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Singular surfaces, mod 2 homology, and hyperbolic volume, II ☆

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ABSTRACT

If g is an integer $\geqslant 2$, and M is a closed simple 3-manifold such that $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group and $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \geqslant \max(3g-1, 6)$, we show that M contains a closed, incompressible surface of genus at most g. As an application we show that if M is a closed orientable hyperbolic 3-manifold such that $\operatorname{Vol} M \leqslant 3.08$, then $\dim_{\mathbb{Z}_2} H_1(M;\mathbb{Z}_2) \leqslant 5$.

1. Introduction

This paper is a sequel to [1]. As in [1], we write $\operatorname{rk}_2 V$ for the dimension of a $\mathbb{Z}_2$-vector space V, and set $\operatorname{rk}_2 X = \operatorname{rk}_2 H_1(X;\mathbb{Z}_2)$ when X is a space of the homotopy type of a finite CW-complex. As in [1], we say that an orientable 3-manifold M is *simple* if M is compact, connected, orientable, irreducible and boundary-irreducible, no subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and M is not a closed manifold with finite fundamental group.

We shall establish the following topological result, which is a refinement of Theorem 8.13 of [1].

Theorem 1.1. *Let g be an integer $\geqslant 2$. Let M be a closed simple 3-manifold such that $\operatorname{rk}_2 M \geqslant \max(3g-1, 6)$ and $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group. Then M contains a closed, incompressible surface of genus at most g.*

Like [1, Theorem 8.13], this result may be regarded as a partial analogue of Dehn's lemma for π_1-injective genus-g surfaces. The difference between the two theorems is that the hypothesis $\operatorname{rk}_2 M \geqslant \max(3g-1, 6)$ assumed in Theorem 1.1 is strictly weaker than the corresponding hypothesis in [1, Theorem 8.13], namely that $\operatorname{rk}_2 M \geqslant 4g-1$.

For the case $g = 2$, Theorem 1.1 is almost sharp: in Section 6 we construct examples of simple 3-manifolds M with $\operatorname{rk}_2 M = 4$ such that $\pi_1(M)$ has a subgroup isomorphic to a genus-2 surface group, but M contains no closed, incompressible surface whatever.

As an application of Theorem 1.1 we shall prove the following theorem relating volume to homology for closed hyperbolic 3-manifolds.

Theorem 1.2. *Let M be a closed orientable hyperbolic 3-manifold such that $\operatorname{Vol} M \leqslant 3.08$. Then $\operatorname{rk}_2 M \leqslant 5$.*

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Theorem 1.2 is a refinement of Theorem 9.6 of [1], and will be deduced from Theorem 1.1 in the same way that [1, Theorem 9.6] was deduced from [1, Theorem 8.13].

In [9], by combining Theorem 1.1 with new geometric results, we will prove that if M is a closed orientable hyperbolic 3-manifold such that $\mathrm{Vol}\, M \leqslant 3.44$, then $\mathrm{rk}_2\, M \leqslant 7$. Further applications of Theorem 1.1 to the study of volume and homology will be given in [6].

The arguments in this paper draw heavily on results from [1]. The improvements that we obtain here depend on a much deeper study of books of I-bundles (see [1, Section 2]) in closed 3-manifolds than the one made in [1]. For all $g \geqslant 2$ this involves new topological ingredients. For $g > 2$ it also involves a surprising application of Fisher's inequality from combinatorics.

Before describing the new ingredients in the proof of Theorem 1.1 we shall briefly review the proof of [1, Theorem 8.13] and explain the role that books of I-bundles play in it. The proof uses a tower of two-sheeted covers analogous to the one used by Shapiro and Whitehead in their proof of Dehn's lemma [17]. The homological hypothesis allows one to construct a good tower (in the sense of [1, Definition 8.4])

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \dots, p_n, M_n, N_n),$$

with base M_0 homeomorphic to M and with some height $n \geqslant 0$, such that N_n contains a connected (non-empty) closed incompressible surface F of genus $\leqslant g$. (Here N_j is a submanifold of M_j for $j = 0, \dots, n$ and $p_j : M_j \to N_{j-1}$ is a two-sheeted covering map for $j = 1, \dots, n$.) The key step is to show, for a given $j > 0$, that if N_j contains a connected closed incompressible surface F of genus $\leqslant g$, then N_{j-1} contains such a surface as well. Certain books of I-bundles arise as obstructions to carrying out this step. Specifically, the arguments of [1] show that this step can be carried out unless N_{j-1} is a closed manifold that contains a submanifold of the form $W = |\mathcal{W}|$, where $\mathcal{W}$ is a book of I-bundles, $\chi(W) \geqslant 2 - 2g$, and the inclusion homomorphism $H_1(W; \mathbb{Z}_2) \to H_1(N_{j-1}; \mathbb{Z}_2)$ is surjective. This situation is then ruled out by estimating ranks of homology groups. Under the hypothesis of [1, Theorem 8.13], one can show that when N_{j-1} is closed we have $\mathrm{rk}_2\, N_{j-1} \geqslant 4g - 2$, while [1, Lemma 2.11] implies that $\mathrm{rk}_2\, W \leqslant 4g - 3$ when $\chi(W) \geqslant 2 - 2g$. Thus the induced map on homology cannot be surjective.

Under the weaker hypothesis of Theorem 1.1 one obtains only a lower bound of $\max(3g - 2, 5)$ for $\mathrm{rk}_2\, N_{j-1}$ when N_{j-1} is closed. So the homological condition given by Lemma 2.11 of [1] does not suffice to overcome the obstruction. Instead, the strategy for carrying out the key step is to first attempt to construct the required incompressible surface by compressing the boundary of a carefully chosen submanifold of W.

It is easy to choose the book of I-bundles $\mathcal{W}$ defining W so that each of its pages has Euler characteristic -1. In this case one can find a sub-book $\mathcal{W}_0$ of $\mathcal{W}$ such that $W_0 = |\mathcal{W}_0|$ has exactly half the Euler characteristic of W. Using classical 3-manifold techniques one can then show that either (a) the inclusion homomorphism $\iota_\sharp : \pi_1(W_0) \to \pi_1(N_{j-1})$ has image of rank at most g, or (b) $\iota_\sharp$ is surjective, or (c) a connected incompressible surface can be obtained from ∂W by doing ambient surgeries in N_{j-1} and selecting a component. One can use Lemma 2.11 of [1] to show that alternative (b) contradicts the lower bound for $\mathrm{rk}_2\, N_{j-1}$. If alternative (c) holds, one has an incompressible surface of genus less than g, which is all that the tower argument requires. If (a) holds, a relative version of the proof of [1, Lemma 2.11] gives an upper bound of $3g - 2$ for $\mathrm{rk}_2\, N_{j-1}$; this contradicts our condition $\mathrm{rk}_2\, N_{j-1} \geqslant \max(3g - 2, 5)$ unless $g > 2$ and $\mathrm{rk}_2\, N_{j-1} = 3g - 2$.

To deal with the latter situation we must exercise even more care in choosing the sub-book $\mathcal{W}_0$. It turns out (see Lemma 4.5) that when $g > 2$ one can choose $\mathcal{W}_0$ in such a way that $H_2(W_0; \mathbb{Z}_2) \neq 0$. In particular it then follows that W_0 is not a handlebody, and the classical 3-manifold argument mentioned above can be modified to show that either (b) or (c) holds, or else (a′) the image of $\iota_\sharp$ has rank at most $g - 1$. One can then improve the upper bound for $\mathrm{rk}_2\, N_{j-1}$ to $3g - 2$ and obtain the required contradiction.

Making the right choice for $\mathcal{W}_0$ in this case requires both a detailed study of the homology of books of I-bundles and an interesting result, Proposition 3.1, about finite-dimensional vector spaces over $\mathbb{Z}_2$. It is in the proof of Proposition 3.1 that we need to apply Fisher's inequality.

Section 2 contains the classical 3-manifold arguments that we mentioned in the outline above, and Section 3 is devoted to the proof of Proposition 3.1. In Section 4 these ingredients are combined with some observations about homology of books of I-bundles to carry out the main step, sketched above, in the proof of Theorem 1.1; the proof of the theorem itself appears in Section 5. Section 6 is devoted to constructing the examples, referred to above, that show that the theorem is almost sharp.

In Section 7 we establish a stronger version of Theorem 1.1, Proposition 7.2, which is particularly well-adapted to the applications to volume estimates, including the proof of Theorem 1.2 and the application in the forthcoming paper [9]. The proof of Theorem 1.2 is given in Section 8.

In general we will use all of the conventions that were used in [1]. In particular, in addition to the notations $\mathrm{rk}_2\, V$ and $\mathrm{rk}_2\, X$, and the definition of a simple manifold, which were explained above, we shall set $\bar{\chi}(X) = -\chi(X)$ when X is a space of the homotopy type of a finite CW-complex (and $\chi(X)$ as usual denotes its Euler characteristic). *Connected* spaces are understood to be in particular non-empty, and *irreducible* 3-manifolds are understood to be in particular connected.

The cardinality of any finite set S will be denoted by $\# S$.

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2. Compressing submanifolds

Recall that a *compressing disk* for a closed surface F in the interior of a 3-manifold M is defined to be a disk $D \subset M$ such that $D \cap F = \partial D$, and such that ∂D does not bound a disk in F.

Lemma 2.1. *Let M be a compact, connected, orientable, simple 3-manifold, and let T be a compressible torus in $\operatorname{int} M$. Then either T bounds a solid torus in $\operatorname{int} M$, or T is contained in a ball in $\operatorname{int} M$.*

Proof. Since M is simple, T is compressible. Fix a compressing disk D for X. Let $E \subset \operatorname{int} M$ be a ball containing D, such that $A \doteq E \cap T \subset \partial E$, and such that A is a regular neighborhood of ∂D in T. Then $\overline{(\partial E) - A}$ has two components D_1 and D_2, both of which are disks, and $D_1 \cup A \cup D_2$ is a sphere, which must bound a ball $B \subset M$. By connectedness we have either $E \cap B = D_1 \cup D_2$ or $E \subset B$. The first alternative implies that $E \cup B \supset T$ is a solid torus, and the second implies that $T \subset B$. □

Definitions 2.2. Let M be a compact, connected, orientable, irreducible 3-manifold. We shall denote by $\mathcal{X}_M$ the set of all compact, connected 3-submanifolds X of $\operatorname{int} M$ such that

(i) no component of ∂X is a 2-sphere, and
(ii) X does not carry $\pi_1(M)$, i.e. the inclusion homomorphism $\pi_1(X) \to \pi_1(M)$ is not surjective.

For any $X \in \mathcal{X}_M$, since ∂X has no 2-sphere components, we have $\bar{\chi}(X) \geqslant 0$. We let $t(X)$ denote the number of components of ∂X that are tori, and we set

$$k(X) = t(X) + 3\bar{\chi}(X) \geqslant 0.$$

For any $X \in \mathcal{X}_M$ we denote by $r(X)$ the rank of the image of the inclusion homomorphism $\pi_1(X) \to \pi_1(M)$. We set

$$i(X) = \bar{\chi}(X) - r(X) \in \mathbb{Z}.$$

If $X \in \mathcal{X}_M$ is given, we define a *compressing disk* for X to be a compressing disk for ∂X. We shall say that D is *internal* or *external* according to whether $D \subset X$ or $D \cap X = \partial D$. We shall say that an internal compressing disk D is *separating* if $X - D$ is connected, and *non-separating* otherwise.

Lemma 2.3. *Let M be a compact, connected, orientable, irreducible 3-manifold, and let $X \in \mathcal{X}_M$ be given. Suppose that every component of ∂X has genus strictly greater than 1, and that X has an internal compressing disk. Then there is an element X' of $\mathcal{X}_M$ such that*

(1) $X' \subset X$,
(2) *every component of $M - X'$ contains at least one component of $M - X$,*
(3) $\bar{\chi}(X') \leqslant \bar{\chi}(X) - 1$,
(4) $k(X') < k(X)$,
(5) $i(X') \leqslant \max(i(X), (i(X) - 1)/2)$, *and*
(6) *if X is not a handlebody then X' is not a handlebody.*

Proof. If X has a non-separating internal compressing disk we fix such a disk and denote it by D. If every internal compressing disk for X is separating we let D denote an arbitrarily chosen internal compressing disk for X. In either case we set $\gamma = \partial D$, and denote by F the component of ∂X that contains γ.

We let E denote a regular neighborhood of D in X. The manifold $Z = \overline{X - E}$ has at most two components. Each component of ∂Z is either a component of ∂X, or a component of the surface obtained from F by surgery on the simple closed curve γ, which is homotopically non-trivial in F. Since every component of ∂X has genus strictly greater than 1, it follows that no component of ∂Z is a 2-sphere, and that ∂Z has at most two torus components. Since $Z \subset X$ and $X \in \mathcal{X}_M$, no component of Z carries $\pi_1(M)$. Hence each component of Z belongs to $\mathcal{X}_M$.

We observe that

$$Z \subset X, \tag{2.3.1}$$

that

$$\text{every component of } M - Z \text{ contains at least one component of } M - X, \tag{2.3.2}$$

and that

$$\bar{\chi}(Z) = \bar{\chi}(X) - 1. \tag{2.3.3}$$

Since ∂Z has at most two torus components, we have

$$t(Y) \leqslant 2 \tag{2.3.4}$$

for every component Y of Z.

We claim:

2.3.5. *There is a component X' of Z such that $i(X') \leqslant \max(i(X), (i(X) - 1)/2)$.*

We first prove 2.3.5 in the case where Z is connected. We shall show that $i(Z) \leqslant i(X)$, which implies 2.3.5 in this case. We fix a base point $\star \in Z$ and let $G, G' \leqslant \pi_1(M, \star)$ denote the respective images of $\pi_1(X, \star)$ and $\pi_1(Z, \star)$ under inclusion. Then G is generated by G' and α, where $\alpha \in \pi_1(M, \star)$ is the homotopy class of a loop in X that crosses D in a single point. Hence $r(X) \leqslant r(Z) + 1$. In view of (2.3.3), it follows that $i(X) \geqslant i(Z)$.

We next prove 2.3.5 in the case where Z is disconnected. Let Y_1 and Y_2 denote the components of Z. We fix a base point $\star \in D$ and let $G, G'_i \leqslant \pi_1(M, \star)$ denote the respective images of $\pi_1(X, \star)$ and $\pi_1(Y_i, \star)$ under inclusion. Then G is generated by G'_1 and G'_2, so that $r(X) \leqslant r(Y_1) + r(Y_2)$. It follows that

$$\begin{aligned} i(Y_1) + i(Y_2) &= \big(\bar{\chi}(Y_1) + \bar{\chi}(Y_2)\big) - \big(r(Y_1) + r(Y_2)\big) \\ &= \big(\bar{\chi}(Y_1) + \bar{\chi}(Y_2)\big) - \big(r(Y_1) + r(Y_2)\big) \\ &\leqslant \bar{\chi}(Z) - r(X) \end{aligned}$$

which in view of (2.3.3) gives

$$i(Y_1) + i(Y_2) \leqslant i(X) - 1.$$

Hence for some $j \in \{1, 2\}$ we have

$$i(Y_j) \leqslant \frac{i(X) - 1}{2}.$$

If we set $X' = Y_j$ with this choice of j, then 2.3.5 follows in this case.

Now let X' denote the component of Z given by 2.3.5. Thus conclusion (5) of the lemma holds with this choice of X'. In view of (2.3.1), conclusion (1) holds as well.

It follows from (2.3.3) that $\bar{\chi}(X) - 1 = \sum_Y \bar{\chi}(Y)$, where Y ranges over the components of Z. Since each component of Z belongs to $\mathcal{X}_M$, we have $\bar{\chi}(Y) \geqslant 0$ for each component Y of Z. Hence $\bar{\chi}(Y) \leqslant \bar{\chi}(X) - 1$ for each component Y of Z. This, together with (2.3.4), implies that $k(Y) < k(X)$ for each component Y of Z. In particular, conclusions (3) and (4) hold with our choice of X'.

Since X' is a component of Z, every component of $M - X'$ contains at least one component of $M - Z$. Combining this observation with (2.3.2) we obtain conclusion (2).

It remains to prove conclusion (6). We shall assume that X' is a handlebody and deduce that X is a handlebody. If Z is connected, so that $X' = Z$, then X is the union of the handlebody Z and the ball E, and $Z \cap E$ is the union of two disjoint disks. Hence X is a handlebody. Now suppose that Z is disconnected, i.e. that $X - D$ is disconnected. Since the handlebody X' is an element of $\mathcal{X}_M$, it must have strictly positive genus. Hence there is a disk $D' \subset X'$ such that $X' - D'$ is connected. After modifying D' by an ambient isotopy in X' we may assume that D' is disjoint from the disk $X' \cap E \subset \partial X'$. Then D' is an internal compressing disk for X, and $X - D'$ is connected. But in this case the choice of D guarantees that $X - D$ is connected, and we have a contradiction. □

Lemma 2.4. *Let M be a compact, connected, orientable, irreducible 3-manifold, and let $X \in \mathcal{X}_M$ be given. Suppose that every component of ∂X has genus strictly greater than 1, and that X has an external compressing disk. Then there is an element X' of $\mathcal{X}_M$ such that*

(1) $\bar{\chi}(X') = \bar{\chi}(X) - 1$,
(2) $k(X') < k(X)$, *and*
(3) $i(X') = i(X) - 1$.

Proof. We fix an external compressing disk D for X, we set $\gamma = \partial D$, and we let E denote a regular neighborhood of D in $\overline{M - X}$. We set $X' = X \cup E$. Note that the inclusion homomorphism $\pi_1(X) \to \pi_1(X')$ is surjective. Hence:

2.4.1. *For any base point $\star \in X$, the inclusion homomorphisms $\pi_1(X, \star) \to \pi_1(M, \star)$ and $\pi_1(X', \star) \to \pi_1(M, \star)$ have the same image.*

Since $X \in \mathcal{X}_M$, it follows from 2.4.1 that X' does not carry $\pi_1(M)$. On the other hand, each component of $\partial X'$ is either a component of ∂X, or a component of the surface obtained from F by surgery on the simple closed curve γ, which is homotopically non-trivial in F. Since every component of ∂X has genus strictly greater than 1, it follows that no component of $\partial X'$ is a 2-sphere, and that at most two of its components are tori. Hence $X' \in \mathcal{X}_M$, and

$$t(X') \leqslant 2. \tag{2.4.2}$$

With this definition of X', it is clear that conclusion (1) of the lemma holds. With (2.4.2), this implies conclusion (2). On the other hand, by 2.4.1 we have

$$r(X') = r(X). \tag{2.4.3}$$

Combining (2.4.3) with conclusion (2), we immediately obtain conclusion (3). □

Definition 2.5. Let $g \geqslant 2$ be an integer and let M be a closed, orientable, irreducible 3-manifold. We shall say that M is *g-small* if M contains no separating, closed, incompressible surface of genus g, and contains no closed incompressible surface of genus $< g$.

Lemma 2.6. *Let c be a positive integer, let M be a compact, connected, orientable, irreducible* 3*-manifold which is* $(c+1)$*-small, and let X be an element of $\mathcal{X}_M$ such that $\bar{\chi}(X) \leqslant c$. Then every component of ∂X is compressible in M.*

Proof. The hypothesis that $\bar{\chi}(X) \leqslant c$ implies that every component of ∂X has genus at most $c+1$, and that if ∂X is disconnected then each of its components has genus at most c. In particular, every component of ∂X is either a separating surface of genus $c+1$, or a surface of genus at most c. Since M is $(c+1)$-small it follows that every component of ∂X is compressible in M. □

Proposition 2.7. *Let M be a compact, connected, orientable, irreducible* 3*-manifold, and let Y be an element of $\mathcal{X}_M$. Set $c = \bar{\chi}(Y)$, and assume that M is* $(c+1)$*-small. Then $i(Y) \geqslant -1$.*

Proof. Suppose that $i(Y) \leqslant -2$. Let $\mathcal{X}_M^* \subset \mathcal{X}_M$ denote the set of all $X \in \mathcal{X}_M$ such that

(i) $\bar{\chi}(X) \leqslant c$ and
(ii) $i(X) \leqslant -2$.

Then $Y \in \mathcal{X}_M^*$ and so $\mathcal{X}_M^* \neq \emptyset$. Let us choose an element $X \in \mathcal{X}_M^*$ such that $k(X) \leqslant k(X')$ for every $X' \in \mathcal{X}_M^*$.

Since X belongs to $\mathcal{X}_M$, it cannot carry $\pi_1(M)$; in particular, $X \neq M$, and so $\partial X \neq \emptyset$. Since $X \in \mathcal{X}_M^*$, we have $\bar{\chi}(X) \leqslant c$. It therefore follows from Lemma 2.6 that every component of ∂X is compressible in M. In particular X has either an internal or an external compressing disk.

We first consider the case in which X has an internal compressing disk, and every component of ∂X has genus > 1. In this case, Lemma 2.3 gives an element X' of $\mathcal{X}_M$ such that $\bar{\chi}(X') \leqslant \bar{\chi}(X) - 1$, $k(X') < k(X)$, and $i(X') \leqslant \max(i(X), (i(X) - 1)/2)$. Since $\bar{\chi}(X) \leqslant c$ and $i(X) \leqslant -2$, it follows that $\bar{\chi}(X') \leqslant c - 1$ and that $i(X') < -1$. In particular, $X' \in \mathcal{X}_M^*$. Since $k(X') < k(X)$, this contradicts our choice of $\mathcal{X}$.

We next turn to the case in which X has an external compressing disk, and every component of ∂X has genus > 1. In this case, Lemma 2.4 gives an element X' of $\mathcal{X}_M$ such that $\bar{\chi}(X') \leqslant \bar{\chi}(X) - 1$, $k(X') < k(X)$, and $i(X') = i(X) - 1$. Since $\bar{\chi}(X) \leqslant c$ and $i(X) \leqslant -2$, it again follows that $\bar{\chi}(X') \leqslant c - 1$ and that $i(X') < -1$. Again it follows that $X' \in \mathcal{X}_M^*$, and since $k(X') < k(X)$, our choice of $\mathcal{X}$ is contradicted.

There remains the case in which some component T of ∂X is a torus. According to Lemma 2.1, T is the boundary of a compact submanifold W of $\operatorname{int} M$ such that either (a) W is a solid torus, or (b) W is contained in a ball in $\operatorname{int} M$. We must have either $X \subset W$ or $X \cap W = T$.

Either of the alternatives (a) or (b) implies that the image of $\pi_1(W)$ under the inclusion to $\pi_1(M)$ is at most cyclic. Hence if $X \subset W$ then $r(X) \leqslant 1$, and hence $i(X) \geqslant -1$. This is a contradiction since $X \in \mathcal{X}_M^*$.

If $X \cap W = T$, we set $X' = X \cup W$. Then $\partial X' = (\partial X) - T$. In particular $\partial X'$ has no sphere components. If $\star$ is a base point in X, either of the alternatives (a) or (b) implies that $\pi_1(X, \star)$ and $\pi_1(X', \star)$ have the same image under the inclusion to $\pi_1(M, \star)$. It follows that X' does not carry $\pi_1(M)$, so that $X \in \mathcal{X}_M$. It also follows that $r(X') = r(X)$. But since $\partial X' = (\partial X) - T$, we have $\bar{\chi}(X') = \bar{\chi}(X)$ and $t(X') = t(X) - 1$. We now deduce that $\bar{\chi}(X') \leqslant c$ and $i(X') = i(X) \leqslant -2$, so that $X' \in \mathcal{X}_M^*$; and that $k(X') = k(X) - 1$. This contradicts our choice of X. □

Proposition 2.8. *Let M be a compact, connected, orientable, irreducible* 3*-manifold, and let Y be an element of $\mathcal{X}_M$. Assume that Y is not a handlebody and that no component of ∂Y is a torus. Set $c = \bar{\chi}(Y)$, and assume that M is* $(c+1)$*-small. Then $i(Y) \geqslant 0$.*

Proof. We define a *Y-special submanifold* of M to be a compact 3-dimensional submanifold W of M such that

- ∂W is a torus contained in $\operatorname{int} Y$,
- $W \not\subset Y$, and
- either W is a solid torus or W is contained in a ball in $\operatorname{int} M$.

We distinguish two cases.

Case I. There is no Y-special submanifold of M. In order to prove that in Case I we have $i(Y) \geqslant 0$, we reason by contradiction. Assume that $i(Y) \leqslant -1$. Let $\mathcal{X}_M^{**} \subset \mathcal{X}_M$ denote the set of all $X \in \mathcal{X}_M$ such that

(i) $X \subset \operatorname{int} Y$,
(ii) every component of $M - X$ contains at least one component of $M - Y$,
(iii) X is not a handlebody,
(iv) $\bar{\chi}(X) \leqslant c$ and
(v) $i(X) \leqslant -1$.

The hypotheses and the assumption that $i(Y) \leqslant -1$, imply that a manifold obtained from Y by removing a half-open collar about ∂Y belongs to $\mathcal{X}_M^{**}$. Hence $\mathcal{X}_M^{**} \neq \emptyset$. Let us choose an element $X \in \mathcal{X}_M^{**}$ such that $k(X) \leqslant k(X')$ for every $X' \in \mathcal{X}_M^{**}$.

Since X belongs to $\mathcal{X}_M$, it cannot carry $\pi_1(M)$; in particular, $X \neq M$, and so $\partial X \neq \emptyset$. Since $X \in \mathcal{X}_M^{**}$, we have $\bar{\chi}(X) \leqslant c$. It therefore follows from Lemma 2.6 that every component of ∂X is compressible in M. In particular, X has either an internal or an external compressing disk.

We first consider the subcase in which X has an internal compressing disk, and every component of ∂X has genus > 1. In this case, there is an element X' of $\mathcal{X}_M$ such that conclusions (1)–(6) of Lemma 2.3 hold. Since $X \in \mathcal{X}_M^{**}$, conclusions (1), (2), (3), (5) and (6) of Lemma 2.3 imply, respectively, that $X' \subset Y$; that every component of $M - X'$ contains at least one component of $M - Y$; that $\bar{\chi}(X') \leqslant c - 1$; that $i(X') \leqslant -1$; and that X' is not a handlebody. Hence $X' \in \mathcal{X}_M^{**}$. But (4) gives $k(X') < k(X)$, and this contradicts our choice of $\mathcal{X}$.

We next turn to the subcase in which X has an external compressing disk, and every component of ∂X has genus > 1. In this case, Lemma 2.4 gives an element X' of $\mathcal{X}_M$ such that $\bar{\chi}(X') \leqslant \bar{\chi}(X) - 1$ and $i(X') = i(X) - 1$. Let us set $c' = \bar{\chi}(X')$. Since $X \in \mathcal{X}_M^{**}$ we have $c' \leqslant \bar{\chi}(X) - 1 \leqslant c - 1$. Since by hypothesis M contains no incompressible closed surfaces of genus $\leqslant c+1$, in particular it contains no incompressible closed surfaces of genus $\leqslant c'+1$. Hence the hypotheses of Proposition 2.7 hold with X and c' in place of Y and c. It follows that $i(X') \geqslant -1$, i.e. that $i(X) \geqslant 0$. But since $X \in \mathcal{X}_M^{**}$ we have $i(X) \leqslant -1$, a contradiction.

The remaining subcase of Case I is the one in which some component T of ∂X is a torus. According to Lemma 2.1, T is the boundary of a compact submanifold W of $\operatorname{int} M$ such that either W is a solid torus, or W is contained in a ball in $\operatorname{int} M$. Since we are in Case I, the submanifold W of M cannot be Y-special. Hence we must have $W \subset Y$.

Since $\partial W = T \subset \partial X$, we must have either $X \subset W$ or $X \cap W = T$. If $X \cap W = T$, then $\operatorname{int} W$ is a component of $M - X$. By condition (ii) in the definition of $\mathcal{X}_M^{**}$, the set W must contain a component of $M - Y$. This is impossible since $W \subset Y$.

Now suppose that $X \subset W$. If X is a proper subset of W then W contains a component of $M - X$, which by condition (ii) in the definition of $\mathcal{X}_M^{**}$ must contain a component of $M - Y$. Again this is impossible since $W \subset Y$. Hence $X = W$. If W is a solid torus, we have a contradiction to condition (iii) in the definition of $\mathcal{X}_M^{**}$. Finally, if X is contained in a ball in $\operatorname{int} M$ we have $r(X) = 0$ and hence $i(X) \geqslant 0$. This contradicts condition (v) in the definition of $\mathcal{X}_M^{**}$.

Case II. There is a Y-special submanifold of M. In this case, we fix a Y-special submanifold W of M, and we set $Y' = Y \cup W$. We also set $F = W \cap \partial Y$. The definition of a Y-special manifold guarantees that $W \not\subset Y$ and hence that $F \neq \emptyset$. But F is a union of components of ∂Y, and by the hypothesis of the proposition, no component of ∂Y is a torus. Hence $\bar{\chi}(F) > 0$.

We have $\partial Y' = (\partial Y) - F$. In particular $\partial Y'$ has no sphere components.

According to the definition of a Y-special submanifold, either W is a solid torus or W is contained in a ball in $\operatorname{int} M$. In either case, if $\star$ is a base point in Y, then $\pi_1(Y, \star)$ and $\pi_1(Y', \star)$ have the same image under the inclusion to $\pi_1(M, \star)$. It follows that Y' does not carry $\pi_1(M)$, so that $Y \in \mathcal{X}_M$. It also follows that $r(Y') = r(Y)$. But since $\partial Y' = (\partial Y) - F$, we have $\bar{\chi}(Y') = \bar{\chi}(Y) - \bar{\chi}(F) < \bar{\chi}(Y)$. It follows that

$$i(Y') < i(Y). \tag{2.8.1}$$

On the other hand, if we set $c'' = \bar{\chi}(Y') < \bar{\chi}(Y) = c$, the hypothesis of the proposition implies that M contains no incompressible closed surface of genus $\leqslant c'' + 1$. Hence by Proposition 2.7 we have $i(Y') \geqslant -1$. In view of (2.8.1) it follows that $i(Y) \geqslant 0$. □

3. An algebraic result

Suppose that V is a finite-dimensional vector space over $\mathbb{Z}_2$ and that $\mathcal{U}$ is a basis of V. Then any element α of V may be written uniquely in the form $\sum_{u \in \mathcal{U}} \lambda_u u$, with $\lambda_u \in \mathbb{Z}_2$ for each $u \in \mathcal{U}$. We denote by $S_{\mathcal{U}}(\alpha)$ the set of elements $u \in \mathcal{U}$ such that $\lambda_u = 1$, and define the *size* of α with respect to the basis $\mathcal{U}$, denoted $\|\alpha\|_{\mathcal{U}}$, to be $\# S_{\mathcal{U}}(\alpha)$. Note that $\|\alpha\|_{\mathcal{U}} = 0$ if and only if $\alpha = 0$.

The purpose of this section is to prove the following result:

Proposition 3.1. *Let m be an integer $\geqslant 2$, let $\mathcal{U}$ be a basis of a $2m$-dimensional vector space V over $\mathbb{Z}_2$, and suppose that H is a subspace of V with dimension at least m. Then there is an element α of H such that $0 < \|\alpha\|_{\mathcal{U}} \leqslant m$.*

Proof. We divide the argument into two cases.

Case I. There is an element β of H such that $\|\beta\|_{\mathcal{U}} \geqslant m+2$. Set $k = \|\beta\|_{\mathcal{U}}$, so that $m+2 \leqslant k \leqslant 2m$. We let L denote the linear subspace of V spanned by $S_{\mathcal{U}}(\beta)$. Thus L consists of all elements $\alpha \in V$ such that $S_{\mathcal{U}}(\alpha) \subset S_{\mathcal{U}}(\beta)$.

We have $\operatorname{rk}_2 L = k$, and so

$$\operatorname{rk}_2(H \cap L) \geqslant \operatorname{rk}_2 H + \operatorname{rk}_2 L - \operatorname{rk}_2 V \geqslant m + k - 2m \geqslant 2.$$

Hence there is an element $\alpha_1 \in H \cap L$ such that $\alpha_1 \neq 0$ and $\alpha_1 \neq \beta$. If we set $\alpha_2 = \beta + \alpha_1$, then $S_{\mathcal{U}}(\alpha_2)$ is the complement of $S_{\mathcal{U}}(\alpha_1)$ relative to $S_{\mathcal{U}}(\beta)$. This implies that

$$\|\alpha_1\|_{\mathcal{U}} + \|\alpha_2\|_{\mathcal{U}} = \|\beta\|_{\mathcal{U}} = k \leqslant 2m,$$

so that $\|\alpha_j\|_{\mathcal{U}} \leqslant m$ for some $j \in \{1, 2\}$. As our choice of α_1 implies that α_1 and α_2 are both non-zero, the conclusion of the proposition follows in this case.

Case II. For every element α of H we have $\|\alpha\|_{\mathcal{U}} \leqslant m+1$. If we are in Case II and the conclusion of the proposition does not hold, then for every $\alpha \in H - \{0\}$ we have $\|\alpha\|_{\mathcal{U}} = m+1$. We shall show this leads to a contradiction.

We consider the collection $\mathcal{S} = \{S_{\mathcal{U}}(\alpha)\colon\ 0 \neq \alpha \in H\}$ of subsets of $\mathcal{U}$. Each set in $\mathcal{S}$ has cardinality exactly $m+1$. If S and T are distinct sets in $\mathcal{S}$ we may write $S = S_{\mathcal{U}}(\alpha)$ and $T = S_{\mathcal{U}}(\beta)$, where α and β are distinct elements of $H - \{0\}$. We then have $\alpha + \beta \in H - \{0\}$, so that $S_{\mathcal{U}}(\alpha+\beta)$ has cardinality $m+1$. But $S_{\mathcal{U}}(\alpha+\beta)$ is the symmetric difference of $S = S_{\mathcal{U}}(\alpha)$ and $T = S_{\mathcal{U}}(\beta)$, so that

$$\begin{aligned} m+1 &= \# S_{\mathcal{U}}(\alpha+\beta) \\ &= \# S + \# T - 2\#(S \cap T) \\ &= 2(m+1) - 2\#(S \cap T), \end{aligned}$$

so that

$$\#(S \cap T) = \frac{m+1}{2} \tag{3.1.1}$$

for any two distinct sets $S, T \in \mathcal{S}$.

Since $\operatorname{rk}_2 H = m \geqslant 2$, there exist distinct elements S and T of $\mathcal{S}$. It therefore follows from (3.1.1) that m is odd. In particular we have $m \geqslant 3$.

We now apply Fisher's inequality [14, Theorem 14.6], which may be stated as follows. Let n and k be positive integers, let $\mathcal{U}$ be a set of cardinality n, and suppose that $\mathcal{X}$ is a collection of subsets of $\mathcal{U}$ such that $\#(S \cap T) = k$ for all distinct sets $S, T \in \mathcal{X}$. Then $\# \mathcal{X} \leqslant n$.

In the present situation, the hypotheses of Fisher's inequality hold with $n = 2m$, $k = (m+1)/2$ and $\mathcal{X} = \mathcal{S}$. But if $d \geqslant m$ is the dimension of H, we have $\# \mathcal{S} = 2^d - 1$. Hence Fisher's inequality gives $2^m - 1 \leqslant 2m$. However, since $m \geqslant 3$ we have $2m < 2^m - 1$. This is the required contradiction. □

4. Homology of books of I-bundles

In this section we will use the notation introduced in [1, Section 2] regarding books of I-bundles. Recall that if $\mathcal{W}$ is a book of I-bundles then $\mathcal{B}_{\mathcal{W}}$ and $\mathcal{P}_{\mathcal{W}}$ denote, respectively the union of all bindings of $\mathcal{W}$ and the union of all its pages; and $|\mathcal{W}|$ denotes the manifold $\mathcal{B}_{\mathcal{W}} \cup \mathcal{P}_{\mathcal{W}}$. Each component of $\mathcal{B}_{\mathcal{W}}$ is a solid torus. Each component P of $\mathcal{P}_{\mathcal{W}}$ is equipped with the structure of an I-bundle over a connected 2-manifold; we denote the associated ∂I-bundle by $\partial_h P$, and the set $\overline{\partial P - \partial_h P}$ by $\partial_v P$.

4.1. Note that if F is any component of $\partial|\mathcal{W}|$, then $\chi(F)$ is the sum of the Euler characteristics of the components of $\partial_h \mathcal{P}$ contained in F. Since by [1, Definition 2.2] every binding of $\mathcal{W}$ meets at least one page, F must contain at least one component of $\partial_h \mathcal{P}$. Hence if every page of $\mathcal{W}$ has strictly negative Euler characteristic, then $\chi(F) < 0$ for every component F of $\partial|\mathcal{W}|$.

Our next result, Proposition 4.3, gives a way of computing $H_2(|\mathcal{W}|, \mathcal{B}_0; \mathbb{Z}_2)$, where $\mathcal{W}$ is a book of I-bundles and $\mathcal{B}_0$ is a union of certain bindings. For this purpose we need some notation.

Notation 4.2. Let $\mathcal{W}$ be a book of I-bundles, and let $\mathcal{B}_0$ be a (possibly empty) union of components of $\mathcal{B} = \mathcal{B}_{\mathcal{W}}$. Let us set $\mathcal{B}_1 = \mathcal{B} - \mathcal{B}_0$. We shall denote by $C_1(\mathcal{W}, \mathcal{B}_0)$ the free $\mathbb{Z}_2$-module generated by the components of $\mathcal{B}_1$, and by $C_2(\mathcal{W}, \mathcal{B}_0)$ the free $\mathbb{Z}_2$-module generated by the pages of $\mathcal{W}$.

(We think of $C_1(\mathcal{W}, \mathcal{B}_0)$ and $C_2(\mathcal{W}, \mathcal{B}_0)$ as being analogous to the groups of 1-chains and 2-chains for the chain complex used to compute the relative homology of a pair of CW-complexes. Here the bindings and pages of $\mathcal{W}$ play the roles of 1-cells and 2-cells respectively. From this point of view it is natural that $C_2(\mathcal{W}, \mathcal{B}_0)$ should be independent of $\mathcal{B}_0$—as is apparent from the formal definition—since $|\mathcal{B}_0|$ contains no pages of $\mathcal{W}$.)

For each component B of $\mathcal{B}_1$, let us denote by $d(B)$ the image in $\mathbb{Z}_2$ of the degree of B; and for each component B of $\mathcal{B}_1$ and each component A of $\mathcal{A}_{\mathcal{W}}$, let us define $\delta_{A,B} \in \mathbb{Z}_2$ to be 1 if $A \subset \partial B$ and 0 otherwise. We define the boundary homomorphism $\Delta_{\mathcal{W},\mathcal{B}_0} : C_2(\mathcal{W}, \mathcal{B}_0) \to C_1(\mathcal{W}, \mathcal{B}_0)$ by setting $\Delta_{\mathcal{W},\mathcal{B}_0}(P) = \sum_{A,B} \delta_{A,B} d(B) B$ for each page P of $\mathcal{W}$, where A ranges over all vertical boundary annuli of P and B ranges over all components of $\mathcal{B}_1$. (Thus the boundary of the 2-chain P is the formal sum of the bindings of odd-valence and odd-degree which are not contained in B_0 and which meet P.) In the case $\mathcal{B}_0 = \emptyset$ we shall write $C_1(\mathcal{W})$, $C_2(\mathcal{W})$ and $\Delta_{\mathcal{W}}$ in place of $C_1(\mathcal{W}, \emptyset)$, $C_2(\mathcal{W}, \emptyset)$ and $\Delta_{\mathcal{W},\emptyset}$.

Proposition 4.3. *Suppose that $\mathcal{W}$ is a book of I-bundles, and that $\mathcal{B}_0$ is a (possibly empty) union of components of $\mathcal{B} = \mathcal{B}_{\mathcal{W}}$. Then $H_2(|\mathcal{W}|, \mathcal{B}_0; \mathbb{Z}_2)$ is isomorphic to the kernel of $\Delta_{\mathcal{W},\mathcal{B}_0} : C_2(\mathcal{W}, \mathcal{B}_0) \to C_1(\mathcal{W}, \mathcal{B}_0)$.*

Proof. In this proof all homology groups will be understood to have coefficients in $\mathbb{Z}_2$. We set $C_1 = C_1(\mathcal{W}, \mathcal{B}_0)$, $C_2 = C_2(\mathcal{W}, \mathcal{B}_0)$, and $\Delta = \Delta_{\mathcal{W},\mathcal{B}_0}$. We define $\mathcal{B}_1$, $d(B)$ and $\delta_{A,B}$ as in Notation 4.2. We set $W = |\mathcal{W}|$. Since $H_2(\mathcal{B}, \mathcal{B}_0) = 0$, we have a natural exact sequence

$$0 \longrightarrow H_2(W, \mathcal{B}_0) \longrightarrow H_2(W, \mathcal{B}) \longrightarrow H_1(\mathcal{B}, \mathcal{B}_0).$$

Hence $H_2(W, \mathcal{B}_0)$ is isomorphic to the kernel of the attaching map $H_2(W, \mathcal{B}) \to H_1(\mathcal{B}, \mathcal{B}_0)$. Setting $\mathcal{P} = \mathcal{P}_{\mathcal{W}}$ and $\mathcal{A} = \mathcal{A}_{\mathcal{W}}$, we have an excision isomorphism $j : H_2(\mathcal{P}, \mathcal{A}) \to H_2(W, \mathcal{B})$, and the domain $H_2(\mathcal{P}, \mathcal{A})$ may be identified with $\bigoplus_P H_2(P, P \cap \mathcal{A})$, where P ranges over the pages of $\mathcal{W}$. Each summand $H_2(P, P \cap \mathcal{A})$ is isomorphic to $H_2(S_P, \partial S_P)$, where S_P denotes the base of the I-bundle P, and therefore has rank 1. If $c_P \in H_2(W, \mathcal{B})$ denotes the image under j of the generator of $H_2(P, P \cap \mathcal{A})$, then the family $(c_P)_P$, indexed by the pages P of $\mathcal{W}$, is a basis for $H_2(W, \mathcal{B})$. Similarly, if for each binding $B \subset \mathcal{B}_1$ we denote by $e_B \in H_1(\mathcal{B}, \mathcal{B}_0)$ the image under inclusion of the generator of the rank-1 vector space $H_1(B)$, then the family $(e_B)_B$, indexed by the bindings B of $\mathcal{B}_1$, is a basis for $H_1(\mathcal{B}, \mathcal{B}_0)$. It is straightforward to check that for each page P of $\mathcal{W}$, the attaching map $H_2(W, \mathcal{B}) \to H_1(\mathcal{B}, \mathcal{B}_0)$ takes c_P to $\sum_{A,B} \delta_{A,B} d(B) e_B$, where A ranges over all vertical boundary annuli of P and B ranges over all components of $\mathcal{B}_1$. The conclusion of the proposition follows. □

Lemma 4.4. *Suppose that $\mathcal{W}$ is a book of I-bundles such that $\bar{\chi}(P) = 1$ for every page P of $\mathcal{W}$, and that $\mathcal{W}_0$ is a connected sub-book of $\mathcal{W}$. Let p_1 denote the number of pages of $\mathcal{W}$ that are not pages of $\mathcal{W}_0$. Then $\operatorname{rk}_2 H_1(|\mathcal{W}|, |\mathcal{W}_0|; \mathbb{Z}_2) \leqslant 2p_1$.*

Proof. We set $W = |\mathcal{W}|$ and $W_0 = |\mathcal{W}_0|$. If we define

$$\chi(W, W_0) = \sum_{i=0}^{2} (-1)^i \operatorname{rk}_2 H_i(W, W_0; \mathbb{Z}_2)$$

then the exact homology sequence of the pair (W, W_0) implies that

$$\chi(W, W_0) = \chi(W) - \chi(W_0).$$

Since each page of $\mathcal{W}$ has Euler characteristic -1, and the bindings of $\mathcal{W}$ and their frontiers are of Euler characteristic 0, we have $\chi(W) - \chi(W_0) = -p_1$. Hence

$$\operatorname{rk}_2 H_0(W, W_0; \mathbb{Z}_2) - \operatorname{rk}_2 H_1(W, W_0; \mathbb{Z}_2) + \operatorname{rk}_2 H_2(W, W_0; \mathbb{Z}_2) = -p_1.$$

Since W and W_0 are connected, we have $H_0(W, W_0; \mathbb{Z}_2) = 0$. To estimate $\operatorname{rk}_2 H_2(W, W_0; \mathbb{Z}_2)$ we consider the sub-book $\mathcal{W}_1$ of $\mathcal{W}$ consisting of all those pages of $\mathcal{W}$ that are not contained in W_0, and all bindings of $\mathcal{W}$ that meet pages not contained in W_0. We set $W_1 = |\mathcal{W}_1|$. We also set $\mathcal{B}_0 = W_0 \cap W_1$, so that $\mathcal{B}_0$ is in particular the union of a certain set of bindings of $\mathcal{W}_1$. It follows from Proposition 4.3 that $H_2(W_1, \mathcal{B}_0; \mathbb{Z}_2)$ is isomorphic to a subspace of $C_2(\mathcal{W}, \mathcal{B}_0)$, the free $\mathbb{Z}_2$-module generated by the pages of $\mathcal{W}$. By definition the dimension of $C_2(\mathcal{W}, \mathcal{B}_0)$ is p_1. Since $H_2(W_1, \mathcal{B}_0; \mathbb{Z}_2)$ is isomorphic to $H_2(W, W_0; \mathbb{Z}_2)$ by excision, we have $\operatorname{rk}_2 H_2(W, W_0; \mathbb{Z}_2) \leqslant p_1$. Therefore

$$-p_1 = -\operatorname{rk}_2 H_1(W, W_0; \mathbb{Z}_2) + \operatorname{rk}_2 H_2(W, W_0; \mathbb{Z}_2) \leqslant -\operatorname{rk}_2 H_1(W, W_0; \mathbb{Z}_2) + p_1,$$

from which the conclusion of the lemma follows. □

Lemma 4.5. *Let $m \geqslant 2$ be an integer. Suppose that $\mathcal{W}$ is a book of I-bundles such that $\bar{\chi}(P) = 1$ for every page P of $\mathcal{W}$, and such that $\bar{\chi}(|\mathcal{W}|) = 2m$. Suppose also that $H_2(|\mathcal{W}|; \mathbb{Z}_2)$ has dimension at least m. Then $\mathcal{W}$ has a connected sub-book $\mathcal{W}_0$ such that $\bar{\chi}(|\mathcal{W}_0|) = m$ and $H_2(|\mathcal{W}_0|; \mathbb{Z}_2) \neq 0$.*

Proof. Since $\bar{\chi}(P) = 1$ for every page P of $\mathcal{W}$, the number of pages of every sub-book $\mathcal{Y}$ of $\mathcal{W}$ is equal to $\bar{\chi}(|\mathcal{Y}|)$. In particular, $\mathcal{W}$ has exactly $2m$ pages. In Notation 4.2 it follows that $\mathrm{rk}_2 C_2(\mathcal{W}) = 2m$.

According to Proposition 4.3, $H_2(|\mathcal{W}|; \mathbb{Z}_2)$ is isomorphic to the kernel H of $\Delta_{\mathcal{W}} : C_2(\mathcal{W}) \to C_1(\mathcal{W})$. The hypothesis of the lemma therefore implies that $\mathrm{rk}_2 H \geqslant m$.

According to the definition of $C_2(\mathcal{W})$ (see Notation 4.2), the set $\mathcal{U}$ of pages of $\mathcal{W}$ is canonically identified with a basis of $C_2(\mathcal{W})$. Since $\mathrm{rk}_2 H \geqslant m$, Proposition 3.1 gives an element α of H such that $0 < \|\alpha\|_{\mathcal{U}} \leqslant m$.

In the notation of Section 3, we set $S = S_{\mathcal{U}}(\alpha)$, so that $0 < \# S \leqslant m$. We define $\mathcal{Z}$ to be the sub-book of $\mathcal{W}$ whose pages are the elements of S, and whose bindings are the bindings of $\mathcal{W}$ that meet pages in the set S. We set $Z = |\mathcal{Z}|$. We have $Z \neq \emptyset$ since $\# S > 0$.

Let $Z_1, \dots, Z_r$ denote the connected components of Z, where $r \geqslant 1$. Then for $i = 1, \dots, r$ we have $Z_i = |\mathcal{Z}_i|$ for some connected sub-book $\mathcal{Z}_i$ of $\mathcal{W}$. We denote by $S_i \subset S$ the set of all pages of $\mathcal{W}$ that belong to $\mathcal{Z}_i$, and we set $\alpha_i = \sum_{u \in S_i} u$, so that $\alpha = \alpha_1 + \cdots + \alpha_r$.

Let X denote the set of bindings of $\mathcal{W}$ that are contained in $\mathcal{Z}$, and for $i = 1, \dots, r$ let X_i denote the set of bindings of $\mathcal{W}$ that are contained in $\mathcal{Z}_i$. Let A and A_i denote, respectively, the subspaces of $C_1(\mathcal{W})$ spanned by X and X_i. Then X is the disjoint union of $X_1, \dots, X_n$ and hence A is the direct sum of $A_1, \dots, A_n$. Since $\alpha \in H$ we have

$$0 = \Delta_{\mathcal{W}}(\alpha) = \sum_{i=1}^{r} \Delta_{\mathcal{W}}(\alpha_i),$$

where $\Delta_{\mathcal{W}}(\alpha_i) \in A_i$ for $i = 1, \dots, r$. Since the sum $A_1 + \cdots + A_r$ is direct, it follows that $\Delta_{\mathcal{W}}(\alpha_i) = 0$ for $i = 1, \dots, r$, so that $\alpha_1, \dots, \alpha_r \in H$. The α_i are non-zero since each component Z_i contains at least one page of $\mathcal{W}$.

We have $\bar{\chi}(Z_1) \leqslant \bar{\chi}(Z) = \|\alpha\|_{\mathcal{U}} \leqslant m$. We set $k = m - \bar{\chi}(Z_1) \geqslant 0$, and recursively define sub-books $\mathcal{Y}_j$ of $\mathcal{W}$ for $0 \leqslant j \leqslant k$, with $\bar{\chi}(|\mathcal{Y}_j|) = \bar{\chi}(Z_1) + j$, as follows. Set $\mathcal{Y}_0 = \mathcal{Z}_1$. If $0 \leqslant j < k$ and $\mathcal{Y}_j$ has been defined, then $\mathcal{Y}_j$ is a proper sub-book of $\mathcal{W}$ since $\bar{\chi}(|\mathcal{Y}_j|) = \bar{\chi}(Z_1) + j < m < 2m = \bar{\chi}(W)$. Since W is connected, $|\mathcal{Y}_j|$ must meet some page P of W which is not a page of $\mathcal{Y}_j$. Define $\mathcal{Y}_{j+1}$ to be the sub-book of $\mathcal{W}$ consisting of the pages and bindings of $\mathcal{Y}_j$ together with the page P and the bindings of $\mathcal{W}$ that meet P. Then

$$\bar{\chi}\big(|\mathcal{Y}_{j+1}|\big) = \bar{\chi}\big(|\mathcal{Y}_j|\big) + \bar{\chi}(P) = \bar{\chi}\big(|\mathcal{Y}_j|\big) + 1 = \bar{\chi}(Z_1) + j + 1,$$

and the recursive definition is complete.

Now set $\mathcal{W}_0 = \mathcal{Y}_k$, so that $\bar{\chi}(\mathcal{W}_0|) = m$. Since the bindings and pages of $\mathcal{W}_0$ are bindings and pages of $\mathcal{W}$, the vector spaces $C_1(\mathcal{W}_0)$ and $C_2(\mathcal{W}_0)$ are naturally identified with subspaces of $C_1(\mathcal{W})$ and $C_2(\mathcal{W})$. The boundary homomorphism $\Delta_{\mathcal{W}_0} : C_2(\mathcal{W}_0) \to C_1(\mathcal{W}_0)$ is the restriction of $\Delta_{\mathcal{W}}$ to $C_2(\mathcal{W}_0)$. Hence the kernel of $\Delta_{\mathcal{W}_0}$ is $H \cap C_1$. The latter subspace contains the non-zero element α_1, and so $\Delta_{\mathcal{W}_0}$ has non-trivial kernel. It now follows from Proposition 4.3 that $H_2(W_0; \mathbb{Z}_2) \neq 0$. □

Lemma 4.6. *Suppose that $\mathcal{W}$ is a book of I-bundles such that $\bar{\chi}(P) > 0$ for every page of $\mathcal{W}$. Then there is a book of I-bundles $\mathcal{W}'$ such that $|\mathcal{W}'| = |\mathcal{W}|$, and such that $\bar{\chi}(P) = 1$ for every page P of $\mathcal{W}$.*

Proof. Set $W = |\mathcal{W}|$ and $\mathcal{P} = \mathcal{P}_{\mathcal{W}}$. Let S denote the base of the I-bundle $\mathcal{P}$, and let $q : \mathcal{P} \to S$ denote the bundle map. Since every component of S has negative Euler characteristic, there is a closed 1-manifold $\mathcal{C} \subset S$ such that every component of $S - \mathcal{C}$ has Euler characteristic -1. Let $\mathcal{N}$ be a regular neighborhood of $\mathcal{C}$ in S. Set $\mathcal{B}' = q^{-1}(\mathcal{N})$ and $\mathcal{P}' = q^{-1}(\overline{S - \mathcal{N}})$. Then $\mathcal{P}'$ inherits an I-bundle structure from $\mathcal{P}$, and we need only set $\mathcal{W}' = (W, \mathcal{B}', \mathcal{P}')$. □

Lemma 4.7. *Let $m \geqslant 1$ be an integer. Suppose that M is a closed, orientable, irreducible 3-manifold which is $(m+1)$-small (Definition 2.5). Suppose that $\mathcal{W}$ is a book of I-bundles with $W \doteq |\mathcal{W}| \subset M$, that $\bar{\chi}(P) = 1$ for every page P of $\mathcal{W}$, and that $\bar{\chi}(|\mathcal{W}|) = 2m$. Suppose also that $H_2(|\mathcal{W}|; \mathbb{Z}_2)$ has dimension at least m. Then $\mathcal{W}$ has a sub-book $\mathcal{W}_0$ such that*

(1) $\bar{\chi}(|\mathcal{W}_0|) = m$, *and*
(2) *the inclusion homomorphism $H_1(|\mathcal{W}_0|; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ is either surjective or has image of rank at most* $\max(m, 2)$.

Proof. We first consider the case $m \geqslant 2$. In this case, according to Lemma 4.5, $\mathcal{W}$ has a connected sub-book $\mathcal{W}_0$ such that $W_0 = |\mathcal{W}_0|$ satisfies $\bar{\chi}(W_0) = m$ and $H_2(W_0; \mathbb{Z}_2) \neq 0$.

If it happens that the inclusion homomorphism $\pi_1(W_0) \to \pi_1(M)$ is surjective, then in particular the inclusion homomorphism $H_1(W_0; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ is surjective, so that the conclusion of the lemma holds.

Now suppose the inclusion homomorphism $\pi_1(W_0) \to \pi_1(M)$ is not surjective. According to 4.1, no component of ∂W_0 is a sphere. Hence, in the notation of Definition 2.2 we have $W_0 \in \mathcal{X}_M$.

The manifold W_0 is not a handlebody, since $H_2(W_0; \mathbb{Z}_2) \neq 0$. According to 4.1, no component of ∂W_0 is a torus. The hypotheses of Proposition 2.8 are now seen to hold with $Y = W_0$ and $c = m$. (The condition that M is $(m+1)$-small is a hypothesis of the present lemma.) It therefore follows from Proposition 2.8 that $i(W_0) \geqslant 0$. According to the definition of $i(X)$ given in Definition 2.2, this means that $r(W_0) \leqslant \bar{\chi}(W_0)$, where $r(W_0)$ is the rank of the inclusion homomorphism

$\pi_1(W_0) \to \pi_1(M)$. In particular, the inclusion homomorphism $H_1(W_0; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ has rank at most $\bar{\chi}(W_0) = m$. This completes the proof of the lemma in the case $m \geqslant 2$.

We now consider the case $m = 1$. In this case we select a page P_0 of $\mathcal{W}$ and define the sub-book $\mathcal{W}_0$ to consist of P_0 and the bindings of $\mathcal{W}$ that meet P_0. Then $W_0 \doteq |\mathcal{W}_0|$ is connected and $\bar{\chi}(W_0) = 1$.

If it happens that the inclusion homomorphism $\pi_1(W_0) \to \pi_1(M)$ is surjective, then in particular the inclusion homomorphism $H_1(W_0; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ is surjective, so that the conclusion of the lemma holds.

Now suppose the inclusion homomorphism $\pi_1(W_0) \to \pi_1(M)$ is not surjective. According to 4.1, no component of ∂W_0 is a sphere. Hence, we have $W_0 \in \mathcal{X}_M$. The hypotheses of Proposition 2.7 are now seen to hold with $Y = W_0$ and $c = 1$. It therefore follows from Proposition 2.7 that $i(W_0) \geqslant -1$, i.e. that $r(W_0) \leqslant \bar{\chi}(W_0) + 1 = 2$. In particular, the inclusion homomorphism $H_1(W_0; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ has rank at most 2. Thus the lemma is proved in all cases. □

Proposition 4.8. *Let $m \geqslant 1$ be an integer. Suppose that M is a closed, orientable, irreducible 3-manifold which is $(m+1)$-small. Suppose that $\mathcal{W}$ is a book of I-bundles with $W \doteq |\mathcal{W}| \subset M$, that $\bar{\chi}(P) > 0$ for every page P of $\mathcal{W}$, and that $\bar{\chi}(|\mathcal{W}|) = 2m$. Then the image of the inclusion homomorphism $H_1(W; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ has dimension at most* $\max(3m, 4)$.

Proof. According to Lemma 4.6 we may assume without loss of generality that $\bar{\chi}(P) = 1$ for every page P of $\mathcal{W}$.

We shall let T denote the image of the inclusion homomorphism $j : H_1(W; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$.

We consider first the case in which $H_2(W; \mathbb{Z}_2)$ has dimension at most $m - 1$. In this case we note that

$$2m = \bar{\chi}(W) = -\operatorname{rk}_2 H_0(W; \mathbb{Z}_2) + \operatorname{rk}_2 H_1(W; \mathbb{Z}_2) - \operatorname{rk}_2 H_2(W; \mathbb{Z}_2) \geqslant -m + \operatorname{rk}_2 W,$$

so that $\operatorname{rk}_2 W \leqslant 3m$. It follows immediately that $\operatorname{rk}_2 T \leqslant 3m$ in this case.

There remains the case in which $H_2(W; \mathbb{Z}_2)$ has dimension at least m. In this case, according to Lemma 4.7, there is a sub-book $\mathcal{W}_0$ of $\mathcal{W}$ such that $\bar{\chi}(|\mathcal{W}_0|) = m$, and such that the inclusion homomorphism $j_0 : H_1(|\mathcal{W}_0|; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ either is surjective or has image of rank at most $\max(m, 2)$.

Set $W_0 = |\mathcal{W}_0|$. By [1, Lemma 2.11], we have

$$\operatorname{rk}_2(W_0) \leqslant 2\bar{\chi}(W) + 1 = 2m + 1.$$

Hence in the subcase where j_0 is surjective, we have $\operatorname{rk}_2 H_1(M; \mathbb{Z}_2) \leqslant \operatorname{rk}_2 H_1(W_0; \mathbb{Z}_2) \leqslant 2m + 1 \leqslant 3m$; since T is a subspace of $H_1(M; \mathbb{Z}_2)$, we in particular have $\operatorname{rk}_2 T \leqslant 3m$ in this subcase.

Finally we consider the subcase in which $T_0 \doteq j_0(H_1(W_0; \mathbb{Z}_2))$ has dimension at most $\max(m, 2)$. Since $\bar{\chi}(W) = 2m$ and $\bar{\chi}(W_0) = m$, and since $\bar{\chi}(P) = 1$ for each page of W, the number of pages of $\mathcal{W}$ that are not pages of $\mathcal{W}_0$ is equal to m. Hence by Lemma 4.4, we have $\operatorname{rk}_2 H_1(W, W_0; \mathbb{Z}_2) \leqslant 2m$.

Let L denote the cokernel of the inclusion homomorphism $H_1(W_0; \mathbb{Z}_2) \to H_1(W; \mathbb{Z}_2)$. The natural surjection from $H_1(W; \mathbb{Z}_2)$ to T induces a surjection from L to T/T_0. Hence

$$\operatorname{rk}_2 T - \operatorname{rk}_2 T_0 = \operatorname{rk}_2(T/T_0) \leqslant \operatorname{rk}_2 L \leqslant \operatorname{rk}_2 H_1(W, W_0; \mathbb{Z}_2) \leqslant 2m.$$

Since $\operatorname{rk}_2 T_0 \leqslant \max(m, 2)$, it follows that

$$\operatorname{rk}_2 T \leqslant 2m + \max(m, 2) = \max(3m, 4),$$

as required. □

5. De-singularizing surfaces

This section is devoted to the proof of Theorem 1.1, which was stated in the Introduction.

Proof of Theorem 1.1. We use the terminology of [1]. Applying [1, Proposition 8.11], we find a good tower

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n),$$

with base M_0 homeomorphic to M and with some height $n \geqslant 0$, such that N_n contains a connected incompressible closed surface F of genus $\leqslant g$. According to the definition of a good tower, ∂N_n is incompressible (and, *a priori*, possibly empty) in M_n. Hence N_n is π_1-injective in M_n. Since F is incompressible in N_n, it follows that it is also incompressible in M_n.

Since M is simple it follows from [1, Lemma 8.12] that all the M_j and N_j are simple.

Let k denote the least integer in $\{0, \ldots, n\}$ for which M_k contains a closed incompressible surface S_k of genus at most g. To prove the theorem it suffices to show that $k = 0$. Let h denote the genus of S_k. Since M_k is simple we have $h \geqslant 2$.

Suppose that $k \geqslant 1$. The minimality of k implies that M_{k-1} contains no closed incompressible surface of genus at most g. In particular:

5.0.1. *M_{k-1} contains no closed incompressible surface of genus at most h.*

From 5.0.1 it follows that, in particular,

5.0.2. *M_{k-1} is h-small.*

We now evoke [1, Proposition 4.4], which states that if $\widetilde{N}$ is a 2-sheeted covering of a simple, compact, orientable 3-manifold N, and if $\widetilde{N}$ contains a closed, incompressible surface of a given genus $h \geqslant 2$, then either (1) N contains a closed, connected, incompressible surface of genus at most h, or (2) N is closed and there is a connected book of I-bundles $\mathcal{W}$ with $W = |\mathcal{W}| \subset N$ such that $\bar{\chi}(W) = 2h - 2$, every page of $\mathcal{W}$ has strictly negative Euler characteristic, and every component of $\overline{N - W}$ is a handlebody. Observe that the hypotheses of [1, Proposition 4.4] hold in the present situation if we set $N = N_{k-1}$ and $\widetilde{N} = M_k$.

Suppose that alternative (1) of the conclusion of [1, Proposition 4.4] holds in the present situation, i.e. that N_{k-1} contains an incompressible closed surface S_{k-1} with genus$(S_{k-1}) \leqslant h \leqslant g$. According to the definition of a good tower, ∂N_{k-1} is an incompressible (and possibly empty) surface in M_{k-1}. Hence N_{k-1} is π_1-injective in M_{k-1}. Since S_{k-1} is incompressible in N_{k-1}, it follows that it is also incompressible in M_{k-1}. This contradicts 5.0.1.

Now suppose that alternative (2) of the conclusion of [1, Proposition 4.4] holds in the present situation, i.e.:

5.0.3. *N_{k-1} is closed and there is a connected book of I-bundles $\mathcal{W}$ with $W = |\mathcal{W}| \subset N_{k-1}$ such that $\bar{\chi}(W) = 2(h-1)$, every page of $\mathcal{W}$ has strictly negative Euler characteristic, and every component of $\overline{N_{k-1} - W}$ is a handlebody.*

Since N_{k-1} is closed we have $N_{k-1} = M_{k-1}$.

It now follows from 5.0.2 and 5.0.3 that the hypotheses of Proposition 4.8 hold with $m = h - 1$, and with M_{k-1} in place of M. Hence by Proposition 4.8, the image of the inclusion homomorphism $H_1(W; \mathbb{Z}_2) \to H_1(M_{k-1}; \mathbb{Z}_2)$ has dimension at most $\max(3h-3, 4)$. On the other hand, since by 5.0.3 every component of $\overline{N_{k-1} - W}$ is a handlebody, the inclusion homomorphism $H_1(W; \mathbb{Z}_2) \to H_1(M_{k-1}; \mathbb{Z}_2)$ is surjective. Hence

$$\operatorname{rk}_2 M_{k-1} \leqslant \max(3h-3, 4) \leqslant \max(3g-3, 4).$$

On the other hand, since by hypothesis we have $\operatorname{rk}_2 M_0 \geqslant \max(3g-1, 6)$, it follows from [1, Lemma 8.5] that for any index j such that $0 \leqslant j \leqslant n$ and such that M_j is closed, we have $\operatorname{rk}_2 M_j \geqslant \max(3g-2, 5)$. This is a contradiction, and the proof is complete. □

6. An example

In this section we investigate the extent to which Theorem 1.1 is sharp. Our discussion focuses on the case $g = 2$ of Theorem 1.1, although the methods can be applied more generally. To show that the theorem is sharp for $g = 2$ one would need an example of a closed simple 3-manifold M with $\operatorname{rk}_2 M = \max(3g-2, 5) = 5$, such that $\pi_1(M)$ contains a genus-2 surface group but M contains no closed, incompressible surface of genus 2. Proposition 6.3 below asserts the existence (and the proof gives an explicit example) of a closed simple 3-manifold M with $\operatorname{rk}_2 M = 4$, such that $\pi_1(M)$ contains a genus-2 surface group but M contains no closed, incompressible surface whatever. We will also show why our construction cannot give a similar example in which $\operatorname{rk}_2 M$ is 5 rather than 4; however, we have no reason to think that such an example does not exist.

Our example is based on a Dehn surgery construction, and we shall use notation that is standard in the study of Dehn surgery. If Q is a compact, orientable 3-manifold whose boundary is a torus, we define a *slope* for Q to be an isotopy class of unoriented simple closed curves in Q. If α and β are slopes, we denote their geometric intersection number by $\Delta(\alpha, \beta)$. We define an *essential surface* in Q to be a π_1-injective, properly embedded, orientable surface which is not boundary-parallel. If S is an essential surface, all its boundary components represent the same slope, called the *boundary slope* of S.

The following result is essentially due to Cooper-Long and Li.

Theorem 6.1. *Let Q be a simple 3-manifold whose boundary is a single torus. Let $S \subset Q$ be an essential surface with two boundary components. Suppose that S is not a fiber or semifiber. Let s denote its boundary slope. Then there is an integer Γ such that for every slope r with $\Delta(r, s) \geqslant \Gamma$, the fundamental group of the Dehn-filled manifold $Q(r)$ contains an isomorphic copy of $\pi_1(T)$, where T is a closed orientable surface with $\chi(T) = \chi(S)$.*

Proof. This follows from the proof of [16, Theorem 1.2]. (See also [4] and [5].) The statement of [16, Theorem 1.2] does not contain the information that $\chi(T) = \chi(S)$, but it follows from the proof because T is constructed from S, as in [12], by adding a singular annulus joining the two boundary components of S. □

Theorem 6.1 will be applied via the following result:

Proposition 6.2. *Let Q be a simple 3-manifold whose boundary is a single torus. Suppose that Q contains no closed incompressible surface of genus > 1. Let $S \subset Q$ be a separating essential surface with $\chi(S) = -2$. Suppose that S is not a semifiber. Then*

(1) *Q has Heegaard genus $\leqslant 4$; and*

(2) *there are infinitely many slopes r such that $M := Q(r)$ has the following properties*:
- $\pi_1(M)$ *contains a genus-2 surface group*;
- *M contains no closed incompressible surface*; *and*
- $\mathrm{rk}_2 M = \mathrm{rk}_2 Q$.

Proof. To prove that conclusion (1) holds we will construct a Heegaard splitting of the form $Q = V \cup W$ where V is a compression body and W is a handlebody of genus 4.

Let A denote the union of three disjoint properly embedded arcs in S such that $S - A$ is simply-connected. Let V be a regular neighborhood of $\partial M \cup A$. Then V is a compression body such that $\partial_- V = \partial M$ and $\partial_+ V$ has genus 4. By adding a 2-handle to V one obtains a regular neighborhood N of $\partial M \cup S$. The frontier of N consists of two surfaces F_1 and F_2 of genus 2. Since Q is simple and contains no incompressible surface of genus > 1, $Q - N$ is a union of two disjoint handlebodies of genus 2. Thus $W = Q - V$ consists of two handlebodies joined by a 1-handle, and hence is a handlebody of genus 4.

To prove (2), we let s denote the boundary slope of S. Let α be an indivisible element of $H_1(\partial M; \mathbb{Z}_2)$ which belongs to the kernel of the inclusion homomorphism $H_1(\partial M; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$. Let us extend α to a basis (α, β) for $H_1(\partial M; \mathbb{Z}_2)$. As there are infinitely many choices for β we may take $\beta \neq s$. For each positive integer n, the primitive homology class $\alpha + 2n\beta$ determines a slope r_n. Since $\alpha + 2n\beta$ lies in the kernel of the inclusion homomorphism $H_1(\partial M; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$, we have $\mathrm{rk}_2 Q(r_n) = \mathrm{rk}_2 Q$ for each n. On the other hand, we have

$$\Delta(r_n, s) \geqslant n\Delta(\beta, s) - \Delta(\alpha, s)$$

for each n. Here $\Delta(\beta, s) \neq 0$ since $\beta \neq s$, and so $\Delta(r_n, s) \to \infty$ as $n \to \infty$. Hence Theorem 6.1 guarantees that for any sufficiently large n the group $\pi_1(Q(r_n))$ contains an isomorphic copy of $\pi_1(T)$, where T is a closed orientable surface with $\chi(T) = \chi(S) = -2$; that is, $\pi_1(Q(r_n))$ contains a genus-2 surface group for all sufficiently large n.

On the other hand, by a theorem of Hatcher [13], there are only finitely many boundary slopes for M. Since Q is simple and contains no closed incompressible surface of genus > 1, the manifold $Q(r)$ cannot contain a closed incompressible surface unless r is a boundary slope. Hence for sufficiently large n the manifold $Q(r_n)$ contains no closed incompressible surface. □

The next result produces our example.

Proposition 6.3. *There exists a simple, closed, orientable* 3*-manifold M with* $\mathrm{rk}_2 M = 4$, *such that* $\pi_1(M)$ *contains a genus-*2 *surface group but M contains no incompressible surface.*

Proof. The Hodgson–Weeks census of cusped hyperbolic 3-manifolds has been extended by Thistlethwaite [18] to include manifolds which have ideal triangulations with eight tetrahedra. We let Θ denote the ideal-triangulated manifold `t12045` in the Thistlethwaite census.

The program SnapPy [8] reports that Θ is hyperbolic with finite volume and one cusp, and that $H_1(Q; \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. Using the program t3m [7] to enumerate spunnormal surfaces, in the sense of [19], with respect to $\mathcal{T}$, one finds a surface Σ_0 with Euler characteristic -1 and one end.

The compact core Q of Θ is a simple manifold with one boundary torus. Truncating Σ_0 gives a properly embedded surface $S_0 \subset Q$ having Euler characteristic -1 and one boundary component. Dehn filling on the boundary slope of S_0 produces a manifold with first homology $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In particular, the boundary curve of S_0 is non-trivial in $H_1(Q; \mathbb{Z})$, so S_0 is a Klein bottle with one disk removed. We let S denote the frontier of a regular neighborhood V of S_0, so that S is an orientable surface with two boundary components and genus 1.

The t3m program reports that Thistlethwaite's triangulation $\mathcal{T}$ of Θ admits a taut structure, in the sense of [15]. The definition of a taut structure involves an assignment of a transverse orientation to every 2-simplex of Q. One of the conditions that these transverse orientations are stipulated to satisfy is that every 3-simplex has two faces for which the transverse orientation is inward and two for which it is outward. In particular each 3-simplex has a distinguished pair of opposite edges, namely the common edge of the two outward faces and the common edge of the two inward faces. Thus there is a distinguished normal quadrilateral type in each 3-simplex, namely the one which is disjoint from the distinguished edges.

The t3m program verifies that for a suitable taut structure on $\mathcal{T}$, the spunnormal surface Σ_0 has the property that all of its quadrilaterals are of distinguished type. It is clear that S may be obtained by truncating a spunnormal surface Σ which has the same quadrilateral types as Σ_0. In particular all the quadrilaterals of Σ are of distinguished type.

An unpublished theorem of Dunfield [11] implies that if an orientable spunnormal surface in a taut ideal triangulation has the property that all its quadrilaterals are of distinguished type, then the properly embedded surface obtained from it by truncation is essential. Thus we see that S is essential.

The surface S separates Q since it is the frontier of V. If S is a semifiber then $W := \overline{Q - V}$ is a twisted I-bundle over a surface with associated ∂I-bundle S, and hence $H_1(W, S; \mathbb{Z}_2) \cong \mathbb{Z}_2$. By excision it follows that $H_1(Q, V; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Since Q and V are connected, it follows from the long exact homology sequence of the pair (Q, V) that $\mathrm{rk}_2 Q \leqslant 1 + \mathrm{rk}_2 V = 1 + \mathrm{rk}_2 S_0 = 3$. This is a contradiction since we have seen that $\mathrm{rk}_2 Q = 4$. Thus we have shown that S is not a semifiber.

The t3m program also verifies that all closed spunnormal surfaces with respect to the ideal triangulation $\mathcal{T}$ bound handlebodies, and hence are compressible. Hence Q has no closed incompressible surfaces.

It now follows from Proposition 6.2 that there are infinitely many distinct Dehn surgeries on Q which produce manifolds M with $\mathrm{rk}_2 M = 4$, such that $\pi_1(M)$ contains a genus-2 surface group but M contains no closed incompressible surface. □

Remark 6.4. The proof of Proposition 6.3 that we have given requires constructing the manifold M by a Dehn filling from a manifold Q satisfying the hypotheses of Proposition 6.2. Conclusion (1) of Proposition 6.2 asserts that any such manifold Q must have a Heegaard splitting of genus at most 4. Since Q has connected boundary, one of the two compression bodies in this splitting will be a handlebody. Thus Q is obtained from a handlebody of genus at most 4 by adding 2-handles. This implies that $\mathrm{rk}_2 Q \leqslant 4$, and hence that $\mathrm{rk}_2 M \leqslant 4$ for any manifold M obtained from Q by a Dehn filling. This is why our method cannot furnish an example with $\mathrm{rk}_2 M = 5$, as it would have to do in order to show that Theorem 1.1 is sharp when $g = 2$.

Remark 6.5. Thurston's Dehn filling theorem implies that the proof of Proposition 6.3 gives infinitely many non-homeomorphic manifolds with the stated properties.

7. Non-fibroid surfaces

In this section we will establish a slightly stronger version of Theorem 1.1, Proposition 7.2, which will be useful for volume estimates.

Definition 7.1. Following the terminology that we introduced in [10], we define a *fibroid* in a closed, orientable topological 3-manifold M to be a closed incompressible surface $S \subset M$ such that each component of the manifold-with-boundary obtained by splitting M along S has the form $|\mathcal{W}|$ for some book of I-bundles $\mathcal{W}$ whose pages are all of negative Euler characteristic.

Proposition 7.2. *Let g be an integer $\geqslant 2$. Let M be a closed simple 3-manifold such that $\mathrm{rk}_2 M \geqslant \max(3g-1, 6)$ and $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group. Then M contains a closed, incompressible surface which has genus at most g and is not a fibroid.*

Proof. According to Theorem 1.1, M contains a closed, incompressible surface of some genus $h \leqslant g$. We may take h to be minimal in the sense that M contains no closed, incompressible surface of genus $< h$. Since M is simple we have $h \geqslant 2$. We distinguish two cases.

Case I. There is a separating closed incompressible surface $S \subset M$ with genus h. We shall show that S is not a fibroid. Let W and W' denote the closures of the components of $M - S$. We have $\bar{\chi}(W) = \bar{\chi}(W') = h - 1$. Suppose that F is a fibroid, so that there are books of I-bundles $\mathcal{W}$ and $\mathcal{W}'$ whose pages are all of negative Euler characteristic, such that $|\mathcal{W}| = W$ and $|\mathcal{W}'| = W'$. It then follows from [1, 2.11] that $\mathrm{rk}_2 W \leqslant 2\chi(W) + 1 = 2h - 1$ and $\mathrm{rk}_2 W' \leqslant 2\chi(W') + 1 = 2h - 1$.

Consider the Mayer–Vietoris exact sequence

$$H_1(F) \xrightarrow{\iota_* + \iota'_*} H_1(W) \oplus H_1(W') \xrightarrow{\alpha} H_1(M) \xrightarrow{\beta} H_0(F) \xrightarrow{\iota_* + \iota'_*} H_0(W) \oplus H_0(W')$$

where coefficients are taken in $\mathbb{Z}_2$, and where ι and ι' are the inclusions of F into W and W'. Since F and W are path-connected, $\iota_* : H_0(F) \to H_0(W)$ is an isomorphism; hence $\beta = 0$, and α is surjective. It is a standard consequence of Poincaré–Lefschetz duality that the dimension of $\iota_*(H_1(F)) \subset H_1(W)$ is equal to the genus h of $F = \partial W$. Hence $(\iota_* + \iota'_*)(H_1(F))$ is a subspace of dimension at least h in $H_1(W) \oplus H_1(W')$. It follows that

$$\begin{aligned} \mathrm{rk}_2 M &\leqslant \mathrm{rk}_2\big(H_1(W; \mathbb{Z}_2) \oplus H_1(W'; \mathbb{Z}_2)\big) - h \\ &\leqslant (2h-1) + (2h-1) - h \\ &= 3h - 2 \\ &\leqslant 3g - 2, \end{aligned}$$

which contradicts the hypothesis.

Case II. There is no separating closed incompressible surface of genus h in M. By our choice of h, there is also no closed incompressible surface of genus $< h$ in M. By definition this means that M is h-small.

Our choice of h also guarantees that there is a closed incompressible surface S of genus h in M. Since we are in Case II, the surface S is non-separating. We shall show that S is not a fibroid.

Fix a regular neighborhood N of S in M, and set $W = \overline{M - N}$. Since S is non-separating, W is connected. We have $\bar{\chi}(W) = 2h - 2$. Suppose that S is a fibroid, so that there is a book of I-bundles $\mathcal{W}$ whose pages are all of negative Euler characteristic, such that $|\mathcal{W}| = W$. Since M is h-small, the hypotheses of Proposition 4.8 are now seen to hold with $m = h - 1$. Hence if T denotes the image of the inclusion homomorphism $H_1(W; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$, it follows from Proposition 4.8 that T has dimension at most $\max(3h - 3, 4)$.

If c is the class in $H_1(M; \mathbb{Z}_2)$ defined by a simple closed curve that crosses S in one point, then $H_1(M; \mathbb{Z}_2)$ is spanned by c and T. It follows that $H_1(M; \mathbb{Z}_2)$ has dimension at most $\max(3h - 2, 5)$. This contradicts the hypothesis. □

8. Volumes

In this section we will establish Theorem 1.2 which was stated in the Introduction. One of the ingredients is a result due to Agol, Storm, and Thurston from [2]. The information from [2] that we need is summarized in Theorem 9.4 of [1], which can be paraphrased as saying that if M is a closed orientable hyperbolic 3-manifold containing a connected incompressible closed surface which is not a fibroid, then $\text{Vol}(M) > 3.66$.

We also recall that a group Γ is said to be k-free, where k is a positive integer, if every subgroup of Γ having rank at most k is a free group. The following result provides the transition between the earlier sections of this paper and the applications to volumes, which include the proofs of Theorem 1.2 and of the corresponding result in [9].

Proposition 8.1. *Let $k \geqslant 3$ be an integer, and let M be a closed orientable simple 3-manifold such that $\text{rk}_2 M \geqslant \max(3k - 4, 6)$. Then either $\pi_1(M)$ is k-free, or M contains a closed incompressible surface of genus at most $k - 1$ which is not a fibroid.*

Proof. First consider the case in which $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group for some g with $1 < g \leqslant k - 1$. The hypothesis then implies that $\text{rk}_2 M \geqslant \max(3g - 1, 6)$, and it follows from Proposition 7.2 that M contains a closed, incompressible surface which is not a fibroid and has genus at most $g \leqslant k - 1$.

Now consider the case in which $\pi_1(M)$ has no subgroup isomorphic to a genus-g surface group for any g with $1 < g \leqslant k - 1$. In this case, since $\text{rk}_2 M \geqslant k + 2$, it follows from [3, Proposition 7.4 and Remark 7.5] that $\pi_1(M)$ is k-free. □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume that $\text{rk}_2 M \geqslant 6$. Then according to Proposition 8.1, either $\pi_1(M)$ is 3-free, or M contains a closed incompressible surface of genus at most 2 which is not a fibroid. If $\pi_1(M)$ is 3-free, it follows from [1, Corollary 9.3] that $\text{Vol}(M) > 3.08$. If M contains a closed incompressible surface which is not a fibroid, it follows from [1, Theorem 9.4] that $\text{Vol}(M) > 3.66$. In either case the hypothesis is contradicted. □

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