{\xi^{-1}}(C_\xi \backslash G) + \nu_{\xi^{-1}}(G) \\ &\leqslant A(C_\xi) - A(G) + \tfrac{1}{2}A(G) \leqslant u - \tfrac{1}{2}A(G). \end{aligned} \tag{3.3.11}$$

Using (3.3.11) and (3.3.8), we get

$$t \leqslant u - I(\alpha, \varepsilon). \tag{3.3.12}$$

We may now apply Lemma 3.2 with $\gamma = \xi$ and $\nu = \nu_{\xi^{-1}}$. Indeed, hypothesis (i) of Lemma 3.2 follows from (3.3.2), and hypothesis (ii) follows from (3.3.3) and (3.3.4). The above definitions of $u$, $t$ now agree with those given in the statement of Lemma 3.2, while the cap $C_0$ defined in the latter statement is $C_\xi$. It follows from Lemma 3.2 that

$$g(\mathrm{e}^{2\Delta}, u, t) \geqslant 1, \tag{3.3.13}$$

where $\Delta = \mathrm{dist}_h(z, \xi \cdot z) = \mathrm{dist}_h(z, \xi^{-1} \cdot z)$. The conclusion of the theorem follows from (3.3.10), (3.3.12) and (3.3.13). □

The second lemma is computational. To maintain the flow of the argument, we delay the proof until Appendix A.

LEMMA A1. *Let $g_x$, $g_u$, $g_t$ denote the partial derivatives of $g$ with respect to the first, second and third variables, respectively. The function $g_t$ is positive everywhere on $\mathscr{G}$. The functions $g_x$ and $g_u$ are positive at every point $(x, u, t) \in \mathscr{G}$ such that $u < \frac{1}{2}$ and $t > \frac{2}{3}u$.*

PROPOSITION 3.4. *Let $(\beta, \varepsilon)$ be any point in $\mathscr{D}$. Let $\xi$ and $\eta$ be two loxodromic isometries of $\mathbb{H}^3$ such that the group $\Gamma$ generated by $\xi$ and $\eta$ is discrete and free on the generators $\xi$ and $\eta$. Suppose that every $\Gamma$-invariant, positive, superharmonic function on $\mathbb{H}^3$ is constant. Let $z$ be a point on the common perpendicular to the axes of $\xi$ and $\eta$. Suppose that $\angle(\xi \cdot z, z, \eta \cdot z) < \beta$. Then we have*

$$\max\{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\} \geqslant \log 3 + \varepsilon.$$

*Proof.* Assume that

$$\max\{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\} < \log 3 + \varepsilon.$$

Set $\alpha = \angle(\xi \cdot z, z, \eta \cdot z)$. According to Lemma 3.3, there exist real numbers $\Delta$, $u_0$ and $t_0$, where $\Delta \in \{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\}$, $u_0 \in [E(\varepsilon), \frac{1}{2} - E(\varepsilon)]$ and $t_0 \in [0, u_0 - I(\alpha, \varepsilon)]$, such that $g(\mathrm{e}^{2\Delta}, t_0, u_0) \geqslant 1$. We write $x_0 = \mathrm{e}^{2\Delta}$.

Let us set $t_1 = u_0 - I(\beta, \varepsilon)$. Since $I$ is monotonically decreasing as a function of the first variable, we have $t_1 \geqslant u_0 - I(\alpha, \varepsilon) \geqslant t_0$. We also have $t_1 \leqslant u_0$ since $I(\beta, \varepsilon)$ is non-negative. It follows that the line segment $\sigma_1 = \{x_0\} \times [t_0, t_1] \times \{u_0\}$ is contained in $\mathcal{G}$. According to Lemma A1, the partial derivative $g_t$ is positive on $\mathcal{G}$ and hence

$$g(x_0, t_1, u_0) \geqslant g(x_0, t_0, u_0) \geqslant 1.$$

Now let us set $x_2 = 9e^{2\varepsilon}$, $u_2 = \frac{1}{2} - E(\varepsilon)$ and $t_2 = f(\beta, \varepsilon) = \frac{1}{2} - E(\varepsilon) - I(\beta, \varepsilon)$. Since $\Delta \in \{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\}$, our assumption implies that $\Delta < \log 3 + \varepsilon$ and hence that $x_0 < x_2$. Since $u_0$ was taken to be $\leqslant \frac{1}{2} - E(\varepsilon)$, we have $u_2 \geqslant u_0$ and $t_2 \geqslant t_1$.

Now consider the line segment $\sigma_2$ from $(x_0, u_0, t_1)$ to $(x_2, u_2, t_2)$. Since $u_2 - t_2 = I(\beta, \varepsilon) = u_0 - t_1$, we have $u - t = I(\beta, \varepsilon)$ for every $(x, u, t) \in \sigma_2$. Since $I(\beta, \varepsilon) \geqslant 0$ it follows that $\sigma_2 \subset \mathcal{G}$. On the other hand, by hypothesis we have $(\beta, \varepsilon) \in \mathcal{D}$; and applying condition (3) of the definition of $\mathcal{D}$ we find that for every $(x, u, t) \in \mathcal{D}$ we have

$$u - t = I(\beta, \varepsilon) < \tfrac{1}{3}E(\varepsilon) \leqslant \tfrac{1}{3}u_0 \leqslant \tfrac{1}{3}u$$

so that $t > \frac{2}{3}u$. Hence by Lemma A1, the partial derivatives $g_x$, $g_u$ and $g_t$ are all positive on $\sigma_2$. Since $x_2 \geqslant x_0$, $u_2 \geqslant u_0$ and $t_2 \geqslant t_1$, it follows that

$$g(x_2, u_2, t_2) \geqslant g(x_0, u_0, t_1) \geqslant 1.$$

However, according to condition (2) of the definition of $\mathcal{D}$, we have

$$g(x_2, u_2, t_2) = g(9e^{2\varepsilon}, \tfrac{1}{2} - E(\varepsilon), f(\beta, \varepsilon)) < 1.$$

This contradiction completes the proof. □

The proof of Theorem 3.1 requires one more proposition, the proof of which is deferred to Appendix C.

Proposition C1. *Let $\xi$ and $\eta$ be two loxodromic isometries of $\mathbb{H}^3$ without any common fixed point. Denote by $L$ the common perpendicular to the axes $A_\xi$ and $A_\eta$ of $\xi$ and $\eta$, respectively. Let $z_0$ be any point of $L$. Then there exist continuous one-parameter families $(\xi_t)_{0 \leqslant t \leqslant 1}$ and $(\eta_t)_{0 \leqslant t \leqslant 1}$ of loxodromic isometries of $\mathbb{H}^3$ with the following properties*:

(i) $\xi_0 = \xi$ *and* $\eta_0 = \eta$;
(ii) *for every $t$ the axes of $\xi_t$ and $\eta_t$ are perpendicular to $L$;*
(iii) *the functions* $t \mapsto \mathrm{dist}(z_0, \xi_t \cdot z_0)$ *and* $t \mapsto \mathrm{dist}(z_0, \eta_t \cdot z_0)$ *are monotonically decreasing on* $[0, 1]$;
(iv) *the function* $t \mapsto \angle(\xi_t \cdot z_0, z_0, \eta_t \cdot z_0)$ *is (weakly) monotonically decreasing on* $[0, 1\}$; *and*
(v) *the isometries $\xi_1$ and $\eta_1$ have the same axis.*

3.5. *Proof of Theorem* 3.1. We argue by contradiction. Assume that

$$\max\{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\} < \log 3 + \varepsilon.$$

Suppose first that $\Gamma$ is topologically tame and is not geometrically finite. Then it follows from [5, Theorem 7.2] that $\mathbb{H}^3$ admits no non-constant positive $\Gamma$-invariant superharmonic functions. The assertion of the theorem now follows from Proposition 3.4.

Now suppose that $\Gamma$ is geometrically finite. Let $L$ denote the common perpendicular to the axes $A_\xi$ and $A_\eta$ of $\xi$ and $\eta$, respectively. Let $(\xi_t)_{0 \leqslant t \leqslant 1}$ and $(\eta_t)_{0 \leqslant t \leqslant 1}$ be one-parameter families having the properties stated in Proposition C1. Let $V$ denote the complex affine variety $PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C})$ endowed with the classical topology, and consider the path $\rho:[0,1] \to V$ defined by $\rho(t) = (\xi_t, \eta_t)$. By property (i) of Proposition C1 and the hypotheses of the theorem, we have that $(\xi_0, \eta_0)$ is a point in the Schottky space $\mathscr{CC} \subset V$, i.e. the group $\Gamma = \langle \xi, \eta \rangle$ is a geometrically finite Kleinian group which is free of rank 2 and has no parabolics. By property (v) of C.1 we have that $\langle \xi_1, \eta_1 \rangle$ is not free of rank 2 and therefore $(\xi_1, \eta_1)$ does not lie in $\mathscr{CC}$. By [12], $\mathscr{CC}$ is an open subset of $V$. Set $t_0 = \inf\{t \in [0,1] | (\xi_t, \eta_t) \notin \mathscr{CC}\}$. It follows that $(\xi_{t_0}, \eta_{t_0})$ is in the frontier of $\mathscr{CC}$. By property (ii) of C.1 , the point $z$ is on the common perpendicular to the axes of $\xi_{t_0}$ and $\eta_{t_0}$. By the hypothesis of the theorem and property (iv) of Proposition C1 we have

$$\angle(\xi_{t_0} \cdot z, z, \eta_{t_0} \cdot z) < \beta.$$

Similarly, our assumption and property (iii) of Proposition C1 imply that

$$\max\{\mathrm{dist}_h(z, \xi_{t_0} \cdot z), \mathrm{dist}_h(z, \eta_{t_0} \cdot z)\} < \log 3 + \varepsilon.$$

By [6, Theorem 8.2] there exists a sequence $(\xi_i, \eta_i)$ in the frontier of $\mathscr{CC}$ in $V$ which converges to $(\xi_{t_0}, \eta_{t_0})$ , and such that for every $i$ the group $\Gamma_i \neq \langle \xi_i, \eta_i \rangle$ is purely loxodromic and free on the generators $\xi_i$ and $\eta_i$, and $\mathbb{H}^3$ admits no non-constant positive $\Gamma_i$-invariant superharmonic functions. For large enough $i$ we have

$$\angle(\xi_i \cdot z, z, \eta_i \cdot z) < \beta \tag{3.5.1}$$

and

$$\max\{\mathrm{dist}_h(z, \xi_i \cdot z), \mathrm{dist}_h(z, \eta_i \cdot z)\} < \log 3 + \varepsilon. \tag{3.5.2}$$

Denote by $L_i$ the common perpendicular to the axes of $\xi_i$ and $\eta_i$. Let $z_i$ denote the foot of the perpendicular from $z$ to $L_i$. The sequence of lines $(L_i)$ converges to the common perpendicular to the axes of $\xi_{t_0}$ and $\eta_{t_0}$, which is $L$ by property (ii) of Proposition C1. Since $z \in L$ we have $\mathrm{dist}_h(z, L_i) \to 0$, and hence $z_i \to z$. It follows from (3.5.1) and (3.5.2) that

$$\angle(\xi_i \cdot z_i, z_i, \eta_i \cdot z_i) < \beta \tag{3.5.3}$$

and

$$\max\{\mathrm{dist}_h(z_i, \xi_i \cdot z_i), \mathrm{dist}_h(z_i, \eta_i \cdot z_i)\} < \log 3 + \varepsilon \tag{3.5.4}$$

for large $i$. Since $(\beta, \varepsilon) \in \mathscr{D}$ and $z_i \in L_i$, and since $\mathbb{H}^3$ admits no non-constant positive $\Gamma_i$-invariant superharmonic functions, it follows from (3.5.3) and Proposition 3.4 that

$$\max\{\mathrm{dist}_h(z_i, \xi_i \cdot z_i), \mathrm{dist}_h(z_i, \eta_i \cdot z_i) \geqslant \log 3 + \varepsilon$$

for large $i$. This contradicts (3.5.4), and the proof of the theorem is thus complete. $\square$

## 4. ANGLES AND DISPLACEMENTS, II

In this section we complete the proof of Theorem 4.1, of which a special case, Corollary 4.9, was used in the proof of Theorem 1.1. Theorem 4.1 is an extension of the result of the previous section to the case where the point $z$ does not lie on the common perpendicular of the axes of $\xi$ and $\eta$. Since the point $z$ is not invariant under the involution that conjugates the generators to their inverses, we have two angles to consider.

We define a function $\phi : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$ by

$$\phi(l, t, s, \alpha) = \frac{2\cosh^2 s - 2\sinh^2 s \cos\alpha - \cosh l - \cos t}{\cosh l - \cos t}.$$

We denote by $\mathscr{V}$ the open subset of $\mathbb{R}^+ \times \mathbb{R}^3$ consisting of all points $(l, t, s, \alpha)$ such that $\phi(l, t, s, \alpha) > 1$. We define a function $\rho : \mathscr{V} \to \mathbb{R}$ by

$$\rho(l, t, s, \alpha) = \cosh^{-1}\phi(l, t, s, \alpha).$$

Recall that the region $\mathscr{D} \subset \mathbb{R}^2$ was defined in the discussion preceding Theorem 3.1.

The main result of the section is the following.

THEOREM 4.1. *Let $\lambda$ and $\theta$ be real numbers with $\lambda > 0$ and $0 \leqslant \theta \leqslant \pi$. Let $(\beta, \varepsilon)$ be a point of the region $\mathscr{D} \subset \mathbb{R}^2$. Let $\xi$ and $\eta$ be isometries which generate a subgroup of* $\mathrm{Isom}_+(\mathbb{H}^3)$ *which is discrete, free of rank 2, purely loxodromic and topologically tame. Suppose that $\xi^{-1}\eta$ has translation length $\geqslant \lambda$ and twist angle $\leqslant \theta$. Let $z$ be a point of $\mathbb{H}^3$ such that*

$$\max\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\} < \log 3 + \varepsilon.$$

*Set $\alpha_1 = \angle(\xi\cdot z, z, \eta\cdot z)$ and $\alpha_{-1} = \angle(\xi^{-1}\cdot z, z, \eta^{-1}\cdot z)$. Then the points $(\lambda, \theta, \log 3 + \varepsilon, \alpha_1)$ and $(\lambda, \theta, \log 3 + \varepsilon, \alpha_{-1})$ are contained in $\mathscr{V}$. If in addition we have $(\lambda, \theta, \log 3 - \varepsilon, \beta) \in \mathscr{V}$, then*

$$\rho(\lambda, \theta, \log 3 + \varepsilon, \alpha_1) + \rho(\lambda, \theta, \log 3 + \varepsilon, \alpha_{-1}) \geqslant 2\rho(\lambda, \theta, \log 3 - \varepsilon, \beta).$$

The proof of Theorem 4.1 requires several lemmas. We begin with two geometric lemmas. The first of these was used in the proof of [6, Proposition 5.2] but was inadvertently omitted from the published paper. The reference to "Proposition 1.15" in the latter paper should have been a reference to the lemma below.

LEMMA 4.2. *Let $\gamma$ be a loxodromic isometry of $\mathbb{H}^3$ with axis $A_\gamma$. Let $L$ be a line which meets $A_\gamma$ orthogonally at a point $w_0$. Suppose that $z$ is any point of $\mathbb{H}^3$ and that $z_0 \in L$ is the orthogonal projection of $z$ to the line $L$. Then*

$$\mathrm{dist}_h(z, \gamma\cdot z) \geqslant \mathrm{dist}_h(z_0, \gamma\cdot z_0).$$

*Proof.* Let $r : \mathbb{H}^3 \to L$ denote the map which sends each point of $\mathbb{H}^3$ to its orthogonal projection in the line $L$, i.e., to the nearest point of $L$. It follows from [11, Lemma 1.3.4] that $r$ is a distance decreasing retraction from $\mathbb{H}^3$ to $L$.

It follows from Lemma C2 that the displacement of a point under the loxodromic isometry $\gamma$ is a monotonically increasing function of its distance from $A_\gamma$. Thus it suffices to show that the distance from $z_0$ to $A_\gamma$ is not larger than the distance from $z$ to $A_\gamma$. Let $w$ denote the point of $A_\gamma$ which is nearest to $z$. Then we must show that $\mathrm{dist}_h(z, w) \geqslant \mathrm{dist}_h(z_0, w_0)$. But we have $z_0 = r(z)$ and $w_0 = r(w)$; this completes the proof since $r$ is distance decreasing. □

LEMMA 4.3. *Let $L$ be a line in $\mathbb{H}^3$ and let $R : \mathbb{H}^3 \to L$ be the function that assigns to each point of $\mathbb{H}^3$ its distance from the line $L$. Then $R$ is a convex function.*

*Proof.* Let $r : \mathbb{H}^3 \to L$ be the nearest point retraction. Let $z$ and $w$ be two points of $\mathbb{H}^3$ and let $m$ be the midpoint of the segment from $z$ to $w$ and let $m' \in L$ denote the midpoint of the segment from $r(z)$ to $r(w)$. We must show that $R(m) \leqslant \frac{1}{2}(R(z) + R(w))$. It follows from [10, Lemma 2] that

$$\mathrm{dist}_h(m', m)) \leqslant \tfrac{1}{2}(\mathrm{dist}_h(r(z), z) + \mathrm{dist}_h(r(w), w)) = \tfrac{1}{2}(R(z) + R(w)).$$

(The quoted result is stated in [10] only for $\mathbb{H}^2$, but the proof works equally well in three dimensions.) But $R(m) = \mathrm{dist}_h(r(m), m)) \leqslant \mathrm{dist}_h(m', m)$ by the definition of $r$. The lemma follows. □

LEMMA 4.4. *Let $l, t, s, \lambda, \theta, s', \alpha_1, \alpha_{-1}$ and $\alpha_0$ be real numbers with $l \geqslant \lambda > 0$, $s > s' > 0$ and $0 \leqslant t \leqslant \theta \leqslant \pi$. Suppose that $(l, t, s, \alpha_1)$, $(l, t, s, \alpha_{-1})$ and $(l, t, s', \alpha_0)$ lie in $\mathscr{V}$ and that*

$$\rho(l, t, s, \alpha_1) + \rho(l, t, s, \alpha_{-1}) \geqslant 2\rho(l, t, s', \alpha_0).$$

*Then the points $(\lambda, \theta, s, \alpha_1)$, $(\lambda, \theta, s, \alpha_{-1})$, and $(\lambda, \theta, s', \alpha_0)$ are contained in $\mathscr{V}$ and we have*

$$\rho(\lambda, \theta, s, \alpha_1) + \rho(\lambda, \theta, s, \alpha_{-1}) \geqslant 2\rho(\lambda, \theta, s', \alpha_0).$$

*Proof.* First note that the function $\phi$ is monotone decreasing in the variable $l$ for $l \geqslant 0$, and monotone increasing in the variable $t$ for $0 \leqslant t \leqslant \pi$. Since the set $\mathscr{V}$ is the region in which $\phi > 1$, and since $\lambda \leqslant l$ and $\theta \geqslant t$, it follows immediately that the points $(\lambda, \theta, s, \alpha_1)$, $(\lambda, \theta, s, \alpha_{-1})$, and $(\lambda, \theta, s', \alpha_0)$ lie in $\mathscr{V}$.

For the rest of the argument we introduce functions $\rho_i(x, y)$ for $i \in \{-1, 0, 1\}$ defined as follows:

$$\rho_1(x, y) = \cosh^{-1}\left(\frac{2\cosh^2 s - 2\sinh^2 s \cos\alpha_1 - \cosh x - \cos y}{\cosh x - \cos y}\right)$$

$$\rho_{-1}(x, y) = \cosh^{-1}\left(\frac{2\cosh^2 s - 2\sinh^2 s \cos\alpha_{-1} - \cosh x - \cos y}{\cosh x - \cos y}\right)$$

$$\rho_0(x, y) = \cosh^{-1}\left(\frac{2\cosh^2 s' - 2\sinh^2 s' \cos\alpha_0 - \cosh x - \cos y}{\cosh x - \cos y}\right).$$

By hypothesis we have that the inequality

$$\rho_1(x, y) + \rho_{-1}(x, y) - 2\rho_0(x, y) \geqslant 0$$

holds with $x = l$, $y = t$. We must show that the same inequality holds with $x = \lambda \leqslant l$, and $y = \theta \geqslant t$. We will consider a linear path $(x(s), y(s))$ for $0 \leqslant s \leqslant 1$ with $(x(0), y(0)) = (l, t)$ and $(x(1), y(1)) = (\lambda, \theta)$. We claim that the quantity

$$w(s) = \rho_1(x(s), y(s)) + \rho_{-1}(x(s), y(s)) - 2\rho_0(x(s), y(s))$$

is non-negative on the entire path. To establish the claim we will show that $w'(s) > 0$ whenever $w(s) = 0$. Since $w(0) \geqslant 0$, this implies $w(0) > 0$ or $w'(0) > 0$. Thus if $s_0$ is the first positive number for which $w(s_0) = 0$ then we must have $w'(s_0) \leqslant 0$. Since this is impossible we must have $w(s) > 0$ for all $s \in (0, 1]$.

Since $x(s)$ decreases monotonically while $y(s)$ increases monotonically we must show that the condition $\rho_1(x, y) + \rho_{-1}(x, y) - 2\rho_0(x, y) = 0$ implies the inequalities

$$\frac{\partial\rho_1}{\partial x} + \frac{\partial\rho_{-1}}{\partial x} - 2\frac{\partial\rho_0}{\partial x} < 0 \quad \text{and} \quad \frac{\partial\rho_1}{\partial y} + \frac{\partial\rho_{-1}}{\partial y} - 2\frac{\partial\rho_0}{\partial y} > 0.$$

Computing partial derivatives with respect to $x$ and $y$ we find

$$\frac{\partial \rho_i}{\partial x} = \frac{-\sinh x}{\cosh x - \cos y}\left(\frac{\cosh \rho_i + 1}{\sinh \rho_i}\right)$$

$$\frac{\partial \rho_i}{\partial x} = \frac{-\sin y}{\cosh x - \cos y}\left(\frac{\cosh \rho_i - 1}{\sinh \rho_i}\right).$$

If we set

$$p(z) = \frac{\cosh z + 1}{\sinh z} \quad \text{and} \quad q(z) = \frac{\cosh z - 1}{\sinh z}$$

then to complete the proof of the claim we must prove that $\rho_1 + \rho_{-1} = 2\rho_0$ implies $p(\rho_1) + p(\rho_{-1}) > 2p(\rho_0)$ and $q(\rho_1) + q(\rho_{-1}) < 2q(\rho_0)$. This is shown by computing second derivatives of $p$ and $q$:

$$p''(z) = \frac{(\cosh z + 1)^2}{\sinh^3 z} > 0 \quad \text{and} \quad q''(z) = \frac{-(\cosh z - 1)^2}{\sinh^3 z} < 0. \qquad \square$$

LEMMA 4.5. *Let $s$, $s_0$, $\lambda$ and $\theta$ be real numbers with $s, s_0, \lambda > 0$ and $0 \leqslant \theta \leqslant \pi$. Let $\xi$ and $\eta$ be loxodromic isometries of $\mathbb{H}^3$ having no common fixed point on the sphere at infinity. Suppose that $\xi^{-1}\eta$ is also loxodromic, and that it has translation length $\geqslant \lambda$ and twist angle $\leqslant \theta$. Let L denote the common perpendicular to the axes of $\xi$ and $\eta$. Let $z$ be any point of $\mathbb{H}^3$, and let $z_0$ denote the foot of the perpendicular from $z$ to L. Suppose that $\operatorname{dist}_h(z, \xi \cdot z) < s$, $\operatorname{dist}_h(z, \eta \cdot z) < s$ and $\operatorname{dist}_h(z_0, \xi \cdot z_0) > s_0$. Set $\alpha_1 = \angle(\xi \cdot z, z, \eta \cdot z)$, $\alpha_{-1} = \angle(\xi^{-1} \cdot z, z, \eta^{-1} \cdot z)$, and $\alpha_0 = \angle(\xi \cdot z_0, z_0, \eta \cdot z_0)$. Suppose that $\cos \alpha_i < (\tanh s_0)/(\tanh s)$ for $i = 1, 0, -1$. Then we have $(\lambda, \theta, s, \alpha_1), (\lambda, \theta, s, \alpha_{-1}) \in \mathscr{V}$. If in addition we have $(\lambda, \theta, s_0, \alpha_0) \in \mathscr{V}$, then*

$$\rho(\lambda, \theta, s, \alpha_1) + \rho(\lambda, \theta, s, \alpha_{-1}) \geqslant 2\rho(\lambda, \theta, s_0, \alpha_0).$$

*Proof.* According to Lemma 4.2 , we have

$$s_0 \leqslant \operatorname{dist}_h(z_0, \xi \cdot z_0) \leqslant \operatorname{dist}_h(z, \xi \cdot z) \leqslant s \tag{4.5.1}$$

$$s_0 \leqslant \operatorname{dist}_h(z_0, \eta \cdot z_0) \leqslant \operatorname{dist}_h(z, \eta \cdot z) \leqslant s. \tag{4.5.2}$$

In particular $s_0 \leqslant s$.

For $i = 1, 0, -1$ we consider the function

$$f_i(x, y) = \cosh x \cosh y - \sinh x \sinh y \cos \alpha_i.$$

We have

$$\frac{\partial}{\partial x} f_i(x, y) = \cosh x \cosh y(\tanh x - \tanh y \cos \alpha_i)$$

and

$$\frac{\partial}{\partial y} f_i(x, y) = \cosh x \cosh y(\tanh y - \tanh x \cos \alpha_i).$$

In view of the hypothesis, it follows that $f_i(x, y)$ is monotone both in $x$ and in $y$ for $x, y \in [s_0, s]$ and for $i = 1, 0, -1$. In particular we have $f_i(s_0, s_0) \leqslant f_i(x, y) \leqslant f_i(s, s)$ for $x, y \in [s_0, s]$; i.e.

$$\begin{aligned} 1 + (1 - \cos \alpha_i)\sinh^2 s_0 &\leqslant \cosh x \cosh y - \sinh x \sinh y \cos \alpha_i \\ &\leqslant 1 + (1 - \cos \alpha_i)\sinh^2 s \end{aligned} \tag{4.5.3}$$

for $i = 1, 0, -1$ and for all $x, y \in [s_0, s]$. Now define $\tau \in \mathrm{Isom}_+(\mathbb{H}^3)$ to be the rotation about $L$ through an angle $\pi$. We have $\tau\xi\tau = \xi^{-1}$ and $\tau\eta\tau = \eta^{-1}$. Hence

$$\alpha_{-1} = \angle(\tau\xi\tau\cdot z, z, \tau\eta\tau\cdot z) = \angle(\xi\tau\cdot z, \tau\cdot z, \eta\tau\cdot z).$$

If we set $z_1 = z$ and $z_{-1} = \tau\cdot z$, it follows from the equation above and the definitions of $\alpha_0$ and $\alpha_1$ that

$$\angle(\xi\cdot z_i, z_i, \eta\cdot z_i) = \alpha_i \tag{4.5.4}$$

for $i = 1, 0, -1$.

We also have $\mathrm{dist}_h(z_{-1}, \xi\cdot z_{-1}) = \mathrm{dist}_h(\tau\cdot z, \xi\tau\cdot z) = \mathrm{dist}_h(z, \tau\xi\tau\cdot z) = \mathrm{dist}_h(z, \xi^{-1}\cdot z) = \mathrm{dist}_h(z_1, \xi\cdot z_1)$. Combining this with (4.5.1), and setting $X_i = \mathrm{dist}_h(z_i, \xi\cdot z_i)$, we get

$$s_0 \leqslant X_i \leqslant s$$

for $i = 1, 0, -1$. Similarly, setting $Y_i = \mathrm{dist}_h(z_i, \eta\cdot z_i)$ and using (4.5.2), we find that

$$s_0 \leqslant Y_i \leqslant s$$

for $i = 1, 0, -1$.

We set $D_i = \mathrm{dist}_h(\xi\cdot z_i, \eta\cdot z_i)$ for $i = 1, 0, -1$, and apply the first hyperbolic law of cosines to the hyperbolic triangle with vertices $z_i$, $\xi\cdot z_i$ and $\eta\cdot z_i$; by (4.5.4), the angle at the vertex $z_i$ is $\alpha_i$. This gives

$$\cosh D_i = \cosh X_i \cosh Y_i - \sinh X_i \sinh Y_i \cos\alpha_i. \tag{4.5.5}$$

Since $X_i, Y_i \in [s_0, s]$, we can combine (4.5.3) and (4.5.5) to conclude that

$$1 + (1 - \cos\alpha_i)\sinh^2 s_0 \leqslant \cosh D_i \leqslant 1 + (1 - \cos\alpha_i)\sinh^2 s \tag{4.5.6}$$

for $i = 1, 0, -1$.

It follows from the definition of $D_i$ that

$$D_i = \mathrm{dist}_h(z_i, \xi^{-1}\eta\cdot z_i).$$

Let $l$ and $t$ denote the translation length and twist angle of $\xi^{-1}\eta$. By hypothesis we have $0 \leqslant \lambda \leqslant l$ and $0 \leqslant t \leqslant \theta \leqslant \pi$. If we define $R_i$ to be the orthogonal distance from $z_i$ to the axis of $\xi^{-1}\eta$, then it follows from the formula for the displacement of point under a loxodromic isometry (see Lemma C2) that

$$\sinh^2 R_i = \frac{\cosh D_i - \cosh l}{\cosh l - \cos t}.$$

Combining this with (4.5.6) we obtain

$$\frac{1 + (1 - \cos\alpha_i)\sinh^2 s_0 - \cosh l}{\cosh l - \cos t} \leqslant \sinh^2 R_i \leqslant \frac{1 + (1 - \cos\alpha_i)\sinh^2 s - \cosh l}{\cosh l - \cos t}.$$

Using the identity $\cosh 2R = 1 + 2\sinh^2 R$ and the definition of $\phi$, we obtain

$$\phi(l, t, s, \alpha_i) \leqslant \cosh 2R_i \leqslant \phi(l, t, s, \alpha_i). \tag{4.5.7}$$

Since $\phi$ is monotone decreasing with respect to the first variable and monotone increasing with respect to the second variable, we have

$$\phi(\lambda, \theta, s, \alpha_i) \geqslant \phi(l, t, s, \alpha_i) \geqslant \cosh 2R_i \geqslant 1.$$

It follows that $(l, t, s, \alpha_i)$ and $(\lambda, \theta, s, \alpha_i)$ lie in $\mathscr{V}$ for $i = 1, 0, -1$. This includes the first assertion of the lemma.

Since $(l, t, s, \alpha_i) \in \mathscr{V}$, we may conclude from (4.5.7) and the monotonicity of the hyperbolic cosine that

$$\rho(l, t, s, \alpha_i) \geqslant 2R_i \tag{4.5.8}$$

for $i = 1, 0, -1$.

It also follows from (4.5.7) that $\cosh 2R_0 = \phi(l, t, s', \alpha_0)$ for some $s' \in [s_0, s]$. This implies that $(l, t, s', \alpha_0) \in \mathscr{V}$ and that

$$\rho(l, t, s', \alpha_0) = 2R_0. \tag{4.5.9}$$

Now let $R: \mathbb{H}^3 \to \mathbb{R}$ denote the function that assigns to each point of $\mathbb{H}^3$ its minimum hyperbolic distance from the axis of $\xi^{-1}\eta$; thus $R(z_i) = R_i$ for $i = 1, 0, -1$. According to Lemma 4.3, $R$ is a convex function on $\mathbb{H}^3$. Since $z_0$ is the midpoint of the segment joining $z_1$ to $z_{-1}$, we have

$$R_1 + R_{-1} \geqslant 2R_0. \tag{4.5.10}$$

Combining (4.5.8), (4.5.9) and (4.5.10), we find that

$$\rho(l, t, s, \alpha_1) + \rho(l, t, s, \alpha_{-1}) \geqslant 2R_1 + 2R_{-1} \geqslant 4R_0 = 2\rho(l, t, s', \alpha_0).$$

Since we have $s' \leqslant s$, $\lambda \leqslant l$ and $t \leqslant \theta \leqslant \pi$, it follows from Lemma 4.4 that $(\lambda, \theta, s, \alpha_1)$, $(\lambda, \theta, s, \alpha_{-1})$, and $(\lambda, \theta, s', \alpha_0)$ lie in $\mathscr{V}$ and that

$$\rho(\lambda, \theta, s, \alpha_1) + \rho(\lambda, \theta, s, \alpha_{-1}) \geqslant 2\rho(\lambda, \theta, s', \alpha_0). \tag{4.5.11}$$

Let us now assume that $(\lambda, \theta, s_0, \alpha_0) \in \mathscr{V}$. Then since $s_0 \leqslant s'$, and since $\phi$ is monotonically increasing with respect to the third variable and the hyperbolic cosine is monotonically increasing on $(0, \infty)$, we have $\rho(\lambda, \theta, s_0, \alpha_0) \leqslant \rho(\lambda, \theta, s', \alpha_0)$. Combining this with (4.5.11) we conclude that

$$\rho(\lambda, \theta, s, \alpha_1) + \rho(\lambda, \theta, s, \alpha_{-1}) \geqslant 2\rho(\lambda, \theta, s_0, \alpha_0). \qquad \square$$

LEMMA 4.6. *Let $\varepsilon$ be a given positive number. Let $\xi$ and $\eta$ be two loxodromic isometries of $\mathbb{H}^3$. Suppose that the group $\Gamma$ generated by $\xi$ and $\eta$ is discrete, free on the generators $\xi$ and $\eta$, and topologically tame. Let $z$ be any point of $\mathbb{H}^3$. Suppose that*

$$\max\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\} < \log 3 + \varepsilon.$$

*Then we have*

$$\min\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\} > \log 3 - \varepsilon.$$

*Proof.* In [1, Theorem 6.1(a)] it was shown that under the hypotheses of the lemma we have

$$\frac{1}{1 + e^{\mathrm{dist}_h(z, \xi\cdot z)}} + \frac{1}{1 + e^{\mathrm{dist}_h(z, \eta\cdot z)}} \leqslant \frac{1}{2}$$

for every $z \in \mathbb{H}^3$. If the conclusion of the lemma is false then one of the two quantities $\mathrm{dist}_h(z, \xi\cdot z)$ and $\mathrm{dist}_h(z, \eta\cdot z)$ is less than $\log 3 - \varepsilon$ while the other is greater than $\log 3 + \varepsilon$.

Hence

$$\frac{1}{1+3e^{\varepsilon}}+\frac{1}{1+3e^{-\varepsilon}} \leqslant \frac{1}{2}. \tag{4.6.1}$$

But the left-hand side of (4.6.1) can be written in the form

$$\frac{1+3\cosh\varepsilon}{5+3\cosh\varepsilon},$$

which is monotonically increasing function of $\varepsilon$ and takes the value $\frac{1}{2}$ at 0. This contradicts (4.6.1) since the given value of $\varepsilon$ is strictly positive. □

LEMMA 4.7. *Let $\varepsilon$ be a positive number less than* $\log 3$. *Let $\xi$ and $\eta$ be isometries which generate a subgroup of* $\mathrm{Isom}_+(\mathbb{H}^3)$ *which is discrete, free of rank* 2, *purely loxodromic and topologically tame. Let $z$ be a point of $\mathbb{H}^3$ such that*

$$\max\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\} < \log 3 + \varepsilon.$$

*Then we have*

$$\cos\angle(\xi\cdot z, z, \eta\cdot z) < \frac{\cosh^2(\log 3+\varepsilon)-\cosh(\log 3-\varepsilon)}{\sinh^2(\log 3-\varepsilon)}.$$

*Proof.* Since $\max\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\} < \log 3 + \varepsilon$ it follows from Lemma 4.6 that $\mathrm{dist}_h(z, \eta\cdot z) > \log 3 - \varepsilon)$ and $\mathrm{dist}_h(z, \xi\cdot z) > \log 3 - \varepsilon)$. On the other hand, since $\xi$ and $\xi^{-1}\eta$ generate the same group as $\xi$ and $\eta$, we may apply Lemma 4.6 with $\xi^{-1}\eta$ in place of $\eta$ to conclude that $\mathrm{dist}_h(z, \xi\eta^{-1}\cdot z) > \log 3 - \varepsilon$. Now consider the triangle with vertices $z$, $\xi\cdot z$ and $\eta\cdot z$. The sides adjacent to the vertex $z$ have lengths $X = \mathrm{dist}_h(z, \xi\cdot z)$ and $Y = \mathrm{dist}_h(z, \eta\cdot z)$ . The third side has length $D = \mathrm{dist}_h(\xi\cdot z, \eta\cdot z) = \mathrm{dist}_h(z, \xi^{-1}\eta\cdot z)$. Since $X$ and $Y$ lie in $(\log 3 - \varepsilon, \log 3 + \varepsilon)$ and since $D > \log 3 - \varepsilon$, the first hyperbolic law of cosines gives

$$\cos\alpha = \frac{\cosh X\cosh Y - \cosh D}{\sinh X\sinh Y} \leqslant \frac{\cosh^2(\log 3+\varepsilon)-\cosh(\log 3-\varepsilon)}{\sinh^2(\log 3-\varepsilon)}. \qquad \square$$

4.8. *Proof of Theorem* 4.1. Let $L$ denote the common perpendicular to the axes of $\xi$ and $\eta$. Let $z_0$ denote the foot of the perpendicular from $z$ to $L$. Let us set $\alpha_0 = \angle(\xi\cdot z_0, z_0, \eta\cdot z_0)$. We must have $\alpha_0 \geqslant \beta$; for if $\alpha_0$ were $< \beta$, then since $(\beta, \varepsilon) \in \mathscr{D}$, it would follow from Theorem 3.1 that $\max\{\mathrm{dist}_h(z, \xi\cdot z), \mathrm{dist}_h(z, \eta\cdot z)\} \geqslant \log 3 + \varepsilon$, a contradiction to the hypothesis of Theorem 4.1.

We shall apply Lemma 4.5, taking $s = \log 3 + \varepsilon$ and $s_0 = \log 3 - \varepsilon$. By hypothesis we have $\mathrm{dist}_h(z, \xi\cdot z) < \log 3 + \varepsilon$. According to Lemma 4.2 we have

$$\mathrm{dist}_h(z_0, \xi\cdot z_0) \leqslant \mathrm{dist}_h(z, \xi\cdot z) < \log 3 + \varepsilon.$$

Hence by Lemma 4.6 we have $\mathrm{dist}_h(z_0, \eta\cdot z_0) > \log 3 - \varepsilon$.

In order to apply Lemma 4.5 we must still check that for $i = 1, 0, -1$ we have $\cos\alpha_i < (\tanh(\log 3 - \varepsilon))/(\tanh(\log 3 + \varepsilon))$. To this end we observe that by Lemma 4.7 we have $\alpha_1 = \cos\angle(\xi\cdot z, z, \eta\cdot z) < a$, where

$$a = \frac{\cosh^2(\log 3+\varepsilon)-\cosh(\log 3-\varepsilon)}{\sinh^2(\log 3-\varepsilon)}.$$

Applying Lemma 4.7 with $\xi^{-1}$ and $\eta^{-1}$ in place of $\xi$ and $\eta$ gives $\alpha_{-1} < a$; and the same lemma, with $z_0$ in place of $z$, gives $\alpha_0 < a$. Now recall that from the definition of $\mathscr{D}$, we have $\varepsilon < \varepsilon_\infty$. Hence $a < a_\infty$, where

$$a_\infty = \frac{\cosh^2(\log 3 + \varepsilon_\infty) - \cosh(\log 3 - \varepsilon_\infty)}{\sinh^2(\log 3 - \varepsilon_\infty)} = 0.80060 \ldots .$$

Thus we have $\alpha_i < a$ for $i = 1, 0, -1$. But

$$\frac{(\tanh(\log 3 - \varepsilon))}{(\tanh(\log 3 + \varepsilon)} > \frac{(\tanh(\log 3 - \varepsilon_\infty))}{(\tanh(\log 3 + \varepsilon_\infty)} = 0.95591 \ldots > a.$$

Hence we indeed have $\cos \alpha_i < (\tanh(\log 3 - \varepsilon))/(\tanh(\log 3 + \varepsilon)$ for $i = 1, 0, -1$.

It now follows from Lemma 4.5 that $(\lambda, \theta, \log 3 + \varepsilon, \alpha_1)$ and $\rho(\lambda, \theta, \log 3 + \varepsilon, \alpha_{-1})$ lie in $\mathscr{V}$. Now suppose that $(\lambda, \theta, \log 3 - \varepsilon, \beta) \in \mathscr{V}$. Since $\alpha_0 \geqslant \beta$, and since the function $\phi$ is monotone increasing in the fourth variable, it follows that $(\lambda, \theta, \log 3 - \varepsilon, \alpha_0) \in \mathscr{V}$ and that

$$\rho(\lambda, \theta, \log 3 - \varepsilon, \alpha_0) \geqslant \rho(\lambda, \theta, \log 3 - \varepsilon, \beta).$$

Since $(\lambda, \theta, \log 3 - \varepsilon, \alpha_0) \in \mathscr{V}$, Lemma 4.5 gives

$$\rho(\lambda, \theta, \log 3 + \varepsilon, \alpha_1) + \rho(\lambda, \theta, \log 3 + \varepsilon, \alpha_{-1}) \geqslant 2\rho(\lambda, \theta, \log 3 - \varepsilon, \alpha_0).$$

Combining the two inequalities above, we conclude that

$$\rho(\lambda, \theta, \log 3 + \varepsilon, \alpha_1) + \rho(\lambda, \theta, \log 3 + \varepsilon, \alpha_{-1}) \geqslant 2\rho(\lambda, \theta, \log 3 - \varepsilon, \beta),$$

and the theorem is proved. □

We now specialize Theorem 4.1 to the case which was needed to prove Theorem 1.1. Recall, from Section 1, that $\varepsilon_0 = 0.0065$, and that

$$\{\alpha \in [0, \pi] \mid \rho(\lambda_0, \pi, \log 3 + \varepsilon_0, \alpha) \geqslant 1\} = [\alpha_{-\infty}, \pi].$$

By comparing the definition of $\sigma$ in Section 1 with the definition of $\rho$ we find that

$$\sigma(\alpha) = \rho(\lambda_0, \pi, \log 3 + \varepsilon_0, \alpha)$$

for $\alpha \in (\alpha_{-\infty}, \pi]$. Also recall from Section 1 that the constant

$$K = 1.30822$$

is defined so that $\rho(\lambda_0, \pi, \log 3 - \varepsilon_0, \beta_0) = \cosh^{-1}\phi(\lambda_0, \pi, \log 3 - \varepsilon_0, \beta_0) > K$. Moreover, the point $(\beta_0, \varepsilon_0)$ lies in the region $\mathscr{D}$ since we have:

(1) $f(\beta_0, \varepsilon_0) = 0.23139 \ldots > 0$;
(2) $g(9e^{2\varepsilon_0}, \frac{1}{2} - E(\varepsilon_0), f(\beta_0, \varepsilon_0)) = 0.98623 \ldots < 1$; and
(3) $I(\beta_0, \varepsilon_0) = 0.03964 \ldots < 0.08292 \ldots = \frac{1}{3}E(\varepsilon_0)$.

Here we have computed $I(\beta_0, \varepsilon_0) = (1/8\pi)\iota(\beta_0, \varepsilon_0)$ using Proposition B1. Thus by specializing Theorem 4.1 we obtain:

COROLLARY 4.9. *Let $\xi$ and $\eta$ be isometries which generate a subgroup of* $\mathrm{Isom}_+(\mathbb{H}^3)$ *which is discrete, free of rank* 2, *purely loxodromic and topologically tame. Suppose that $\xi^{-1}\eta$ has translation length $\geqslant \lambda_0$. Let $z$ be a point of $\mathbb{H}^3$ such that*

$$\max\{\mathrm{dist}_h(z, \xi \cdot z), \mathrm{dist}_h(z, \eta \cdot z)\} < \log 3 + \varepsilon_0.$$

*Set* $\alpha_1 = \angle(\xi\cdot z, z, \eta\cdot z)$ *and* $\alpha_{-1} = \angle(\xi^{-1}\cdot z, z, \eta^{-1}\cdot z)$. *Then* $\alpha_1$ *and* $\alpha_{-1}$ *lie in the interval* $[\alpha_{-\infty}, \pi]$ *and*

$$\sigma(\alpha_1) + \sigma(\alpha_{-1}) > 2K.$$

We conclude this section by proving another result that was used in the proof of Theorem 1.1. It is similar in flavor to the above results, although independent of them. Recall that we define the function $\omega(\lambda, \Delta)$ for $\Delta > \lambda > 0$ by

$$\omega(\lambda, \Delta) = \cos^{-1}\left(1 - \frac{2(\cosh\lambda - 1)}{\cosh\Delta - 1}\right).$$

PROPOSITION 4.10. *Let* $\lambda$ *and* $\Delta$ *be positive real numbers. Let* $\xi$ *be a loxodromic isometry of* $\mathbb{H}^3$ *with translation length* $\geqslant\lambda$, *and let* $z$ *be a point of* $\mathbb{H}^3$ *such that* $\operatorname{dist}_h(z, \xi\cdot z) < \Delta$. *Then we have* $\angle(\xi^{-1}\cdot z, z, \xi\cdot z) > \omega(\lambda, \Delta)$.

*Proof.* We let $l$ and $\theta$ denote the translation length and twist angle of $\xi$, so that $\xi^2$ has length $2l$ and twist angle $2\theta$. We set $D = \operatorname{dist}_h(z, \xi\cdot z)$ and $D' = \operatorname{dist}_h(z, \xi^2\cdot z)$, and we denote by $R$ the distance from $z$ to the axis of $\xi$ (which is also the axis of $\xi^2$). In Appendix C we derive a formula for the displacement of a point of $\mathbb{H}^3$ under the action of a loxodromic isometry. According to Lemma C2 we have

$$\sinh^2 R = \frac{\cosh D - \cosh l}{\cosh l - \cos\theta}$$

and likewise

$$\sinh^2 R = \frac{\cosh D' - \cosh 2l}{\cosh 2l - \cos 2\theta}.$$

Hence

$$\cosh D' = \frac{\cosh D - \cosh l}{\cosh l - \cos\theta}(\cosh 2l - \cos 2\theta) + \cosh 2l. \tag{4.10.1}$$

Now consider the triangle with vertices $z$, $\xi\cdot z$ and $\xi^{-1}\cdot z$. The sides adjoining the vertex $z$ have length $D$, and the third side has length $\operatorname{dist}_h(\xi^{-1}\cdot z, \xi\cdot z) = \operatorname{dist}_h(z, \xi^2\cdot z) = D'$. Setting $w = \angle(\xi^{-1}\cdot z, z, \xi\cdot z)$ and applying the first hyperbolic law of cosines we find that

$$\cosh D' = \cosh^2 D - \sinh^2 D\cos w = 1 + \sinh^2 D(1 - \cos w).$$

Combining this with (4.10.1) we obtain

$$\begin{aligned}
1 - \cos w &= \frac{1}{\sinh^2 D}\left(\frac{\cosh D - \cosh l}{\cosh l - \cos\theta}(\cosh 2l - \cos 2\theta) + \cosh 2l - 1\right)\\
&= \frac{2}{\sinh^2 D}((\cosh l + \cos\theta)(\cosh D - \cosh l) + \cosh^2 l - 1)\\
&\geqslant \frac{2}{\sinh^2 D}((\cosh l - 1)(\cosh D - \cosh l) + \cosh^2 l - 1)\\
&= \frac{2}{\sinh^2 D}(\cosh D + 1)(\cosh l - 1) = \frac{2(\cosh l - 1)}{\cosh D - 1}.
\end{aligned}$$

Since $0 < D < \Delta$ and $l \geqslant \lambda$, it follows that

$$1 - \cos w > \frac{2(\cosh \lambda - 1)}{\cosh \Delta - 1},$$

which is equivalent to the conclusion of the proposition. $\square$

## 5. GEOGRAPHY

This section is devoted to the proof of the proposition about configurations of eight points on the sphere which was used in the proof of Theorem 1.1. Recall that the constants $\delta_0 = 0.71497\pi$, $K = 1.30822$ and $\alpha_{-\infty} = 0.80060\ldots$ were defined in Section 1. Recall also that the function $\sigma$ has domain $[\alpha_{-\infty}, \pi]$. By inspection of the definition of $\sigma$ in Section 1 one sees that for every $x > \alpha_{-\infty}$ we have

$$\sigma(x) = \cosh^{-1}(A - B\cos(x)) \tag{5.0.2}$$

where the constants $A$ and $B$ are defined by

$$A = \frac{2\cosh^2(\log 3 + \varepsilon_0) - \cosh \lambda_0 + 1}{\cosh \lambda_0 + 1} = 1.98717\ldots$$

$$B = \frac{2\sinh^2(\log 3 + \varepsilon_0)}{\cosh \lambda_0 + 1} = 1.41781\ldots.$$

The main result of this section, which was quoted in Section 1, is the following.

PROPOSITION 5.1. *Suppose that we are given an indexed family*

$$(P_{(i,u)})_{(i,u)\in\{0,1,2,3\}\times\{-1,1\}}$$

*of points in $S^2$. Assume that for any two indices $(i, u)$ and $(j, v)$ in $\{0, 1, 2, 3\} \times \{-1, 1\}$ with $i \neq j$, we have $\operatorname{dist}_s(P_{(i,u)}, P_{(j,v)}) > \alpha_{-\infty}$. Then either*

(i) *there is an element $i$ of $\{0, 1, 2, 3\}$ such that $\operatorname{dist}_s(P_{(i,1)}, P_{(i,-1)}) \leqslant \delta_0$, or*
(ii) *there exist indices $(i, u)$ and $(j, v)$ in $\{0, 1, 2, 3\} \times \{-1, 1\}$, with $i \neq j$, such that*

$$\sigma(\operatorname{dist}_s(P_{(i,u)}, P_{(j,v)})) + \sigma(\operatorname{dist}_s(P_{(i,-u)}, P_{(j,-v)})) \leqslant 2K.$$

We will need a formula for the distance between two points on the unit sphere. We use the conventions established in the introduction. If $P$ and $P'$ are points of $S^2$, the spherical distance $\operatorname{dist}_s(P, P')$ is equal to the angle between the (unit) position vectors of $P$ and $P'$ in $\mathbb{R}^3$. If $P, P' \notin \{N, S\}$, so that $\Theta(P, P')$ is defined, we find by writing down the inner product and using the identity $\cos(\theta - \theta') = \cos \Theta(P, P')$ that

$$\cos \operatorname{dist}_s(P, P') = \cos \lambda(P) \cos \lambda(P') \cos \Theta(P, P') + \sin \lambda(P) \sin \lambda(P').$$

Proposition 5.1 is deduced by a combinatorial argument from three estimates which are stated below as Lemmas 5.2, 5.7 and 5.8.

LEMMA 5.2. *Let $P$, $P'$, $Q$ and $Q'$ be points of $S^2$. Suppose that*

(a) *$Q$ and $Q'$ lie on a common meridian and $0 < \lambda(Q) = -\lambda(Q')$,*
(b) *$\operatorname{dist}_s(P, P') > \delta_0$ and $\operatorname{dist}_s(Q, Q') = 2\lambda(Q) > \delta_0$,*

(c) $\mathrm{dist}_s(P, Q)$, $\mathrm{dist}_s(P', Q')$, $\mathrm{dist}_s(P', Q))$ *and* $\mathrm{dist}_s(P, Q')$ *are all* $\geqslant \alpha_{-\infty}$,
(c′) $\sigma(\mathrm{dist}_s(P, Q)) + \sigma(\mathrm{dist}_s(P', Q')) > 2K$, *and*
(c″) $\sigma(\mathrm{dist}_s(P', Q)) + \sigma(\mathrm{dist}_s(P, Q')) > 2K$.

*Then* $|\lambda(P)|$ *and* $|\lambda(P')|$ *are less than* $\pi/6$.

Lemma 5.2 will follow from the next three lemmas.

LEMMA 5.3. *Let* $a$ *and* $b$ *be positive real numbers. If* $|a - b| < 1$ *then the function* $\cosh^{-1}(a - b\cos x)$ *has negative second derivative at each point in the interior of its domain.*

*Proof.* We may assume that $a + b > 1$ since otherwise the domain has empty interior. One checks that the second derivative is given by

$$\frac{-q(\cos x)}{((a - b\cos x)^2 - 1)^{3/2}}$$

where $q$ is the quadratic polynomial function given by

$$q(t) = ab^2t^2 - (a^2b + b^3 - b)t + ab^2.$$

Thus it suffices to check that the polynomial $q$ is everywhere positive. Clearly $q(0) > 0$, and the discriminant of $q$ is

$$b^2(1 - (a + b)^2)(1 - (a - b)^2),$$

which is negative since $a + b > 1$ and $|a - b| < 1$. □

We let $A'$ denote the function given by $A'(x, y) = A - 2B\sin x \sin y$.

LEMMA 5.4. *Let* $P$, $Q$, *and* $Q'$ *be points on* $S^2$. *Assume that* $Q$ *and* $Q'$ *lie on the same meridian and that* $0 \leqslant \lambda(Q) = -\lambda(Q')$. *Assume also that* $\mathrm{dist}_s(P, Q) > \alpha_{-\infty}$. *Then*

$$\sigma(\mathrm{dist}_s(P, Q)) = \cosh^{-1}(A'(\lambda(P), \lambda(Q)) - B\cos \mathrm{dist}_s(P, Q')).$$

*Proof.* This follows directly from (5.0.2) and the identity

$$\cos(\mathrm{dist}_s(P, Q)) = \cos(\mathrm{dist}_s(P, Q')) + 2\sin\lambda(P)\sin\lambda(Q).$$

The identity is an immediate consequence of the spherical distance formula, given that $\lambda(Q') = -\lambda(Q)$. □

LEMMA 5.5. *Let* $P$, $Q$, *and* $Q'$ *be points on* $S^2$. *Assume that* $Q$ and $Q'$ *lie on the same meridian and that* $\delta_0/2 \leqslant \lambda(Q) = -\lambda(Q')$. *Assume also that* $\mathrm{dist}_s(P, Q) \geqslant \alpha_{-\infty}$ *and that* $\lambda(P) \geqslant \pi/6$. *Then* $2\pi - \delta_0 - \mathrm{dist}_s(P, Q') \geqslant \alpha_{-\infty}$ *and*

$$\sigma(\mathrm{dist}_s(P, Q)) + \sigma(2\pi - \delta_0 - \mathrm{dist}_s(P, Q')) < 2K.$$

*Proof.* Our hypothesis on the latitudes of $Q$ and $Q'$ implies that $\mathrm{dist}_s(Q, Q') > \delta_0$. Since the perimeter of a spherical triangle is at most $2\pi$ we have

$$2\pi - \delta_0 - \mathrm{dist}_s(P, Q') \geqslant 2\pi - \mathrm{dist}_s(Q, Q') - \mathrm{dist}_s(P, Q') \geqslant \mathrm{dist}_s(P, Q) \geqslant \alpha_{-\infty}.$$

In particular, the left-hand side of the inequality in the conclusion is defined.

It follows from the spherical distance formula that if the configuration of the three points $P$, $Q$ and $Q'$ is modified by moving these points along their meridians toward the equator while preserving the condition $\lambda(Q) = -\lambda(Q')$, then $\mathrm{dist}_s(P, Q)$ increases while $\mathrm{dist}_s(P, Q')$ decreases. Hence it suffices to consider the case where $\lambda(P) = \pi/6$ and $\lambda(Q) = \delta_0/2$. Then the expression $\sigma(\mathrm{dist}_s(P, Q)) + \sigma(2\pi - \delta_0 - \mathrm{dist}_s(P, Q'))$ is defined and, by (5.0.2) and Lemma 5.4, is equal to

$$\cosh^{-1}(A'(\lambda(P), \lambda(Q)) - B\cos \mathrm{dist}_s(P, Q')) + \cosh^{-1}(A - B\cos(2\pi - \delta_0 - \mathrm{dist}_s(P, Q'))).$$

We will use $A'$ to denote the quantity $A'(\pi/6, \delta_0/2) = 0.70911\ldots$. It follows from Lemma 5.4 that the left-hand side of the inequality in the conclusion of the lemma is given by $f(\mathrm{dist}_s(P, Q'))$, where

$$f(x) = \cosh^{-1}(A' - B\cos(x)) + \cosh^{-1}(A - B\cos(2\pi - \delta_0 - x)).$$

Thus it suffices to show that $f(x) < 2K$ for all $x$ in the intersection of the domain of $f$ with the interval $[0, \pi]$. (This intersection is easily seen to be an interval, which we will denote $J$.) Note that $f''(x) < 0$ for all $x$ in $J$, since Lemma 5.3 implies that each of the summands of $f$ has negative second derivative on the interior of its domain.

To complete the proof we give a numerical estimate of the maximum value of $f$ on $J$. To make the estimate we will exhibit two points $x_1$ and $x_2$ with $x_1 < x_2$ such that $f'(x_1) > 0$, and $f'(x_2) < 0$. By the concavity of $f$ we then know that $f(x) < f(x_1) + f'(x_1)(x_2 - x_1)$ for all $x$ in $J$.

We set

$$x_1 = 2.3; \qquad x_2 = 2.4.$$

We then have

$$f(x_1) = 2.52535\ldots; \qquad f'(x_1) = 0.09792\ldots; \qquad f'(x_2) = 0.11091\ldots;$$

$$f(x_1) + f'(x_1)(x_2 - x_1) = 2.53541\ldots \leqslant 2K = 2.61644.$$

This completes the proof of the lemma. □

5.6. *Proof of Lemma* 5.2. Because of the symmetry of the hypotheses, it suffices to show that $|\lambda(P)| < \pi/6$. Assume to the contrary that $|\lambda(P)| \geqslant \pi/6$. Then in view of hypotheses (a)–(c), we may apply Lemma 5.5 to deduce that $2\pi - \delta_0 - \mathrm{dist}_s(P, Q') \geqslant \alpha_{-\infty}$ and that

$$\sigma(\mathrm{dist}_s(P, Q)) + \sigma(2\pi - \delta_0 - \mathrm{dist}_s(P, Q')) < 2K.$$

On the other hand, using hypothesis (b) and the fact that a spherical triangle has perimeter at most $2\pi$, we find that

$$2\pi - \delta_0 - \mathrm{dist}_s(P, Q') > 2\pi - \mathrm{dist}(P, P') - \mathrm{dist}_s(P, Q') \geqslant \mathrm{dist}(P', Q').$$

Hence

$$\sigma(\mathrm{dist}_s(P, Q)) + \sigma(\mathrm{dist}_s(P', Q')) < 2K.$$

But this contradicts hypothesis (c′). □

We need two more lemmas for the proof of Proposition 5.1.

LEMMA 5.7. *Let $0 < \Delta < \pi$ and $0 < \Lambda < \pi/2$ be constants. Suppose that $R$ and $R'$ are points of $S^2$ such that $|\lambda(R)|$ and $|\lambda(R')|$ are less than $\Lambda$, and* $\operatorname{dist}_s(R, R') \geqslant \Delta$. *Then*

$$\cos \Theta(R, R') < \frac{\cos(\Delta) + \sin^2 \Lambda}{\cos^2 \Lambda}.$$

*In particular, if $|\lambda(R)|$ and $|\lambda(R')|$ are less than $\pi/6$ and* $\operatorname{dist}_s(R, R') > \delta_0$ *then* $\Theta(R, R') > 2\pi/3$.

*Proof.* We have

$$\cos \Theta(R, R') = \frac{\cos \operatorname{dist}_s(R, R') - \sin \lambda(R) \sin \lambda(R')}{\cos \lambda(R) \cos \lambda(R')} \leqslant \frac{\cos \Delta - \sin \lambda(R) \sin \lambda(R')}{\cos \lambda(R) \cos \lambda(R')}.$$

We consider the function

$$f(x, y) = \frac{\cos \Delta - \sin x \sin y}{\cos x \cos y} = \cos \Delta \sec x \sec y - \tan x \tan y.$$

One checks by elementary calculus that $f$ is differentiable in a neighborhood of the rectangle $[-\Lambda, \Lambda] \times [-\Lambda, \Lambda]$, that $f$ has only one critical point in the interior of the rectangle, which is a saddle located at the origin, and that the maxima occur at the two points $\pm(-\Lambda, \Lambda)$. This gives the first conclusion.

Direct computation shows that

$$\frac{\cos \delta_0 + \sin^2 \pi/6}{\cos^2 \pi/6} = -0.50022\ldots < -\tfrac{1}{2}.$$

The second conclusion follows. □

LEMMA 5.8. *Let $P$, $P'$, $Q$ and $Q'$ be points of $S^2 \setminus \{N, P\}$. Suppose that both* $\operatorname{dist}_s(P, Q)$ *and* $\operatorname{dist}_s(P'Q')$ *are greater than $\alpha_{-\infty}$, and that*

$$\sigma(\operatorname{dist}_s(P, Q)) + \sigma(\operatorname{dist}_s(P', Q')) > 2K.$$

*Suppose further that $|\lambda(P)|$, $|\lambda(Q)|$, $|\lambda(P')|$ and $|\lambda(Q')|$ are all less than $\pi/6$. Then*

$$\Theta(P, Q) + \Theta(P', Q') > \tfrac{2}{3}\pi.$$

*Proof.* Applying Lemma 5.7 with $P$ and $Q$ playing the roles of $P$ and $P'$ in the latter result, and taking $\Lambda = \pi/6$ and $\Delta = \operatorname{dist}_s(P, Q)$, we obtain

$$\cos \operatorname{dist}_s(P, Q) \geqslant \tfrac{3}{4}\cos \Theta(P, Q) - \tfrac{1}{4}.$$

Similarly,

$$\cos \operatorname{dist}_s(P', Q') \geqslant \tfrac{3}{4} \cos \Theta(P', Q') - \tfrac{1}{4}.$$

Recall that $\sigma(x) = \cosh^{-1}(A - B \cos x)$. Since $B > 0$ we have that $\sigma$ is monotone increasing on the intersection of its domain with the interval $[0, \pi]$. Thus by the inequalities above,

$$\sigma(\operatorname{dist}_s(P, Q)) \leqslant \cosh^{-1}((A + \tfrac{1}{4}B) - \tfrac{3}{4}B \cos \Theta(P, Q))$$

and

$$\sigma(\operatorname{dist}_s(P', Q')) \leqslant \cosh^{-1}((A + \tfrac{1}{4} B) - \tfrac{3}{4} B \cos \Theta(P', Q')).$$

We define a function on the interval $[0, \pi]$ by $f(x) = \cosh^{-1}((A + \frac{1}{4}B) - \frac{3}{4}B\cos x)$. Note that $A - \frac{1}{2}B > 1$, so this definition makes sense. The inequalities above show that

$$f(\Theta(P, Q)) + f(\Theta(P', Q')) \geqslant \sigma(\mathrm{dist}_s(P, Q)) + \sigma(\mathrm{dist}_s(P', Q')) \geqslant 2K. \tag{5.8.1}$$

The function $f$ is monotonically increasing, and we have $f(0) = 0.72972\ldots$ and $f(\pi) = 1.89609\ldots$ while $2K - f(0) = 1.88671\ldots$ and $2K - f(\frac{2\pi}{3}) = 0.89958\ldots$. It follows that $f^{-1}(2K - f(x))$ is defined for all $x$ in $[0, \frac{2\pi}{3}]$, so we may define a monotonically increasing function $g$ on the interval $[0, \frac{2\pi}{3}]$ by $g(x) = \frac{2\pi}{3} - f^{-1}(2K - f(x))$.

We now assume, contrary to the conclusion of the lemma, that

$$\Theta(P, Q) + \Theta(P', Q') \leqslant \tfrac{2\pi}{3}.$$

By symmetry we may also assume that $\Theta(P, Q) \leqslant \frac{\pi}{3}$.

From the inequality (5.8.1) and the monotonicity of $f$ we conclude that

$$2K - f(\Theta(P, Q)) \leqslant f(\Theta(P', Q')) \leqslant f(\tfrac{2\pi}{3} - \Theta(P, Q).$$

Since $2K - f(\Theta(P, Q))$ is in the domain of the increasing function $f^{-1}$ we have

$$f^{-1}(2K - f(\Theta(P, Q))) \leqslant \tfrac{2\pi}{3} - \Theta(P, Q)$$

and hence that

$$g(\Theta(P, Q)) = \tfrac{2\pi}{3} - f^{-1}(2K - f(\Theta(P, Q))) \geqslant \Theta(P, Q).$$

Since the function $g$ is increasing and

$$g(\tfrac{2\pi}{3}) = 1.54878\ldots \in [\tfrac{\pi}{3}, \tfrac{2\pi}{3}] \subset [\Theta(P, Q), \tfrac{2\pi}{3}],$$

we have shown (under our assumption) that $g$ maps the interval $[\Theta(P, Q), \frac{2\pi}{3}]$ into itself. In particular for every positive integer $k$ the $k$th iterate $g^k(\pi/3)$ is positive. But this leads to a contradiction because direct computation shows that $g^5(\frac{\pi}{3}) = -0.30973\ldots < 0$. □

5.9. *Proof of Proposition* 5.1. Suppose that we are given an indexed family

$$(P_{(i,u)})_{(i,u) \in \{0,1,2,3\} \times \{-1,1\}}$$

of points in $S^2$ for which conclusions (i) and (ii) of 5.1 both fail to hold. Thus the $(P_{(i,u)})$ satisfy the following conditions:

(-i) for every $i \in \{0, 1, 2, 3\}$ we have $\mathrm{dist}_s(P_{(i,1)}, P_{(i,-1)})) > \delta_0$; and
(-ii) for every two indices $(i, u)$ and $(j, v)$ in $\{0, 1, 2, 3\} \times \{-1, 1\}$ such that $i \neq j$, we have

$$\sigma(\mathrm{dist}_s(P_{(i,u)}, P_{(j,v)})) + \sigma(\mathrm{dist}_s(P_{(i,-u)}, P_{(j,-v)})) > 2K.$$

We shall derive a contradiction.

After a rotation of the sphere we may assume that $P_{(0,1)}$ and $P_{(0,-1)}$ lie on a common meridian, that $\lambda(P_{(0,1)}) \geqslant 0$ and that $\lambda(P_{(0,-1)}) = -\lambda(P_{(0,-1)})$. Since conditions (-i) and (-ii) are open, we may assume after perturbing the remaining $(P_{(i,u)})$ that none of the six points $P_{(1,\pm 1)}, P_{(2,\pm 1)}, P_{(3,\pm 1)}$ has latitude 0 and that no two of these six points lie on a common meridian. In particular these six points are distinct.

Let us consider any index $i \in \{1, 2, 3\}$. We wish to apply Lemma 5.2, taking $P = P_{(i,1)}$, $P' = P_{(i,-1)}$, $Q = P_{(0,1)}$ and $Q' = P_{(0,-1)}$. We have arranged that $P_{(0,1)}$ and $P_{(0,-1)}$ lie on a common meridian, that $\lambda(P_{(0,1)}) \geqslant 0$ and that $\lambda(P_{(0,-1)}) = -\lambda(P_{(0,-1)})$. Thus hypothesis (a) of 5.2 holds. Hypothesis (b) follows from condition (-i) above, while hypothesis (c) follows directly from the hypotheses of the proposition. Hypothesis (c′) of 5.2 is simply condition (-ii) above with the given $i \in \{1, 2, 3\}$ and with $u = 1, j = 0$ and $v = 1$. Likewise, hypothesis (c″) is condition (-ii) with the given $i$ and with $u = 1, j = 0$ and $v = -1$. Thus, for each $(i, u) \in \{1, 2, 3\} \times \{-1, 1\}$, we have $|\lambda(P_{i,u})| < \pi/6$.

Now consider any two indices $(i, u), (j, v) \in \{1, 2, 3\} \times \{-1, 1\}$ such that $i \neq j$. We shall apply Lemma 5.8 taking $P = P_{(i,u)}$, $Q = P_{(j,v)}$, $P' = P_{(i,-u)}$, $Q' = P_{(j,-v)}$. According to (-ii) we have

$$\sigma(\mathrm{dist}_s(P_{(i,u)}, P_{(j,v)}) + \sigma(\mathrm{dist}_s(P_{(i,-u)}, P_{j,-v)}) > 2K$$

$$\sigma(\mathrm{dist}_s(P_{(i,u}, P_{(j,-v)}) + \sigma(\mathrm{dist}_s(P_{(i,-u)}, P_{(j,v)}) > 2K.$$

On the other hand, we have that $\lambda(P_{(i,u)})$, $\lambda(P_{(j,v)})$, $\lambda(P_{(i,-u)})$, and $\lambda(P_{(j,-v)})$ are all less than $\pi/6$. Thus the hypotheses of 5.8 all hold, and it follows that

$$\Theta(P_{(i,u)}, P_{(j,v)}) + \Theta(P_{(i,-u)}, P_{(j,-v)}) > \tfrac{2\pi}{3}. \tag{5.9.1}$$

For each $(i, u) \in \{1, 2, 3\} \times \{-1, 1\}$, the hypothesis of Lemma 5.7 holds if we set $P = P_{(i,u)}$ and $Q = Q_{(i,-u)}$. Hence we have

$$\Theta(P_{(i,u)}, P_{(i,-u)}) > \tfrac{2\pi}{3}. \tag{5.92}$$

Recall that $\ell$ is the retraction with meridian fibers from $S^2\backslash\{N, S\}$ to the equator. We set $\zeta_{(i,u)} = \ell(P_{(i,u)})$ for each $(i, u) \in \{1, 2, 3\} \times \{-1, 1\}$. The six points $\zeta_{(1,\pm 1)}, \zeta_{(2,\pm 1)}, \zeta_{(3,\pm 1)}$ are distinct because no two of the six points $P_{(1,\pm 1)}, P_{(2,\pm 1)}, P_{(3,\pm 1)}$ lie on a common meridian. The six-element set $Z = \{\zeta_{(i,u)} : (i, u) \in \{1, 2, 3\} \times \{-1, 1\}\}$ has two natural fixed-point-free involutions. The first, which we denote $\tau_1$, is defined by $\tau_1(\zeta_{(i,u)}) = \zeta_{(i,-u)}$. To define the second involution, which we denote by $\tau_2$, we observe that since $Z$ is a six-element subset of $S^1$, there exists for each $\zeta \in Z$ a unique $\zeta' \in Z$ such that each component of $S^1\backslash\{\zeta, \zeta'\}$ contains exactly two elements of $Z\backslash\{\zeta, \zeta'\}$. We set $\tau_2(\zeta) = \zeta'$. We consider two cases.

*Case* 1. *The involutions $\tau_1$ and $\tau_2$ coincide.* In this case we may assume, after relabeling some of the $P_{(i,u)}$ if necessary, that as one circumnavigates $S^1$ in the counterclockwise sense beginning with $\zeta_{(1,1)}$, the $\zeta_{i,u}$ appear in the order

$$\zeta_{(1,1)}, \zeta_{(2,1)}, \zeta_{(3,1)}, \zeta_{(1,-1)}, \zeta_{(2,-1)}, \zeta_{(3,-1)}.$$

(The relabeling may involve permuting the indices $i = 1, 2, 3$ and, for certain values of $i$, interchanging the labels of $P_{(i,-1)}$ and $P_{(i,1)}$.)

To unify the notation we set $P_{(4,u)} = P_{(1,-u)}$ and $\zeta_{4,u} = \zeta_{(1,-u)}$ for $u = \pm 1$. The points of the set $Z$ divide $S^1$ into six arcs $A_{(1,\pm 1)}$ , $A_{(2,\pm 1)}, A_{(3,\pm 1)}$, where $A_{(i,u)}$ has endpoints $\zeta_{(i,u)}$ and $\zeta_{(i+1,u)}$ for $i = 1, 2, 3$. The length of the arc $A_{(i,u)}$ is greater than or equal to the circular distance between its endpoints, which by definition is $\Theta(P_{(i,u)}, P_{(i+1,u)})$. Hence

$$\begin{aligned} 2\pi &= \sum_{i=1}^{3} \text{length } A_{(i,1)} + \sum_{i=1}^{3} \text{length } A_{(i,-1)} \\ &\geqslant \sum_{i=1}^{3} \Theta(P_{(i,1)}, P_{(i+1,1)}) + \sum_{i=1}^{3} \Theta(P_{(i,-1)}, P_{(i+1,-1)}) \\ &= \sum_{i=1}^{3} (\Theta(P_{(i,1)}, P_{(i+1,1)}) + \Theta(P_{(i,-1)}, P_{(i+1,-1)}). \end{aligned}$$

But it follows from (5.9.1) that

$$\Theta(P_{(i,1)}, P_{(i+1,1)}) + \Theta(P_{(i,-1)}, P_{(i+1,-1)}) > \tfrac{2\pi}{3}$$

for $i = 1, 2, 3$. Thus we have a contradiction in this case.

*Case* 2. *The involutions* $\tau_1$ *and* $\tau_2$ *are distinct.* In this case we may assume, after relabeling the $P_{(i,u)}$ if necessary, that $\tau_1(\zeta_{(1,1)}) = (\zeta_{(1,-1)})$ is distinct from $\tau_2(\zeta_{(1,1)})$; that is, some component $A$ of $S^1 \setminus \{\zeta_{(1,1)}, \zeta_{(1,-1)}\}$ contains at least three points of $Z \setminus \{\zeta_{(1,1)}, \zeta_{(1,-1)}\}$. Hence for some $p \in \{2, 3\}$, the arc $A$ contains $\zeta_{(p,1)}$ and $\zeta_{(p,-1)}$. After further relabeling the $P_{(i,u)}$ if necessary we may assume that $A$ contains $\zeta_{(2,1)}$ and $\zeta_{(2,-1)}$. Then the four element set $Z' = \{\zeta_{(i,u)} \colon i \in \{1, 2\}, u \in \{-1, 1\}\}$ divides $S^1$ into four arcs $B_1, B_2, C_1, C_{-1}$, where $B_i$ has endpoints $P_{(i,1)}$ and $P_{(i,-1)}$, and $C_u$ has endpoints $P_{(1,u)}$ and $P_{(2,u)}$.

Reasoning as in Case 1, we find that

$$\begin{aligned} 2\pi &= \text{length } B_1 + \text{length } B_2 + \text{length } C_1 + \text{length } C_{-1} \\ &\geqslant \Theta(P_{(1,1)}, P_{(1,-1)}) + \Theta(P_{(2,1)}, P_{(2,-1)}) + \Theta(P_{(1,1)}, P_{(2,1)}) + \Theta(P_{(1,-1)}, P_{(2,-1)}). \end{aligned}$$

But we have

$$\Theta(P_{(1,1)}, P_{(2,1)}) + \Theta(P_{(1,-1)}, P_{(2,-1)}) > \tfrac{2\pi}{3}$$

by (5.9.1), and by (5.9.2) each of the terms $\Theta(P_{(1,1)}, P_{(1,-1)})$ and $\Theta(P_{(2,1)}, P_{(2,-1)})$ is $> \frac{2\pi}{3}$.

Thus we have a contradiction in this case as well. □

*Acknowledgements*—We thank John Smillie for suggesting the idea that led to the proof of Theorem 3.1. The first and third authors were partially supported by a grant from the National Science Foundation.

## REFERENCES

1. Anderson, J. W., Canary, R., Culler, M. and Shalen, P. B., Free kleinian groups and volumes of hyperbolic 3-manifolds. *Journal of Differential Geometry* (to appear).
2. Beardon, A. F., The geometry of discrete groups. Springer, Berlin, 1983.
3. Benedetti, R. and Petronio, C., *Lectures on Hyperbolic Geometry*. Springer, Berlin, 1992.
4. Böröczky, K., Packing of spheres in spaces of constant curvature. *Acta Mathematica Academy of Science Hungary*, 1978, **32**, 243–261.
5. Canary, R. D., Ends of hyperbolic 3-manifolds. *Journal of the American Mathematical Soceity*, 1993, **6**, 1–35.
6. Culler, M. and Shalen, P. B., Paradoxical decompositions, 2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds. *Journal of the American Mathematical Soceity*, 1992, **5**, 231–288.
7. Culler, M. and Shalen, P. B., Hyperbolic volume and mod *p* homology. *Commentarii Mathematici Helvetici*, 1993, **68**, 494–509.
8. Culler, M. and Shalen, P. B., Volumes of hyperbolic Haken manifolds, I. *Inventionnes Mathematicae*, 1994, **118**, 285–329.
9. Culler, M. and Shalen, P. B., The volume of a hyperbolic 3-manifold with Betti number 2. *Proceedings of the American Mathematical Society*, 1994, **120**, 1281–1288.
10. Douady, A., L'espace de Teichmüller. In *Travaux de Thurston, Astérisque*, Vol. 66–67, pp. 127–137. Société Mathématique de France, 1984.
11. Epstein, D. B. A. and Marden, A., Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces. In: *Analytical and Geometrical Aspects of Hyperbolic Spaces*, ed. D. B. A. Epstein, LMS Lecture Note Series, Vol. 111. Cambridge University Press, Cambridge, 1987, pp. 113–253.
12. Marden, A., The geometry of finitely generated Kleinian groups. *Annals of Mathematics*, (2) 1974, **99**, 383–462.
13. Meyerhoff, R., Sphere-packing and volume in hyperbolic 3-space. *Commentarii Mathematici Helvetici*, 1986, **61**, 271–278.

14. O'Neill, B., *Elementary Differential Geometry*. Academic Press, New York, 1966.
15. Weeks, J., Hyperbolic structures on three-manifolds, Thesis, Princeton University, 1985.

*Department of Mathematics*
*University of Illinois*
*Chicago, IL 60607*
*U.S.A.*

*Department of Mathematics*
*Cornell University*
*Ithaca, NY 14853*
*U.S.A.*

*Department of Mathematics*
*University of Illinois*
*Chicago, IL 60607*
*U.S.A.*

## APPENDIX A. MONOTONICITY OF $g$

In this appendix we prove the following monotonicity statement, which was needed in the proof of Theorem 3.1. The definition of the function $g$ and the domain $\mathscr{G}$ appear immediately before the statement of Theorem 3.1.

LEMMA A1. *Let $g_x$, $g_u$, $g_t$ denote the partial derivatives of $g$ with respect to the first, second and third variables, respectively. The function $g_t$ is positive everywhere on $\mathscr{G}$. The functions $g_x$ and $g_u$ are positive at every point $(x, u, t) \in \mathscr{G}$ such that $u < \frac{1}{2}$ and $t > \frac{2}{3}u$.*

*Proof.* On the set $\mathscr{G}$ we define functions $A = A(x, t)$, $B = B(x, u)$ and $C = C(x, u, t)$ by $A = 1 + t(x - 1)$, $B = 1 + u(x - 1)$ and $C = 1 + (2u - t)(x - 1)$. Since $x > 1$ and $0 < t < u$ for $(x, u, t) \in \mathscr{G}$, we have $0 < A < B < C$ on $\mathscr{G}$. The definition of $g$ may be rewritten in the form

$$g(x, u, t) = \frac{tx}{A} + \frac{(u - t)x}{BC} + u.$$

Differentiating with respect to $t$, we find that

$$g_t(x, u, t) = \frac{x}{A^2} - \frac{x}{B^2}$$

and the right-hand side is positive on $\mathscr{G}$ since $0 < A < B$. This proves the first assertion.

Retaining the assumptions $(x, u, t) \in \mathscr{G}$, $u < \frac{1}{2}$ and $t > \frac{2}{3}u$, we now differentiate with respect to $u$. We obtain

$$g_u(x, u, t) = 1 + \frac{x}{B^2C^2}(BC - (u - t)(x - 1)(2B + C)).$$

In order to show that $g_u(x, u, t) > 0$, it certainly suffices to show that

$$BC - (u - t)(x - 1)(2B + C) > 0.$$

Again using that $0 < B < C$, that $x > 1$ and that $t < u$, we find that

$$BC - (u-t)\,(x-1)(2B + C) > C(B - 3(u-t)(x-1)).$$

Now, again using the definition of $B$ and the inequalities $x > 1$ and $\frac{3}{2}t < u$, we find that

$$B - 3(u-t)(x-1) = 1 + (x-1)(3t - 2u) > 0.$$

Thus $g_u(x, u, t)$ is indeed positive.

Differentiating with respect to $x$ we find that $g_x$ is a rational function in $x$, $u$ and $t$ with denominator $(ABC)^2$. We introduce a new variable $v = t/u$ and define

$$h(x, u, v) = (A(x, u, uv)B(x, u, uv)C(x, u, uv))^2 g_x(x, u, uv).$$

It suffices to show that $h(x, u, v) > 0$ for all $x > 0$ and for all $(u, v)$ in the rectangle $R = \{(u, v) | 0 < u < \frac{1}{2}, \frac{2}{3} < v < 1\}$.

We find, with the assistance of a symbolic computation program, that

$$h(x, u, v) = \sum_{i=0}^{4} p_i(u, v) x^i$$

where

$$p_0 = u(1-u)(vu + 1 - 2u)(1 - vu)(1 - (2v - v^2)(2u - u^2))$$

$$p_1 = 4vu^2(2-v)(u-1)^2(1-vu)(vu + 1 - 2u)$$

$$p_2 = u^3((-6v^4 + 24v^3 - 24v^2)u^3 + (6v^4 - 24v^3 + 12v^2 + 24v)u^2$$
$$+ (-2v^4 + 8v^3 + 8v^2 - 32v)u + (-8v^2 + 16v - 2))$$

$$p_3 = 4vu^4(2-v)(1-vu)(vu + 1 - 2u)$$

$$p_4 = vu^5(2-v)(2 - (2v - v^2)(1+u)).$$

Using the inequalities $2v - v^2 < 1$ and $2u - u^2 < \frac{3}{4}$ one checks easily that $p_0$, $p_1$, $p_3$ and $p_4$ are positive for $(u, v) \in R$. It remains to show that $p_2$ is positive for $(u, v) \in R$. For this it suffices to show that

$$q(u, v) = (-6v^4 + 24v^3 - 24v^2)u^3 + (6v^4 - 24v^3 + 12v^2 + 24v)u^2$$
$$+ (-2v^4 + 8v^3 + 8v^2 - 32v)u + (-8v^2 + 16v - 2)$$

is positive. We will show that $q_u(u, v) < 0$ and $q_v(u, v) > 0$ for $(u, v) \in R$ and thus that $q(u, v) > q(\frac{1}{2}, \frac{2}{3}) = \frac{19}{324}$ for $(u, v) \in R$.

Differentiating, we find that

$$q_u(u, v) = -2v(2-v)(2 + v - 3vu)(3vu + (4 - v - 6u)).$$

Thus by inspection $q_u < 0$ for $(u, v) \in R$.

Finally we have

$$q_v = (-3v^3 + 9v^2 - 16v)u^3 + (3v^2 - 9v^2 + 3v + 24)u^2$$
$$+ (-v^3 + 3v^2 + 2v - 2)u + (2 - 2v).$$

One checks that the coefficient of $u^3$ is negative for $\frac{2}{3} < v < 1$ and hence, since $u^3 < u^2$, we have

$$q_v(u, v) \geqslant (24 - 13v)u^2 + (3v^2 - v^3)u + (2 - 2v)(1 - u).$$

The right hand side of this inequality is positive for $v \leqslant 1$ and $0 < u < \frac{1}{2}$. This completes the proof of the lemma. □

## APPENDIX B. INTERSECTIONS OF SPHERICAL CAPS

In this appendix we use the Gauss–Bonnet theorem to derive a formula for the area of the intersection of two spherical caps. This formula was needed in the proof of Corollary 4.9. Recall that $\imath(\alpha, r_1, r_2)$ denotes the area of the intersection of two spherical caps $C(P_1, r_1)$ and $C(P_2, r_2)$, where $P_1$ and $P_2$ are two points of $S^2$ such that $\mathrm{dist}_s(P_1, P_2) = \alpha$.

We shall denote by $\mathscr{U}$ the region in $\mathbb{R}^3$ consisting of all points $(x_1, x_2, x_3)$ such that $0 < x_i < \pi$ for $i = 1, 2, 3$ and such that $x_1 + x_2 > x_3$, $x_2 + x_3 > x_1$ and $x_3 + x_1 > x_2$. For any $(x_1, x_2, x_3) \in \mathscr{U}$ we have

$$-1 < \frac{\cos x_3 - \cos x_1 \cos x_2}{\sin x_1 \sin x_2} < 1.$$

Hence we may define a function $J: \mathscr{U} \to (0, \pi)$ by

$$J(x_1, x_2, x_3) = \arccos\left(\frac{\cos x_3 - \cos x_1 \cos x_2}{\sin x_1 \sin x_2}\right).$$

Note that the region $\mathscr{U}$ is by definition invariant under permutations of the coordinates of $\mathbb{R}^3$; hence for any point $(x_1, x_2, x_3) \in \mathscr{U}$ and any permutation $\pi \in \mathscr{S}_3$, we have $(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \in \mathscr{U}$, so that $J(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ is defined.

We shall prove:

PROPOSITION B1. *Let $\alpha, r_1, r_2$ be numbers in the interval $(0, \pi)$. If $r_1 + r_2 \leqslant \alpha$ then $\imath(\alpha, r_1, r_2) = 0$. If $r_1 + \alpha \leqslant r_2$ or $r_2 + \alpha \leqslant r_1$ then $\imath(\alpha, r_1, r_2) = |\cos r_1 - \cos r_2|$. Finally, if $(r_1, r_2, \alpha) \in \mathscr{U}$ then*

$$\imath(r_1, r_2, \alpha) = 2\pi - 2(J(r_1, \alpha, r_2)\cos r_1 + J(r_2, \alpha, r_1)\cos r_2 + J(r_1, r_2, \alpha)).$$

Suppose that $P_1, P_2$ are points in $S^2$ and that $r_1, r_2$ are numbers in $(0, \pi)$. Let us set $C_i = C(P_i, r_i)$ and $\alpha = \mathrm{dist}_s(P_1, P_2)$. It is clear that $\mathrm{Int}\, C_2 \cap \mathrm{Int}\, C_1 = \emptyset$ if and only if $r_1 + r_2 \leqslant \alpha$. Furthermore, we have $C_1 \subset C_2$ if and only if $r_1 + \alpha \leqslant r_2$; in this case, since $C_i$ has area $2\pi(1 - \cos r)$, we have $\imath(r_1, r_2, \alpha) = 2\pi(\cos r_2 - \cos r_1)$. Similarly, we have $C_2 \subset C_1$ if and only if $r_2 + \alpha \leqslant r_1$, and in this case $\imath(r_1, r_2, \alpha) = 2\pi(\cos r_2 - \cos r_1)$ and likewise $C_2 \subset C_1$ if and only if $r_2 + \alpha \leqslant r_1$. This establishes the first two assertions of Proposition B1, and also shows that the boundary circles of $C_1$ and $C_2$ cross if and only if $(r_1, r_2, \alpha) \in \mathscr{U}$. In this case it is clear that $A_1 = \partial C_1 \cap C_2$ and $A_2 = \partial C_2 \cap C_1$ are arcs and that $C_1 \cap C_2$ is a topological disk whose boundary $A_1 \cup A_2$ consists of two smooth arcs. We shall refer to such a configuration as a *digon*.

The rest of the section is devoted to the proof of the final assertion of Proposition B1. The proof depends on Lemmas B2 and B3 below.

At certain points in the proof it will be necessary to keep track of orientations. We shall always give $S^2$ the orientation induced from the restriction to the unit ball of the standard orientation of $\mathbb{R}^3$. When we consider a spherical cap $C(P, r)$ we shall always give it the orientation obtained by restricting the orientation of $S^2$. This orientation of $C(P, r)$ induces an orientation of $\partial C(P, r)$.

LEMMA B2. *Let $P_1$, $P_2$ be points in $S^2$, let $r_1$ and $r_2$ be numbers, set $\alpha = \mathrm{dist}_s(P_1, P_2)$, and suppose that $(r_1, r_2, \alpha) \in \mathcal{U}$. Then length of the arc $\partial C(P_1, r_1) \cap C(P_2, r_2)$ is equal to $2J(r_1, \alpha, r_2) \sin r_1$. Furthermore, the exterior angles of the digon $C(P_1, r_1)$ are both equal to $J(r_1, r_2, \alpha)$.*

*Proof.* Set $C_i = C(P_i, r_i)$. Let $Q$ and $Q'$ denote the endpoints of $A$. It is clear that the exterior angles of $C_1 \cap C_2$ at $Q$ and $Q'$ have the same value $\varepsilon$. Let $p_i$, $q$ and $q'$ denote the position vectors of $P_i$ and $Q$ in $\mathbb{R}^3$. We have

$$\cos r_2 = \cos \mathrm{dist}_s(Q, P_2) = q \cdot P_i \tag{B.2.1}$$

where $\cdot$ denotes the scalar product.

Let $\partial C_i$ be given the orientation induced from the orientation of $C_i$, and let $v_i$ denote the unit tangent vector to $C_i$ at $Q$ which is positive with respect to this orientation. We have

$$v_1 \cdot v_2 = \cos \varepsilon. \tag{B.2.2}$$

Note that $q$ is orthogonal to $S^2$ and hence to the $v_i$, and that $p_i$ is orthogonal to $C_i$ and hence to $v_i$. The angle between $p_i$ and $q$ is $\mathrm{dist}_s(P_i, Q) = r_i$. In view of our orientation conventions, it follows that

$$v_i = (\csc r_i) p_i \times q \tag{B.2.3}$$

where $\times$ denotes the vector product.

After a rotation of the sphere, we may assume that $P_1$ is the north pole and that $P_2$ has longitude 0 and polar angle $\alpha$. Thus $p_1 = (0, 0, 1)$ and $p_2 = (\sin\alpha, 0, \cos\alpha)$. The endpoints of the arc $A = \partial C_1 \cap C_2$ have polar angle $r_1$, and their longitudes are opposite in sign and equal in absolute value. Thus we have

$$q = (\sin r_1 \cos\theta, \sin r_1 \sin\theta, \cos r_1),$$

$$Q' = (\sin r_1 \cos\theta, \; -\sin r_1 \sin\theta, \cos r_1).$$

After a reflection of the sphere we may assume that $\theta > 0$. From (B.2.1) we obtain

$$\cos r_2 = (\sin r_1 \cos\theta, \sin r_1 \sin\theta, \cos r_1) \cdot (\sin\alpha, 0, \cos\alpha)$$

which implies that

$$\theta = J(r_1, \alpha, r_2). \tag{B.2.4}$$

Since the circle $\partial C(P_1, r_1)$ has Euclidean radius $\sin r_1$, and since the arc $A$ subtends an angle of $2\theta = 2J(r_1, \alpha, r_2)$ in this circle, the length of $A$ is $2J(r_1, \alpha, r_2) \sin r_1$. This is the first assertion of the lemma. From (B.2.3) we have

$$v_1 = \frac{1}{\sin r_1}(a_2, \; -a_1, 0)$$

$$v_2 = \frac{1}{\sin r_2}(a_2 \cos\alpha, \; -a_1 \cos\alpha + a_3 \sin\alpha, \; -a_2 \sin\alpha).$$

Computing the scalar product of the $v_i$ from these expressions and substituting the value of $\theta$ given by (B.2.4) we find that

$$v_1 \cdot v_2 = \frac{\cos\alpha - \cos r_1 \cos r_2}{\sin r_1 \sin r_2}$$

which by (B.2.2) implies that $\varepsilon = J(r_1, r_2, \alpha)$. □

The next lemma gives the geodesic curvature of the boundary of a spherical cap. Recall that geodesic curvature is a signed quantity and that its sign depends on both an orientation of the curve and an orientation of the ambient surface. We have already fixed an orientation of $S^2$.

LEMMA B3. *Let $P$ be any point $P \in S^2$ and let any $r$ be any number in the interval $(0, \pi)$. Let us regard the circle $\partial C(P, r)$, as an oriented curve in $S^2$ with the orientation induced from that of $C(P, r)$. Then $\partial C(P, r)$ has constant geodesic curvature* $\cot r$.

*Proof.* Since the stabilizer of $C = C(P, r)$ in the group of orientation-preserving isometries of $S^2$ acts transitively on the points of $\partial C$, the geodesic curvature of $\partial C$ is a constant $\kappa$. Since $\partial C$ is a Euclidean circle of radius $\sin r$, its length is $2\pi \sin r$. On the other hand, $C(P, r)$ has area $2\pi(1 - \cos r)$ and constant Gaussian curvature $K = 1$. According to the Gauss–Bonnet theorem we have

$$2\pi = 2\chi(C(P, r)) = \int_C K + \int_{\partial C} \kappa = 2\pi(1 - \cos r) + 2\pi\kappa \sin r$$

from which the conclusion follows. □

B.4. *Proof of Proposition B*1. By the remarks following the statement of the Proposition, we need only consider the case $(\alpha, r_1, r_2) \in \mathcal{U}$. Let $P_1$ and $P_2$ denote points of $S^2$ with $\mathrm{dist}_s(P_1, P_2) = \alpha$. Again by the remarks following the statement, the boundaries of $C_1 = C(P_1, r_1)$ and $C_2 = C(P_2, r_2)$ cross, and $G = C_1 \cap C_2$ is a digon. Its boundary consists of two circular arcs $A_1 = \partial C_1 \cap C_2$ and $A_2 = \partial C_2 \cap C_1$, which by Lemma B2 have lengths $l_1 = 2J(r_1, \alpha, r_2) \sin r_1$ and $l_2 = 2J(r_2, \alpha, r_1) \sin r_2$ respectively. Let us fix an orientation on $S^2$, so that $G$ and the $C_i$ inherit orientations. We give $A_i$ the orientation induced from $G$, which is the same as the one induced from $C_i$. By Lemma B3, $A_i \subset \partial C_i$ has constant geodesic curvature $\kappa_i = \cot r_i$. The digon $G$ has constant Gaussian curvature $K = 1$, and its exterior angles are equal to $\varepsilon = J(r_1, r_2, \alpha)$ by Lemma B2. By the Gauss–Bonnet theorem we have

$$\begin{aligned} 2\pi &= 2\pi\chi(G) = K\,\mathrm{area}(G) + l_1\kappa_1 + l_2\kappa_2 + 2\varepsilon \\ &= \iota(\alpha, r_1, r_2) + 2J(r_1, \alpha, r_2)\cos r_1 + 2J(r_2, \alpha, r_1)\cos r_2 + 2J(r_1, r_2, \alpha), \end{aligned}$$

which implies the conclusion of the proposition. □

## APPENDIX C. A PATH OF 2-GENERATOR GROUPS

In this appendix we prove the remaining result that was required for the proof of Theorem 3.1. We give a construction of certain 1-parameter families of 2-generator subgroups of $\mathrm{Isom}_+(\mathbb{H}^3)$; these paths were used in the proof of Theorem 3.1 for the reduction to the case of groups which admit no non-constant invariant super-harmonic functions. In this appendix we also derive a formula, which is included in Lemma C2, for the displacement of a point of $\mathbb{H}^3$ under a loxodromic isometry. This formula was used in Lemma 4.2 and Proposition 4.10 as well as in [9].

PROPOSITION C1. *Let $\xi$ and $\eta$ be two loxodromic isometries of $\mathbb{H}^3$ without any common fixed point. Denote by L the common perpendicular to the axes $A_\xi$ and $A_\eta$ of $\xi$ and $\eta$, respectively. Let $z_0$ be any point of L. Then there exist continuous one-parameter families $(\xi_t)_{0 \leqslant t \leqslant 1}$ and $(\eta_t)_{0 \leqslant t \leqslant 1}$ of loxodromic isometries of $\mathbb{H}^3$ with the following properties:*

(i) *$\xi_0 = \xi$ and $\eta_0 = \eta$;*
(ii) *for every t the axes of $\xi_t$ and $\eta_t$ are perpendicular to L;*
(iii) *the functions $t \mapsto \operatorname{dist}(z_0, \xi_t \cdot z_0)$ and $t \mapsto \operatorname{dist}(z_0, \eta_t \cdot z_0)$ are monotonically decreasing on $[0, 1]$;*
(iv) *the function $t \mapsto \angle(\xi_t \cdot z_0, z_0, \eta_t \cdot z_0)$ is monotonically decreasing on $[0, 1]$; and*
(v) *the isometries $\xi_1$ and $\eta_1$ have the same axis.*

The proof of Proposition C1 depends on the following three lemmas, C2–C4. The first assertion of Lemma C2 was used, but not proved, in [9].

LEMMA C2. *Let $\gamma$ be a loxodromic isometry of $\mathbb{H}^3$ with axis A, translation length l and twist angle $\theta$. Let z be any point of $\mathbb{H}^3$, let z′ denote the foot of the perpendicular from z to A, and let R denote the perpendicular distance from z to A. Then we have*

$$\cosh \operatorname{dist}_h(z, \gamma \cdot z) = \Delta(R)$$

*where $\Delta = \Delta_{l,\theta}$ is the function defined on $[0, \infty)$ by*

$$\Delta(R) = \cosh^{-1}(\cosh l + \sinh^2 R(\cosh l - \cos \theta)).$$

*We also have*

$$\angle(z', z, \gamma \cdot z) = \alpha(R)$$

*where $\alpha = \alpha_{l,\theta}$ is the function defined on $[0, \infty)$ by*

$$\alpha(R) = \cos^{-1} \frac{(\sinh R)(\cosh R)(\cosh l - \cos \theta)}{\sqrt{(\cosh l + \sinh^2 R(\cosh l - \cos \theta))^2 - 1}}.$$

*Furthermore, $\Delta(R)$ is strictly monotone increasing on $[0, \infty)$ and tends to $\infty$ as $R \to \infty$, while $\alpha(R)$ is strictly monotone decreasing on $[0, \infty)$ and tends to 0 as $R \to \infty$.*

*Proof.* We identify $\mathbb{H}^3$ conformally with the upper half-space $\mathbb{R}^2 \times \mathbb{R}^+$ in such a way that $A = \{0\} \times \mathbb{R}^+$, and so that $\gamma$ is given by $\gamma \cdot (w, t) = (e^{l + i\theta} w, e^l t)$. We may also suppose the identification to have been made in such a way that $z_0 = (s, 1)$ for some positive real number $s$. The distance from any point $(w, t) \in \mathbb{H}^3$ to A is given by the formula

$$\sinh \operatorname{dist}_h((w, t), A) = |w|/t. \tag{C.2.1}$$

Here the right-hand side is the tangent of the angle between the ray $\bar{A}$ and the ray which has origin 0 and passes through $(w, t)$. Formula (C.2.1) is the 3-dimensional analogue of formula (7.20.3) of [2], and follows from applying the latter formula to the hyperbolic plane spanned by A and $(w, t)$. In the same way, formula (7.2.1(ii)) of [2] implies that the distance between any two points $(w, t)$ and $(w', t')$ of $\mathbb{H}^3$ is given by

$$\cosh \operatorname{dist}_h((w, t), (w', t')) = 1 + \frac{|w - w'|^2 + (t - t')^2}{2tt'}. \tag{C.2.2}$$

It follows from (C.2.2) that $\sinh R = \sinh \operatorname{dist}_h((s, 1), A) = s$. Now let us apply (C.2.2) to the points $z_0 = (s, 1)$ and $\gamma \cdot z_0 = (e^{l+i\theta}s, e^l)$. This gives

$$\begin{aligned}\cosh \operatorname{dist}(z_0, \gamma \cdot z_0) &= 1 + \frac{s^2|e^{l+i\theta} - 1|^2 + (e^l - 1)^2}{2e^l} \\ &= (\cosh l - \cos \theta)s^2 + \cosh l = \cosh \Delta(R)\end{aligned}$$

and the first assertion of the lemma follows.

To prove the second assertion, we first apply the hyperbolic Pythagorean theorem to the right triangle with vertices $z'$, $\gamma \cdot z'$ and $\gamma \cdot z$. The lengths of the sides adjacent to the right angle are $\operatorname{dist}_h(z', \gamma \cdot z') = l$ and $\operatorname{dist}_h(\gamma \cdot z', \gamma \cdot z) = R$. Hence, setting $c = \operatorname{dist}_h(z', \gamma \cdot z)$, we have

$$\cosh c = \cosh l \cosh R.$$

Now we apply the first hyperbolic law of cosines to the triangle with vertices $z'$, $z$ and $\gamma \cdot z$, whose side lengths are $\operatorname{dist}_h(z, \gamma \cdot z) = \Delta(R)$, $\operatorname{dist}_h(\gamma \cdot z, z') = c$ and $\operatorname{dist}_h(z', z) = R$. This gives

$$\begin{aligned}\cos \angle(z', z, \gamma \cdot z) &= \frac{\cosh \Delta(R) \cosh R - \cosh c}{\sinh \Delta(R) \sinh \mathrm{R}} \\ &= \frac{\cosh \Delta(R)(\cosh R - \cosh l)}{\sinh \Delta(R) \sinh \mathrm{R}} \\ &= \cos \alpha(R),\end{aligned}$$

and the second assertion follows.

It is clear from the definition of the function $\Delta(R)$ that it is strictly monotone increasing and tends to $\infty$ with $R$. In order to establish the properties of $\alpha(R)$, we set $A = \cosh l - \cos \theta$ and $B = \cosh l$. Note that $B \geqslant A > 0$ and $B > 1$. For each $R \in [0, \infty)$ we have $\alpha(R) = \cosh^{-1} \sqrt{g(\sinh R)}$, where $g$ is defined on $[0, \infty)$ by

$$g(x) = \frac{A(x + 1)x}{(Ax + B)^2 - 1}.$$

It is clear that $g(x)$ tends to 1 as $x \to \infty$, and hence that $\alpha(R)$ tends to 0 as $R \to \infty$. To complete the proof, we need only show that $g$ is strictly monotone increasing on $[0, \infty)$. Differentiating $g$ we find that

$$((Ax + B)^2 - 1)^2 A^{-1} g'(x) = 2A(B - A)x^2 + 2(B^2 - 1)x + B^2 - 1$$

where the right-hand side is strictly positive for $x > 0$ since $B \geqslant A > 0$ and $B > 1$. Hence $g'(x) > 0$ for every $x > 0$. $\square$

In the statement of the next lemma, we fix a line $L \subset \mathbb{H}^3$ and a point $z_0 \in L$. We identify $\hat{\mathbb{H}}^3 = \mathbb{H}^3 \cup S_\infty$ conformally with the closed unit ball in $\mathbb{R}^3$ in such a way that $z_0$ is the origin and $L$ is the vertical axis. In particular, $S_\infty$ is identified with $S^2$ in such a way that the end points of $L$ are the north and south poles $N$ and $S$.

We denote by $G_L$ the subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$ consisting of all isometries that stabilize the line $L$ and preserve orientation on $L$. Then $G_L$ is a Lie subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$ and in particular it has the structure of a manifold. The non-trivial elements of $G_L$ are precisely the loxodromic and elliptic isometries of $\mathbb{H}^3$ having $L$ as axis.

LEMMA C3. *Let $\xi$ be a loxodromic isometry of $\mathbb{H}^3$ whose axis meets $L$ perpendicularly at a point $z_0$. Let us define a map $f_\xi: G_L \to S_\infty = S^2$ as follows: for each $\gamma \in G_L$ we define $f_\xi(\gamma) \in S_\infty$ to be the endpoint of the ray from $z_0$ to $\gamma\, \xi\gamma^{-1} \cdot z_0$. Then $f_\xi$ maps $G_L$ homeomorphically onto $S^2 \backslash \{N, S\}$. Furthermore, there is a strictly monotone increasing function $h_\xi: [0, \infty) \to [0, \pi/2)$ such that for every $\gamma \in G_L$ we have*

$$|\lambda(f_\xi(\gamma))| = h_\xi(\mathrm{dist}_h(z_0, \gamma\xi\gamma^{-1} \cdot z_0)).$$

*Finally, for each point $\zeta$ on the equator of $S^2 = S_\infty$, the isometry $f_\xi^{-1}(\zeta) \in G_L$ maps the axis of $\xi$ onto the line which passes through $z_0$ and has $\zeta$ as an end point.*

*Proof.* Let us fix a transverse orientation for the line $L$. For each $w = e^{2\pi i\theta} \in S^1$, let $\tau_w$ denote the isometry which fixes $L$ pointwise and whose restriction to each plane orthogonal to $L$ is a rotation through an angle $\theta$ which is counterclockwise in terms of the chosen transverse orientation. For each $r \in \mathbb{R}\backslash\{0\}$ let $h_r$ denote the hyperbolic isometry with axis $L$ having translation length $|r|$, and having $S$ or $N$ as its attracting fixed point according as $r$ is positive or negative. (To say that $h_r$ is hyperbolic means that it leaves each plane through $L$ invariant.) Let $h_r$ denote the identity map on $\mathbb{H}^3$. Then $(r, w) \mapsto \tau_w \circ h_r$ is a homeomorphism of $\mathbb{R} \times S^1$ onto $G_L$. In particular $G_L$ is a topological 2-manifold homeomorphic to $\mathbb{R} \times S^1$.

Let $l$ and $\theta$ denote the translation length and twist angle of $\xi$, and let $A_\xi$ denote its axis. Let $\Delta = \Delta_{l,\theta}$ and $\alpha = \alpha_{l,\theta}$ be defined as in the statement of Lemma C2. Consider any $r \geqslant 0$ and any $w \in S^1$, and set $\gamma = \tau_w \circ h_r$. The isometry $\gamma\xi\gamma^{-1}$ has axis $\gamma \cdot A_\xi$, and its translation length and twist angle are $l$ and $\theta$. The point $z_0$ lies at a distance $r$ from $\gamma \cdot A_\xi$. The foot of the perpendicular from $z_0$ to $\gamma \cdot A_\xi$ is $\gamma \cdot z_0$. Hence by Lemma C2 we have $\angle(\gamma \cdot z_0, z_0, \gamma\xi\gamma^{-1} \cdot z_0) = \alpha(r)$. On the other hand, since $r \geqslant 0$, the point $\gamma \cdot z_0$ lies on the ray $L_+ \subset L$ which has origin $z_0$ and endpoint $N$. Hence

$$\lambda(f_\xi(\gamma)) = \tfrac{\pi}{2} - \angle(N, z_0, \gamma\xi\gamma^{-1} \cdot z_0) = \tfrac{\pi}{2} - \angle(\gamma \cdot z_0, z_0, \gamma\xi\gamma^{-1} \cdot z_0) = \tfrac{\pi}{2} - \alpha(r).$$

Similarly, for $r \leqslant 0$ we find that $\lambda(f_\xi(\gamma)) = \alpha(|r|) - \frac{\pi}{2}$. Thus for all $w \in S^1$ and $r \in \mathbb{R}$ we have

$$\lambda(f_\xi(\tau_w \circ h_r)) = \beta(r) \tag{C.3.1}$$

where $\beta(r) = |\frac{\pi}{2} - \alpha(|r|)|$.

By Lemma C2 the function $\alpha(R)$ decreases monotonically from $\pi/2$ to 0 as $R$ varies from 0 to $\infty$. Hence $\beta(r)$ increases montonically from 0 to $\pi/2$ as $r$ increases from $-\infty$ to $\infty$. It therefore follows from (C.3.1) that $f_\xi: G_L \to S^2\backslash\{N, S\}$ is a proper map.

We claim that $f_\xi$ is also one-to-one. Indeed, suppose that for some $\gamma = (\tau_w \circ h_r)$ and $\gamma' = (\tau_{w'} \circ h_{r'})$ we have $f_\xi(\gamma) = f_\xi(\gamma')$. By (C.3.1) it follows that $\beta(r) = \beta(r')$. By the monotonicity of $\beta$ it follows that $r = r'$. Hence $(\gamma') = \tau_{w'/w} \circ \gamma$. Since $\tau_{w'/w}$ fixes $z_0 \in L$ we have $\gamma'\xi\gamma'^{-1} \cdot z_0 = \tau_{w'/w} \cdot (\gamma\xi\gamma^{-1} \cdot z_0)$. In view of the definition of $f_\xi$, this implies that $f_\xi(\gamma') = \tau_{w'/w} \cdot f_\xi(\gamma)$. Since we have assumed that $f_\xi(\gamma') = f_\xi(\gamma)$, we conclude that $w = w'$. This shows that $f$ is one-to-one.

Since $f$ is a proper one-to-one map between the 2-manifolds $G_L$ and $S^2\backslash\{N, S\}$, it is a homeomorphism of $G_L$ onto $S^2 - \{N, S\}$. This is the first assertion of the lemma. To prove the second assertion, we first note that for any $\gamma = \tau_w \circ h_r$ we have $|\lambda(f_\xi(\gamma))| = \frac{\pi}{2} - \alpha(r)$ by virtue of (C.3.1). On the other hand, since $z_0$ is at a distance $r$ from the axis of $\gamma\xi\gamma^{-1}$, which has twist angle $\theta$ and translation length $l$, it follows from Lemma C2 that $\mathrm{dist}_h(z_0, \gamma\xi\gamma^{-1} \cdot z_0) = \Delta(|r|)$. Since, by Lemma C2, the function $\Delta(R)$ increases monotonically from $l$ to $\infty$ as $R$ increases from 0 to $\infty$, and $\frac{\pi}{2} - \alpha(R)$ increases from 0 to $\frac{\pi}{2}$ as $R$ increases from 0 to $\infty$, the second assertion of Lemma C3 follows if we set $h_\xi = \alpha \circ \Delta^{-1}$.

Finally, suppose that $\zeta$ is a point on the equator of $S^2 = S^\infty$, and let $B$ denote the line which passes through $z_0$ and has $\zeta$ as an end point. Since $\zeta$ is on the equator, $B$ is perpendicular to $L$ at $z_0$. As $A$ is also perpendicular to $L$ at $z_0$, we have $\tau_w \cdot A = B$ for some $w \in S^1$. We may suppose $w$ to be chosen so that the ray $A_+ \subset A$ which has origin $z_0$ and passes through $\xi \cdot z_0$ is transformed by $\tau_w$ onto the ray $B_+ \subset B$ which has origin $z_0$ and end point $\zeta$. Then $\tau_w\xi\tau_w^{-1} \cdot z_0 = \tau_w\xi \cdot z_0 \in B_+$, and hence $f_\xi(\tau_w) = \zeta$. In view of the first assertion of the lemma we may therefore write $f_\xi^{-1}(\zeta) = \tau_w$. In particular we have $f_\xi^{-1}(\zeta)(A) = B$. This is the final assertion of the lemma. ☐

LEMMA C4. *For any two points $P$ and $Q$ of $S^2\backslash\{N, S\}$, there exist two continuous paths $p, q[0, 1] \to S^2\backslash\{N, S\}$ with the following properties*:

(i) $p(0) = P$ *and* $q(0) = Q$;
(ii) *the functions* $\lambda \circ p$ and $\lambda \circ q$ *are (weakly) monotonically decreasing on* $[0, 1]$;
(iii) *the function* $t \mapsto \mathrm{dist}_s(p(t), q(t))$ *is (weakly) monotonically decreasing on* $[0, 1]$; *and*
(iv) *the points* $p(1)$ *and* $q(1)$ *coincide and lie on the equator.*

*Proof.* We may assume without loss of generality that $\theta(Q) = 0$ and that $0 \leqslant \theta(P) \leqslant \pi$. We define $p(t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ by setting $\theta(p(t)) = (1 - 2t)\theta(P)$ and $\lambda(p(t)) = \lambda(P)$. For $\frac{1}{2} \leqslant t \leqslant 1$ we define $p(t)$ by $\theta(p(t)) = 0$ and $\lambda(p(t)) = (2 - 2t)\lambda(P)$. We define $q(t) = Q$ for $0 \leqslant t \leqslant \frac{1}{2}$, and for $\frac{1}{2} \leqslant t \leqslant 1$ we define $q(t)$ by $\theta(q(t)) = 0$ and $\lambda(q(t)) = (2 - 2t)\lambda(Q)$.

Conclusions (i), (ii) and (iv) of the lemma are now clear. It is also clear that the function $t \mapsto \Theta(p(t), q(t))$ is monotone decreasing on $[0, 1]$. Since by the spherical distance formula we have

$$\cos \mathrm{dist}_s(p(t), q(t)) = \cos \lambda(p(t)) \cos \lambda(q(t)) \cos \Theta(p(t), q(t)) + \sin \lambda(p(t)) \sin \lambda(q(t))$$

it follows readily that $t \mapsto \cos \mathrm{dist}_s(p(t), q(t))$ is monotone increasing on $[0, 1]$, so that conclusion (iii) holds as well. ☐

C.5. *Proof of Proposition* C1. According to Lemma C3 we have homeomorphisms $f_\xi$ and $f_\eta$ from $G_L$ to $S^2\backslash\{N, S\}$. We set $P = f_\xi(I)$ and $Q = f_\eta(I)$, where $I$ denotes the identity element of $G_L$. With these choices of $P$ and $Q$, let $p$ and $q$ be paths satisfying conclusions (i)–(iv) of Lemma C4. For each $t \in [0, 1]$ we set $\gamma_t = f_\xi^{-1}(p(t))$ and $\delta_t = f_\eta^{-1}(q(t))$. We set $\xi_t = \gamma_t\xi\gamma_t^{-1}$ and $\eta_t = \gamma_t\eta\gamma_t^{-1}$. It is clear that $(\xi_t)_{0 \leqslant t \leqslant 1}$ and $(\eta_t)_{0 \leqslant t \leqslant 1}$ are continuous one-parameter families of isometries of $\mathbb{H}^3$.

As $\xi$ and $\eta$ are loxodromic with axes $A_\xi$ and $A_\eta$, the isometries $\xi_t$ and $\eta_t$ are also loxodromic and have axes $A_{\xi_t} = \gamma_t \cdot A_\xi$ and $A_{\eta_t} = \delta_t \cdot A_\eta$. Since $A_\xi$ and $A_\eta$ are perpendicular to $L$ and since $\gamma_t, \delta_t \in G_L$, it follows that $A_{\xi_t}$ and $A_{\eta_t}$ are also perpendicular to $L$. This is conclusion (ii) of Proposition C1. We have $\gamma_0 = f_\xi^{-1}(P) = I$ and hence $\xi_0 = \xi$; similarly $\eta_0 = \eta$. This is conclusion (i) of Proposition C1.

For any $t \in [0, 1]$, using the function $\Lambda_\xi$ given by Lemma C3, we have

$$\Lambda_\xi(\mathrm{dist}(z_0, \xi_t \cdot z_0)) = \Lambda_\xi(\mathrm{dist}(z_0, \gamma_t\xi\gamma_t^{-1} \cdot z_0)) = |\lambda(f_\xi(\gamma t))| = |\lambda(p(t))|.$$

Since $\Lambda_\xi$ is strictly monotone increasing by Lemma C3 and $\lambda \circ p$ is monotonically decreasing by conclusion (iii) of Lemma C4, it follows that $t \mapsto \operatorname{dist}(z_0, \xi_t \cdot z_0)$ is monotonically decreasing on $[0, 1]$; the same argument shows that and $t \mapsto \operatorname{dist}(z_0, \eta_t \cdot z_0)$ is monotonically decreasing on $[0, 1]$. This is conclusion (iii) of Proposition C1.

For any $t \in [0, 1]$ we have

$$\angle(\xi_t \cdot z_0, z_0, \eta_t \cdot z_0) = \angle(\gamma_t \xi \gamma_t^{-1} \cdot z_0, z_0, \gamma_t \eta \gamma_t^{-1} \cdot z_0).$$

In view of the definition of $f_\xi$ and $f_\eta$ the right-hand side of the equation above is equal to the spherical distance between $f_\xi(\gamma_t) = p(t)$ and $f_\eta(\gamma_t) = q(t)$. Hence conclusion (iv) of Proposition C1 follows from conclusion (iii) of Lemma C4.

Finally, according to conclusion (iv) of Lemma C4, we have $p(1) = q(1) = \zeta$, where $\zeta$ is some point of the equator of $S^2$. Thus we have $\gamma_1 = f_\xi^{-1}(\zeta)$, and the last sentence of Lemma C3 implies that $A_{\xi_1} = \gamma_1 \cdot A_\xi$ is the line passing through $z_0$ and having $\zeta$ as an end point. The same argument shows that $A_{\eta_1}$ is the very same line, and thus conclusion (v) of Proposition C1 holds. □