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SINGULAR SURFACES, MOD 2 HOMOLOGY, AND HYPERBOLIC VOLUME, I

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ABSTRACT. If M is a simple, closed, orientable 3-manifold such that $\pi_1(M)$ contains a genus-g surface group, and if $H_1(M;\mathbb{Z}_2)$ has rank at least $4g-1$, we show that M contains an embedded closed incompressible surface of genus at most g. As an application we show that if M is a closed orientable hyperbolic 3-manifold of volume at most 3.08, then the rank of $H_1(M;\mathbb{Z}_2)$ is at most 6.

1. INTRODUCTION AND GENERAL CONVENTIONS

Let M be any closed, orientable, hyperbolic 3-manifold. The volume of M is known to be an extremely powerful topological invariant, but its relationship to more classical topological invariants remains elusive. The main geometrical result of this paper, Theorem 9.6, asserts that if $\operatorname{Vol} M \leq 3.08$, then $H_1(M,\mathbb{Z}_2)$ has rank at most 6.

The Weeks-Hodgson census of closed hyperbolic 3-manifolds [19] contains two examples, `m135(-1,3)` and `m135(1,3)`, for which the volume is < 3.08 and the rank of the first homology with $\mathbb{Z}_2$ coefficients is 3. (They are both of volume $2.666745\ldots$, and they have integer first homology isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$, respectively.) There are no examples in that census for which the volume is < 3.08 and the rank of the first homology with $\mathbb{Z}_2$ coefficients is ≥ 4. Thus there is still a substantial gap between our results and the known examples. However, the bound on the rank of $H_1(M;\mathbb{Z}_2)$ given in this paper seems to be better by orders of magnitude than what could be readily deduced by previously available methods.

The proof of Theorem 9.6 relies on a purely topological result, Theorem 8.13, which states that if M is a closed 3-manifold which is simple (see Definition 1.10), if $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group for a given integer g, and if the rank of $H_1(M;\mathbb{Z}_2)$ is at least $4g-1$, then M contains a connected incompressible closed surface of genus g. This may be regarded as a partial analogue of Dehn's lemma for π_1-injective genus-g surfaces.

Theorem 9.6 will be proved in Section 9 by combining Theorem 8.13 with a number of deep geometric results. These include the Marden tameness conjecture,

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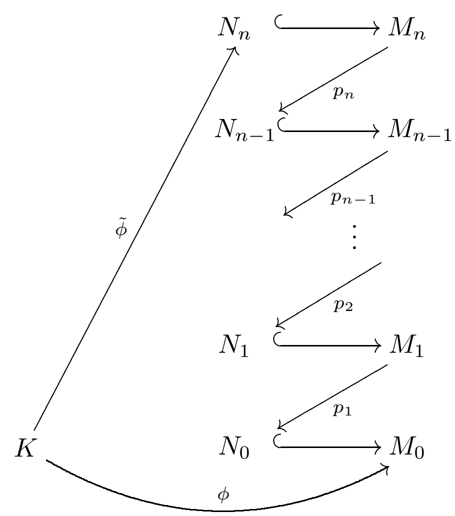
recently established by Agol [1] and by Calegari-Gabai [4]; a co-volume estimate for 3-tame, 3-free Kleinian groups due to Anderson, Canary, Culler and Shalen [3, Proposition 8.1]; and a volume estimate for hyperbolic Haken manifolds recently proved by Agol, Storm and Thurston [2].

The results of [2] depend in turn on estimates developed by Perelman in his work [12] on geometrization of 3-manifolds.

By refining the methods of this paper one can obtain improvements of Theorems 8.13 and 9.6. In particular, in the case $g = 2$, the lower bound of 7 for the rank of $H_1(M; \mathbb{Z}_2)$ in the hypothesis of Theorem 8.13 can be replaced by 6, and the upper bound of 6 in the conclusion of Theorem 9.6 can be replaced by 5. The relevant refinements will be explored systematically in [5].

Our strategy for proving Theorem 8.13 is based on the method of two-sheeted coverings used by Shapiro and Whitehead in their proof [14] of Dehn's lemma. (This method was inspired by Papakyriakopoulos's tower construction [11], and was systematized by Stallings [16].) We consider a π_1-injective genus-g singular surface in the 3-manifold M, i.e. a map $\phi : K \to M$, where K is a closed orientable genus-g surface and $\phi_\sharp$ is injective. One can construct a "tower"



where the M_j are simple (Definitions 1.10) 3-manifolds, N_j is a simple 3-dimensional submanifold of M_j for $j = 0, \dots, n$, the $p_j : M_j \to N_{j-1}$ are two-sheeted covering maps, $\tilde{\phi}_* : H_1(K; \mathbb{Z}_2) \to H_1(N_n; \mathbb{Z}_2)$ is surjective, and the diagram commutes up to homotopy. In general this diagram may contain both closed and bounded manifolds, but we use ideas from [13] to construct the tower in such a way that if $H_1(M, \mathbb{Z}_2)$ has rank $\geq 4g - 1$, then $H_1(M_j, \mathbb{Z}_2)$ has rank $\geq 4g - 2$ whenever M_j is closed. We also use ideas developed in [9], based on Simon's results [15] on compactification of covering spaces, to construct the tower in such a way that the (possibly empty and possibly disconnected) surface ∂N_j is incompressible in M_j for each $j \leq n$.

The manifold N_n always has non-empty boundary. This is obvious if $\partial M_n \neq \emptyset$. If M_n is closed, then $H_1(M_n; \mathbb{Z}_2)$ has rank at least $4g - 2$, whereas the surjectivity of $\tilde{\phi}_* : H_1(K;\mathbb{Z}_2) \to H_1(N_n;\mathbb{Z}_2)$ implies that the rank of $H_1(N_n;\mathbb{Z}_2)$ is at most $2g$. It follows that in this case N_n is a proper submanifold of M_n, and hence $\partial N_n \neq \emptyset$.

We in fact show, using elementary arguments based on Poincaré-Lefschetz duality, that if the map $\tilde{\phi}_* : H_2(K;\mathbb{Z}) \to H_2(N_n;\mathbb{Z})$ is trivial, then ∂N_n has a component F of genus at most g. In the case where $\tilde{\phi}_* : H_2(K;\mathbb{Z}) \to H_2(N_n;\mathbb{Z})$ is non-trivial, we use Gabai's results from [7] to show that N_n contains a non-separating incompressible closed surface F of genus at most g.

The rest of the proof consists of showing that if a given M_j, with $0 < j \leq n$, contains a closed incompressible surface of genus at most g, then N_{j-1} also contains such a surface. The surface in N_{j-1} will be incompressible in M_{j-1}, as well as in N_{j-1}, because ∂N_{j-1} is incompressible in M_{j-1}. It is at this step that we need to know that closed manifolds in the tower have first homology with $\mathbb{Z}_2$-coefficients of rank at least $4g-2$. Indeed, it is a consequence of Proposition 4.4 that if a 2-sheeted covering of a simple compact 3-manifold N contains a closed incompressible surface of genus at most g, then N itself must contain such a surface, unless N is closed and $H_1(N;\mathbb{Z}_2)$ has rank at most $4g-3$.

Proposition 4.4 involves the notion of a "book of I-bundles" which we define formally in Definitions 2.2. Books of I-bundles in PL 3-manifolds arise naturally as neighborhoods of "books of surfaces" (Definition 2.6). We may think of a book of surfaces as being constructed from a 2-manifold with boundary $\hat{\Pi}$, whose components have Euler characteristic ≤ 0, and a closed 1-manifold Ψ, by attaching $\partial\hat{\Pi}$ to Ψ by a covering map. The components of Ψ and $\Pi = \operatorname{int}\hat{\Pi}$ are, respectively, "bindings" and "pages." A book of I-bundles comes equipped with a corresponding decomposition into "pages" which are I-bundles over surfaces, and "bindings" which are solid tori. (In the informal discussion that we give in this Introduction, the extra structure defined by the decomposition will be suppressed from the notation.)

With these notions as background we shall now sketch the proof of Proposition 4.4. An incompressible surface F in a two-sheeted covering space of N, if it is in general position, projects to N via a map which has only double-curve singularities. After routine modifications one obtains a map from F to N with the additional property that its double curves are homotopically non-trivial. In particular, the image of such a map is a book of surfaces X. A regular neighborhood W of X in N is then a book of I-bundles, which has Euler characteristic $\geq 2-2g$ if F has genus at most g. Using the simplicity of N one can then produce a book of I-bundles V with $W \subset V \subset N$ and $\chi(W) \geq 2-2g$, such that each page of W has strictly negative Euler characteristic. (This step is handled by Lemma 2.5.)

We now distinguish two cases. In the case where some page P_0 of V has the property that $P_0 \cap \partial V$ is contained in a single component of ∂V, we show that by splitting bindings of the book of surfaces X, one can produce an embedded (possibly disconnected) closed, orientable surface S which is homologically non-trivial in N. Ambient surgery on S in N then produces a non-empty incompressible surface whose components have genus at most g. In the case where no such page P_0 exists, a Euler characteristic calculation shows that the boundary components of V have genus at most g. In this case, ambient surgery on ∂V produces a non-empty incompressible surface whose components have genus at most g. We show that this surface is non-empty unless V carries $\pi_1(N)$. But for a book of I-bundles V whose Euler characteristic is at least $2-2g$ and whose pages are all of negative

Euler characteristic, one can show that $H_1(V;\mathbb{Z}_2)$ has rank at most $4g-3$ (this is included in Lemma 2.11); so in the case where V carries $\pi_1(N)$, the rank of $H_1(N;\mathbb{Z}_2)$ is at most $4g-3$.

The details and background needed for the proof of Proposition 4.4 occupy Sections 2—4. Section 5 provides the combinatorial background needed to construct the tower, while Sections 6 and 7 provide the homological background. The application of Gabai's results mentioned above appears in Section 7. The material on towers proper, and the proof of the main topological theorem and its corollary, are given in Section 8, and the geometric applications are given in Section 9.

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The rest of this Introduction will be devoted to indicating some conventions that will be used in the rest of the paper.

1.1. In general, if X and Y are subsets of a set, we denote by $X \setminus Y$ the set of elements of X that do not belong to Y. In the case where we know that $Y \subset X$ and wish to emphasize this, we will write $X - Y$ for $X \setminus Y$.

1.2. A *manifold* may have a boundary. If M is a manifold, we shall denote the boundary of M by ∂M and its interior $M - \partial M$ by $\operatorname{int} M$.

In many of our results about manifolds of dimension ≤ 3 we do not specify a category. These results may be interpreted in the category in which manifolds are topological, PL or smooth, and submanifolds are, respectively, locally flat, PL or smooth. These three categories are equivalent in these low dimensions as far as classification is concerned. In much of the paper the proofs are done in the PL category, but the applications to hyperbolic manifolds in Section 9 are carried out in the smooth category.

1.3. A (possibly disconnected) codimension-1 submanifold S of a manifold M is said to be *separating* if M can be written as the union of two 3-dimensional submanifolds M_1 and M_2 such that $M_1 \cap M_2 = S$.

1.4. We shall say that a map of topological spaces $f : \mathcal{X} \to \mathcal{Y}$ is π_1-*injective* if for every path component X of $\mathcal{X}$, the map $f|X$ induces an injection from $\pi_1(X)$ to $\pi_1(Y)$, where Y is the path component of $\mathcal{Y}$ containing $f(X)$. We shall say that a subset A of a topological space X is π_1-*injective* in X if the inclusion map $A \to X$ is π_1-injective.

1.5. If X is a space having the homotopy type of a finite CW complex, the Euler characteristic of X will be denoted by $\chi(X)$. We have $\chi(X) = \sum_{j\in\mathbb{Z}} \dim_F H_j(X;F)$ for *any* field F: the sum is independent of F by virtue of the standard observation that it is equal to $\sum_{j\in\mathbb{Z}}(-1)^j c_j$, where c_j denotes the number of j-cells in a finite CW complex homotopy equivalent to X.

We shall often write $\bar{\chi}(X)$ as shorthand for $-\chi(X)$.

1.6. If x is a point of a compact PL space X, there exist a finite simplicial complex K and a PL homeomorphism $h : X \to |K|$ such that $v = h(x)$ is a vertex of K. If L denotes the link of v in K, then the PL homeomorphism type of the space $|L|$ depends only on X and x, not on the choice of K and h. We shall refer to L as the *link of x in X*, with the understanding that it is defined only up to PL homeomorphism.

1.7. Suppose that x is a point of a compact PL space X and that $n \geq 0$ is an integer. The link of x is PL homeomorphic to S^{n-1} if and only if x is an n-manifold point of X, i.e. some neighborhood of x is piecewise-linearly homeomorphic to $\mathbb{R}^n$.

If X is a compact PL space of dimension at most 2, we shall denote by $\Pi(X)$ the set of all 2-manifold points of X. Note that $\Pi(X)$ is an open subset of X, and with its induced PL structure it is a PL 2-manifold. Furthermore, $X - \Pi(X)$ is a compact PL subspace of X.

1.8. Let F be a properly embedded orientable surface in an orientable 3-manifold M. We define a *compressing disk* for F in M to be a disk $D \subset M$ such that $D \cap F = \partial D$ and such that ∂D is not the boundary of a disk in F. It is a standard consequence of the loop theorem that F is π_1-injective in M if and only if there is no compressing disk for F in M.

A closed orientable surface S contained in the interior of an orientable 3-manifold M will be termed *incompressible* if S is π_1-injective in M and no component of S is a 2-sphere. (We have avoided using the term "incompressible" for surfaces that are not closed.)

1.9. An *essential arc* in a 2-manifold F is a properly embedded arc in F which is not the frontier of a disk.

Definitions 1.10. A 3-manifold M will be termed *irreducible* if every 2-sphere in M bounds a ball in M. We shall say that M is *boundary-irreducible* if ∂M is π_1-injective in M, or equivalently if, for every properly embedded disk $D \subset M$, the simple closed curve ∂D bounds a disk in ∂M. We shall say that a 3-manifold M is *simple* if (i) M is compact, connected, orientable, irreducible and boundary-irreducible; (ii) no subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$; and (iii) M is not a closed manifold with finite fundamental group.

1.11. It is a theorem due to Meeks, Simon and Yau [10] that a covering space of an irreducible orientable 3-manifold is always irreducible. Given this result, it follows formally from our definition of simplicity that if a compact, orientable 3-manifold M is simple, then every connected finite-sheeted covering of M is also simple.

1.12. The unit interval $[0, 1]$ will often be denoted by I.

By an *I-bundle* we shall always mean a compact space equipped with a specific locally trivial fibration over some (often unnamed) base space, in which the fibers are homeomorphic to $[0, 1]$. (The reader is referred to [8, Chapter 10] for a general discussion of 3-dimensional I-bundles.)

By a *Seifert fibered manifold* we shall always mean a compact 3-manifold equipped with a specific Seifert fibration.

In particular, the notion of a *fiber* of an I-bundle or a Seifert fibered manifold is well defined, although the fiber projection and base space will often not be explicitly named. A compact subset of an I-bundle or Seifert fibered space will be called *horizontal* if it meets each fiber in one point. A compact set will be called *vertical* if it is a union of fibers.

If $\mathcal{P}$ is an I-bundle, we define the *horizontal boundary* of $\mathcal{P}$ to be the subset of $\mathcal{P}$ consisting of all endpoints of fibers of $\mathcal{P}$. We shall denote the horizontal boundary of $\mathcal{P}$ by $\partial_h \mathcal{P}$.

In the case where the base of the I-bundle $\mathcal{P}$ is a 2-manifold F (so that $\mathcal{P}$ is a 3-manifold), we define the *vertical boundary* of $\mathcal{P}$ to be $p^{-1}(\partial F)$, where $p : \mathcal{P} \to F$

denotes the bundle map. Note that in this case we have $\partial\mathcal{P} = \partial_v\mathcal{P} \cup \partial_h\mathcal{P}$, and if $\mathcal{P}$ is orientable, then $\partial_v\mathcal{P}$ is always a finite disjoint union of annuli.

1.13. The *rank* of a finitely generated group Γ is the cardinality of a minimal generating set for Γ. In particular, the trivial group has rank 0.

A group Γ is said to be *freely indecomposable* if Γ is not the free product of two non-trivial subgroups.

1.14. If V is a finite-dimensional vector space over $\mathbb{Z}_2$, then the dimension of V will be denoted $\mathrm{rk}_2\, V$. If X is a topological space, we will set $\mathrm{rk}_2\, X = \mathrm{rk}_2\, H_1(X;\mathbb{Z}_2)$.

2. Books of I-bundles

Definition 2.1. A *generalized book of I-bundles* is a triple $\mathcal{W} = (W, \mathcal{B}, \mathcal{P})$, where W is a (possibly empty) compact, orientable 3-manifold, and $\mathcal{B}, \mathcal{P} \subset W$ are submanifolds such that

- $\mathcal{B}$ is a (possibly disconnected) Seifert fibered space,
- $\mathcal{P}$ is an I-bundle over a (possibly disconnected) 2-manifold, and every component of $\mathcal{P}$ has Euler characteristic ≤ 0,
- $W = \mathcal{B} \cup \mathcal{P}$,
- $\mathcal{B} \cap \mathcal{P}$ is the vertical boundary of $\mathcal{P}$, and
- $\mathcal{B} \cap \mathcal{P}$ is vertical in the Seifert fibration of $\mathcal{B}$.

We shall denote W, $\mathcal{B}$ and $\mathcal{P}$ by $|\mathcal{W}|$, $\mathcal{B}_{\mathcal{W}}$ and $\mathcal{P}_{\mathcal{W}}$, respectively. The components of $\mathcal{B}_{\mathcal{W}}$ will be called *bindings* of $\mathcal{W}$, and the components of $\mathcal{P}_{\mathcal{W}}$ will be called its *pages*. The submanifold $\mathcal{P} \cap \mathcal{B}$, whose components are properly embedded annuli in W, will be denoted $\mathcal{A}_{\mathcal{W}}$.

If B is a binding of a generalized book of I-bundles $\mathcal{W}$, we define the *valence* of B to be the number of components of $\mathcal{A}_{\mathcal{W}}$ that are contained in ∂B.

A generalized book of I-bundles $\mathcal{W}$ will be termed *connected* if the manifold $|\mathcal{W}|$ is connected. Likewise, $\mathcal{W}$ will be termed *boundary-irreducible* if $|\mathcal{W}|$ is boundary-irreducible.

Definitions 2.2. A *book of I-bundles* is a generalized book of I-bundles $\mathcal{W}$ such that

- $|\mathcal{W}| \neq \emptyset$,
- each binding of $\mathcal{W}$ is a solid torus, and
- each binding of $\mathcal{W}$ meets at least one page of $\mathcal{W}$.

If B is a binding of a book of I-bundles $\mathcal{W}$, there is a unique integer $d > 0$ such that for every component A of $\mathcal{A}_{\mathcal{W}}$ contained in ∂B, the image of the inclusion homomorphism $H_1(A;\mathbb{Z}) \to H_1(B;\mathbb{Z})$ has index d in $H_1(B;\mathbb{Z})$. We shall call d the *degree* of the binding B.

Lemma 2.3. *Suppose that $\mathcal{W}$ is a generalized book of I-bundles. Then there is a generalized book of I-bundles $\mathcal{W}_0$ such that*

(1) $|\mathcal{W}_0| = |\mathcal{W}|$,
(2) every page of $\mathcal{W}_0$ has strictly negative Euler characteristic, and
(3) every page of $\mathcal{W}_0$ is a page of $\mathcal{W}$.

Proof. Set $W = |\mathcal{W}|$, $\mathcal{B} = \mathcal{B}_{\mathcal{W}}$ and $\mathcal{P} = \mathcal{P}_{\mathcal{W}}$. Let $\mathcal{Q}$ denote the union of all components P of $\mathcal{P}$ such that $\chi(P) = 0$. Then $\mathcal{Q}$ is an I-bundle over a compact surface A whose components are annuli and Möbius bands, and $\mathcal{Q} \cap \mathcal{B}$ is the induced

I-bundle over ∂A. Hence every component Q of $\mathcal{Q}$ is a solid torus, and $Q \cap \mathcal{B}$ consists of either a single annulus of degree 2 in Q or of two annuli of degree 1 in Q. Since every such annulus is also vertical in the Seifert fibration of $\mathcal{B}$, it follows that this Seifert fibration may be extended to a Seifert fibration of the manifold $\mathcal{B}_0 = \mathcal{B} \cup \mathcal{Q}$, in such a way that each component of $\mathcal{Q}$ contains either no singular fiber, or exactly one singular fiber of order 2. Furthermore, since every component of $\mathcal{Q}$ meets $\mathcal{B}$, every component of $\mathcal{B}_0$ contains a component of $\mathcal{B}$.

The manifold $\mathcal{P}_0 = \mathcal{P} - \mathcal{Q}$ is a union of components of $\mathcal{P}$ and therefore inherits an I-bundle structure. It is now clear that $\mathcal{W}_0 = (W, \mathcal{B}_0, \mathcal{P}_0)$ is a generalized book of I-bundles. It follows from the definition of $\mathcal{Q}$ that $\mathcal{W}_0$ satisfies conclusions (2) and (3) of the lemma. □

Lemma 2.4. *Suppose that $\hat{B}$ is a connected, Seifert-fibered submanifold of a simple, closed, orientable* 3*-manifold M. Then either*

(1) $\hat{B}$ is a solid torus, or
(2) $\hat{B}$ is contained in a ball in M, or
(3) some component of $M - \operatorname{int} \hat{B}$ is a solid torus.

Proof. Since M is simple and $\hat{B}$ is Seifert-fibered, we have $\hat{B} \neq M$, i.e. $\partial \hat{B} \neq \emptyset$. Since the components of $\partial \hat{B}$ are tori and M is simple, $\partial \hat{B}$ cannot be π_1-injective in M. Hence there is a compressing disk for $\partial \hat{B}$ in M. If $D \subset \hat{B}$, then $\hat{B}$ is a boundary-reducible Seifert fibered space, and hence (1) holds. The other possibility is that $D \cap \hat{B} = \partial D$. In this case, let V denote a regular neighborhood of D relative to $M - \operatorname{int} \hat{B}$. The boundary of the manifold $\hat{B} \cup V$ has a unique sphere component S. Since M is irreducible, S bounds a ball $\Delta \subset M$. We must have either $\Delta \supset \hat{B}$, which gives conclusion (2), or $\operatorname{int} \Delta \cap \hat{B} = \emptyset$. In the latter case, $\Delta \cup V$ is a solid torus component of $M - \operatorname{int} \hat{B}$, and so (3) holds. □

Lemma 2.5. *Suppose that M is a simple, closed, orientable* 3*-manifold and that $\mathcal{W}$ is a connected generalized book of I-bundles such that $W = |\mathcal{W}| \subset M$. Suppose that $\chi(W) < 0$ and that $\mathcal{P}_{\mathcal{W}}$ is π_1-injective in M. Then there is a connected book of I-bundles $\mathcal{V}$ with $V = |\mathcal{V}| \subset M$ such that*

(1) $V \supset W$,
(2) $\bar{\chi}(V) = \bar{\chi}(W)$,
(3) $\chi(P) < 0$ for every page P of $\mathcal{V}$,
(4) ∂V is a union of components of ∂W,
(5) every component of $\overline{V - W}$ is a solid torus,
(6) every page of $\mathcal{V}$ is a page of $\mathcal{W}$, and
(7) for each page P of $\mathcal{V}$ we have $P \cap \partial V = P \cap \partial W$.

Proof. Let $\mathcal{W}_0 = (W, \mathcal{B}, \mathcal{P})$ be a generalized book of I-bundles satisfying conditions (1)–(3) of Lemma 2.3. Since each page of $\mathcal{W}_0$ is also a page of $\mathcal{W}$, the hypothesis implies that each page of $\mathcal{W}_0$ is π_1-injective in M.

Let B be any binding of $\mathcal{W}_0$. We will show that the Seifert fibers of B are homotopically non-trivial in M. Since $\mathcal{W}_0$ is connected and $\chi(|\mathcal{W}_0|) < 0$, the binding B must meet some page P of $\mathcal{W}_0$. Let A be one of the annulus components of $B \cap P$. Then A is a component of the vertical boundary of P and, since $\chi(P) < 0$, it follows that A is π_1-injective in P. Since P is π_1-injective in M, it follows that A is also π_1-injective in M. Recalling that the annulus A is saturated in the Seifert fibration of B, we may conclude that each Seifert fiber of B is homotopically non-trivial in M.

Now for any binding B of $\mathcal{W}_0$ let us define $\hat{B}$ to be the union of B with all of the solid torus components of $\overline{M-B}$. We will show that $\hat{B}$ is a Seifert fibered submanifold of M such that $\hat{B} \cap \mathcal{W}_0 = B$. If J is any solid torus component of $\overline{M-B}$, then no page of $\mathcal{V}_0$ can be contained in J, since the pages are π_1-injective in M and have negative Euler characteristic. Thus $\operatorname{int} J$ must be disjoint from all of the pages of $\mathcal{W}_0$. This implies that $\hat{B} \cap \mathcal{W}_0 = B$. If $F \subset \partial J$ is a fiber of the Seifert fibered space B, then, since F is homotopically non-trivial in M, the simple closed curve $F \subset \partial J$ cannot be a meridian curve for the solid torus J. It follows that the Seifert fibration of B may be extended to a Seifert fibration of $B = B \cup J$, and hence that $\hat{B}$ admits a Seifert fibration.

Next we will show that $\hat{B}$ is, in fact, a solid torus. We know that $\hat{B}$ must satisfy one of the conditions (1)—(3) of Lemma 2.4. Condition (3) of Lemma 2.4 is ruled out since, by construction, no component of $M - \operatorname{int} \hat{B}$ is a solid torus. The fact that the Seifert fibers of B are homotopically non-trivial in M implies that the inclusion homomorphism $\pi_1(B) \to \pi_1(M)$ has non-trivial image and thus B cannot be contained in a ball in M. This rules out condition (2) of Lemma 2.4. Thus we conclude that condition (1) of Lemma 2.4 must hold, i.e that $\hat{B}$ is a solid torus.

Since each binding of $\mathcal{W}$ must meet some page, and since no page can be contained in a solid torus, we have that if B_1 and B_2 are distinct bindings of $\mathcal{W}_0$, then $\hat{B}_1$ is disjoint from $\hat{B}_2$. We define $\mathcal{B}'$ to be the union of the solid tori $\hat{B}$ as B ranges over all bindings of $\mathcal{W}_0$, and we set $V = \mathcal{B}' \cup \mathcal{P}$. We have $\mathcal{B}' \cap |\mathcal{W}_0| = \mathcal{B}$. It follows that $\mathcal{V} = (V, \mathcal{B}', \mathcal{P})$ is a book of I-bundles and that every page of $\mathcal{V}$ has strictly negative Euler characteristic.

We shall now complete the proof by observing that V satisfies conclusions (1)—(7) of the present lemma. Conclusions (1), (4) and (5) are immediate from the construction of V, and they imply conclusion (2). The pages of $\mathcal{V}$ are the same as the pages of $\mathcal{W}_0$, and each page of $\mathcal{W}_0$ is a page of $\mathcal{W}$ and has negative Euler characteristic. Hence conclusions (3) and (6) hold. Since ∂W is the union of ∂V with a collection of tori that are disjoint from all pages, it follows that $P \cap \partial V = P \cap \partial W$ for every page P of $\mathcal{V}$. This is conclusion (7). □

Recall that in Subsection 1.7 we defined $\Pi(X) \subset X$ to be the set of 2-manifold points in an arbitrary compact PL space X of dimension at most 2, and we observed that $X - \Pi(X)$ is a compact PL subset of X. It follows that $\Pi(X)$ has the homotopy type of a compact PL space. In particular $\chi(\pi)$ is a well-defined integer for every component π of $\Pi(X)$.

Definition 2.6. We define a *book of surfaces* to be a compact PL space X such that

(1) the link of every point of $x \in X$ is PL homeomorphic to the suspension of some non-empty finite set Z_x, and
(2) for every component π of $\Pi(X)$ we have $\chi(\pi) \leq 0$.

The cardinality of the set Z_x appearing in condition (1) is clearly uniquely determined by the point x. It will be called the *order* of x.

2.7. Note that a point x in a book of surfaces X has order 2 if and only if $x \in \Pi(X)$.

It also follows from the definition that if X is a book of surfaces, the set $X - \Pi(X)$ is a compact PL 1-manifold, which will be denoted by $\Psi(X)$. The components of $\Psi(X)$ and $\Pi(X)$ may be, respectively, thought of as *bindings* and *pages* of X.

We also observe that if M is a PL 3-manifold and if S_1 and S_2 are closed surfaces in $\operatorname{int} M$ which meet transversally, then $S_1 \cup S_2$ is a book of surfaces.

Lemma 2.8. *If X is a book of surfaces, there exist a (possibly disconnected) compact PL surface F and a PL map $r : F \to X$ such that*

(1) $r|\operatorname{int} F$ is a homeomorphism of $\operatorname{int} F$ onto $\Pi(X)$, and

(2) $r|\partial F$ is a covering map from ∂F to $\Psi(X)$.

Proof. Let us identify X with $|K|$, where $|K|$ is some finite simplicial complex. After subdividing K if necessary we may assume that for every closed simplex Δ of K the set $\Delta \cap \Psi(X)$ is a (possibly empty) closed face of Δ.

Let $\mathcal{D}$ denote the abstract disjoint union of all the closed 2-simplices of X, and let $i : \mathcal{D} \to X$ denote the map which is the inclusion on each closed 2-simplex. For each point $z \in \mathcal{D}$ let Δ_z denote the closed 2-simplex containing z. We define a relation $\sim$ on $\mathcal{D}$ by writing $z \sim w$ if and only if (i) $\Delta_z \cap \Delta_w \not\subset \Psi(X)$ and (ii) $i(z) = i(w)$. It is straightforward to show that $\sim$ is an equivalence relation. The quotient space $F = \Delta/\sim$ inherits a PL structure from $\mathcal{D}$. The definition of $\sim$ implies that there is a unique map $r : F \to X$ such that $r \circ q = i$, where $q : \mathcal{D} \to F$ is the quotient map, and that r maps $E = r^{-1}\Pi(X)$ homeomorphically onto $\Pi(X)$.

If x is a point of $\Psi(X)$, then since X is a book of surfaces, there exist a neighborhood A of x in $\Psi(X)$ and a neighborhood V of x in X, such that A is a PL arc, V is a union of PL disks $D_1 \cup \cdots \cup D_m$, where m is the order of x in X, and $D_i \cap D_j = A$ whenever $i \neq j$. The definition of $\sim$ implies that $r^{-1}(V)$ is a disjoint union of PL disks $\widetilde{D}_1, \ldots, \widetilde{D}_m$ such that r maps $\widetilde{D}_i$ homeomorphically onto D_i for $i = 1, \ldots, m$. Hence F is a PL surface with interior E and boundary $r^{-1}(\Psi(X))$, and $r|\partial F : \partial F \to \Psi(X)$ is a covering map. □

Lemma 2.9. *Suppose that M is an orientable PL 3-manifold and that $X \subset \operatorname{int} M$ is a book of surfaces. Then there is a book of I-bundles $\mathcal{W}$ such that*

(1) $|\mathcal{W}| = W$ is a regular neighborhood of X;

(2) $|\mathcal{B}_{\mathcal{W}}|$ is a regular neighborhood of $\psi(X)$;

(3) for every page P of $\mathcal{W}$, the set $X \cap P$ is a section of the I-bundle P; and

(4) $\mathcal{P}_{\mathcal{W}}$ is a regular neighborhood in M of a deformation retract of $\Pi(X)$.

Proof. Let $\mathcal{B}$ be a regular neighborhood of $\Psi(X)$ in M such that $N = \mathcal{B} \cap X$ is a regular neigborhood of $\Psi(X)$ in the PL space X. Every component of $\mathcal{B}$ is a solid torus. Since $\Pi(X)$ is an open 2-manifold, $Y = X \cap \overline{M - \mathcal{B}}$ is a compact 2-manifold and a deformation retract of $\Pi(X)$. In particular, in view of condition (2) in the definition of a book of surfaces, every component of Y has Euler characteristic ≤ 0. Let $\mathcal{P}$ be a regular neighborhood of Y in $\overline{M - N}$. Then $W = \mathcal{B} \cup \mathcal{P}$ is a regular neighborhood of X in M. We may give $\mathcal{P}$ the structure of an I-bundle over Y in such a way that Y is identified with a section of the bundle. We have $\mathcal{P} \cap \mathcal{B} = \partial_v \mathcal{P}$, and $\chi(P) \leq 0$ for every component P of $\mathcal{P}$.

Let F be the surface, and $r : F \to X$ the map, given by Lemma 2.8. We have $N = r(C)$, where C is a collar neighborhood of ∂F in F. Now if A is any component of $\partial_v \mathcal{B}$, then $A \cap Y$ is a component of ∂Y and therefore cobounds an annulus component of C with some component $\widetilde{\psi}_A$ of ∂F. It follows from Lemma 2.8 that $r|\widetilde{\psi}_A$ is a covering map of some degree d_A to some component ψ_A of $\psi(X)$. The annulus A lies in the boundary of the component B_A of $\mathcal{P}$ containing ψ_A, and the (unsigned) degree of A in the solid torus B_A is d_A. In particular, every component of $\partial_v\mathcal{P}$ has non-zero degree in the component of $\mathcal{B}$ containing it. This implies that $\mathcal{W} = (W, \mathcal{B}, \mathcal{P})$ is a book of I-bundles.

Each page P of $\mathcal{P}$ was constructed as an I-bundle over a component Y_0 of Y, where Y_0 is identified with a section of the bundle. Since $Y_0 = X \cap P$, conclusion (3) of the lemma follows. Conclusions (1), (2) and (4) are immediate from the construction of $\mathcal{W}$. $\square$

Lemma 2.10. *Suppose that $\mathcal{W}$ is a book of I-bundles, and let p denote the number of pages of $\mathcal{W}$. Then*

$$\operatorname{rk}_2 H_2(|\mathcal{W}|; \mathbb{Z}_2) \leq p.$$

Proof. It is most natural to prove a very mild generalization: if $\mathcal{W}$ is a generalized book of I-bundles whose bindings are all solid tori, and if p denotes the number of pages of $\mathcal{W}$, then $\operatorname{rk}_2 H_2(|\mathcal{W}|; \mathbb{Z}_2) \leq p$. We set $W = |\mathcal{W}|$ and use induction on p. If $p = 0$, then the components of W are solid tori and hence $\operatorname{rk}_2 H_2(W) = 0$. If $p > 0$, choose a page P of $\mathcal{W}$ and set $W' = \overline{W - P}$ and $\mathcal{P}' = \mathcal{P}_{\mathcal{W}} - P$. Then $\mathcal{P}'$ inherits an I-bundle structure from $\mathcal{P}$, and $\mathcal{W}' = (W', \mathcal{B}, \mathcal{P}')$ is a book of I-bundles with $p - 1$ pages. By the induction hypothesis we have $\operatorname{rk}_2 H_2(W') \leq p - 1$. On the other hand, if F denotes the base surface of the I-bundle P, we have

$$H_2(W, W'; \mathbb{Z}_2) \cong H_2(P, \partial_v P; \mathbb{Z}_2) \cong H_2(F, \partial F; \mathbb{Z}_2)$$

and hence $\operatorname{rk}_2 H_2(W, W') = 1$. It follows that

$$\operatorname{rk}_2 H_2(W) \leq \operatorname{rk}_2 H_2(W') + \operatorname{rk}_2 H_2(W, W') \leq p.$$

$\square$

Lemma 2.11. *If $\mathcal{W}$ is a book of I-bundles, and if every page of $\mathcal{W}$ has strictly negative Euler characteristic, we have*

$$\operatorname{rk}_2(|\mathcal{W}|) \leq 2\bar{\chi}(|\mathcal{W}|) + 1.$$

Proof. Set $W = |\mathcal{W}|$. By hypothesis we have $\bar{\chi}(P) \geq 1$ for every page P of $\mathcal{W}$. Hence if $P_1, \ldots, P_p$ denote the pages of $\mathcal{W}$, we have

$$\bar{\chi}(W) = \sum_{i=1}^{p} \bar{\chi}(P_i) \geq p.$$

According to Lemma 2.10 we have

$$\operatorname{rk}_2 H_2(W; \mathbb{Z}_2) \leq p \leq \bar{\chi}(W).$$

Now W is a connected 3-manifold with non-empty boundary. Hence $\operatorname{rk}_2 H_0(W; \mathbb{Z}_2)$ $= 1$, and $H_j(W; \mathbb{Z}_2) = 0$ for each $j > 2$. In view of Subsection 1.5, we have

$$\bar{\chi}(W) = \operatorname{rk}_2 H_1(W; \mathbb{Z}_2) - \operatorname{rk}_2 H_2(W; \mathbb{Z}_2) - 1.$$

Hence

$$\mathrm{rk}_2(W) = \mathrm{rk}_2\, H_1(W;\mathbb{Z}_2) = \bar{\chi}(W) + \mathrm{rk}_2\, H_2(W;\mathbb{Z}_2) + 1 \leq 2\bar{\chi}(W) + 1.$$

□

3. Compressing submanifolds

Definition 3.1. If $\mathcal{S}$ is a closed (possibly empty or disconnected) surface, we define a non-negative integer $\kappa(\mathcal{S})$ by

$$\kappa(V) = \sum_{S}(1 + \mathrm{genus}(S)^2),$$

where S ranges over the components of $\mathcal{S}$.

Lemma 3.2. *Let $\mathcal{S}$ be a closed (possibly empty or disconnected) surface, let $A \subset \mathcal{S}$ be a homotopically non-trivial annulus, and let $\mathcal{S}'$ be the surface obtained from the bounded surface $\overline{\mathcal{S} - A}$ by attaching disks D_1 and D_2 to its two boundary components. Then $\kappa(\mathcal{S}') < \kappa(\mathcal{S})$.*

Proof. Let us index the components of $\mathcal{S}$ as $S_0, \ldots, S_n$, where $n \geq 0$ and $A \subset S_0$. If $S_0 - A$ is connected, the components of $\mathcal{S}'$ are $S_0', S_1, \ldots, S_n$, where $S_0' = (S_0 - A) \cup D_1 \cup D_2$. We then have $\mathrm{genus}\, S_0' = (\mathrm{genus}\, S_0) - 1$, so that $\kappa(\mathcal{S}) < \kappa(\mathcal{S}')$.

If $S_0 - A$ is disconnected, then $(S_0 - A) \cup D_1 \cup D_2$ has two components S_0' and S_0''. If we denote the respective genera of S_0, S_0' and S_0'' by g, g' and g'', we have $g = g' + g''$. Also, since A is homotopically non-trivial in S_0, both g' and g'' are strictly positive. It follows that $(1 + (g')^2) + (1 + (g'')^2) < 1 + g^2$, and we again deduce that $\kappa(\mathcal{S}) < \kappa(\mathcal{S}')$. □

3.3. Recall that a connected 3-manifold H is called a *compression body* if it can be constructed from a product $T \times [-1, 1]$, where T is a connected, closed, orientable 2-manifold, by attaching finitely many 2- and 3-handles to $T \times \{-1\}$. One defines $\partial_+ H$ to be the submanifold $T \times \{1\}$ of ∂H, and one defines $\partial_- H$ to be $\partial H - \partial_+ H$.

3.4. If H is a connected compression body, it is clear that $\partial_+ H$ is connected and that for each component F of $\partial_- H$ we have $\mathrm{genus}(F) \leq \mathrm{genus}(\partial_+ H)$.

3.5. It is a standard observation that a connected compression body H with $\partial_- H = \emptyset$ is a handlebody.

3.6. Another standard observation is that any connected compression body H with $\partial_H \neq \emptyset$ can be constructed from a product $S \times [-1, 1]$, where S is a possibly disconnected, closed, orientable 2-manifold, by attaching 1-handles to $S \times \{1\}$. One then has $\partial_- H = S \times \{-1\}$.

An immediate consequence of this observation is that if H is a connected compression body, then $\partial_- H$ is π_1-injective in H.

3.7. More generally, we shall define a *compression body* to be a compact, possibly disconnected 3-manifold $\mathcal{H}$ such that each component of $\mathcal{H}$ is a compression body in the sense defined above. We define $\partial_+ \mathcal{H} = \bigcup_H \partial_+ H$ and $\partial_- \mathcal{H} = \bigcup_H \partial_- H$, where H ranges over the components of $\mathcal{H}$.

Proposition 3.8. *Let N be a compact orientable, irreducible 3-manifold, and let V be a compact, connected, non-empty 3-submanifold of $\mathrm{int}\, N$. Suppose that $\overline{N - V}$*

is π_1-injective in N. Then at least one of the following conditions holds:

(1) V is contained in a ball in N; or

(2) $\partial V \neq \emptyset$, and there exists a connected, incompressible closed surface in N whose genus is at most the maximum of the genera of the components of ∂V; or

(3) N is closed, and every component of $\overline{N - V}$ is a handlebody.

Proof. First note that if $V = N$, then conclusion (3) holds. (The hypothesis $V \subset \operatorname{int} N$ implies that N is closed, and the other assertion of (3) is vacuously true.) Hence we may assume that $V \neq N$, so that $\partial V \neq \emptyset$.

Let $\mathcal{C}$ denote the set of all (possibly disconnected) compression bodies $\mathcal{H} \subset N$ such that $\mathcal{H} \cap V = \partial_+ \mathcal{H} = \partial V$. Note that a regular neighborhood of ∂V relative to $\overline{N - V}$ is an element of $\mathcal{C}$, and hence that $\mathcal{C} \neq \emptyset$. Let us fix an element $\mathcal{H}$ of $\mathcal{C}$ such that (in the notation of Subsection 3.1) we have $\kappa(\partial_- \mathcal{H}) \leq \kappa(\partial_- \mathcal{H}')$ for every $\mathcal{H}' \in \mathcal{C}$.

Note that $V \cup \mathcal{H}$ is connected since $\mathcal{H} \in \mathcal{C}$.

Consider first the case in which $\partial_- \mathcal{H} = \emptyset$. In this case, it follows from Subsection 3.5 that every component of $\mathcal{H}$ is a handlebody, and we have $\partial \mathcal{H} = \partial_+ \mathcal{H} = \partial V$. Since N is connected and $\partial V \neq \emptyset$, we must have $\mathcal{H} = \overline{N - V}$. In particular N must be closed. Thus conclusion (3) of the proposition holds in this case.

Now consider the case in which some component S of $\partial_- \mathcal{H}$ is a 2-sphere. By irreducibility, S bounds a ball $B \subset N$. Since $V \cup \mathcal{H}$ is connected, we have either $V \cup \mathcal{H} \subset B$ or $B \cap (V \cup \mathcal{H}) = \partial B$. If $V \cup \mathcal{H} \subset B$, then in particular conclusion (1) of the proposition holds. If $B \cap (V \cup \mathcal{H}) = \partial B$, then $\mathcal{H}' \doteq \mathcal{H} \cup B$ is obtained from $\mathcal{H}$ by attaching a 3-handle to $\partial_- \mathcal{H}$, and hence $\mathcal{H}' \in \mathcal{C}$ (cf. Subsection 3.3). But we have $\partial_- \mathcal{H}' = \partial_- \mathcal{H} - S$, and it follows from Definition 3.1 that $\kappa(\mathcal{H}') = \kappa(\mathcal{H}) - 1$. This contradicts the minimality of $\kappa(\mathcal{H})$.

There remains the case in which $\partial_- \mathcal{H} \neq \emptyset$, and every component of $\partial_- \mathcal{H}$ has positive genus. Let us fix a component Z of $\overline{N - V}$ which contains at least one component of $\partial_- \mathcal{H}$. Let us set $F = Z \cap \partial_- \mathcal{H}$. Then F is a non-empty (and possibly disconnected) closed surface in $\operatorname{int} Z$, and each component of F has positive genus. We claim that F is incompressible in Z.

Suppose to the contrary that F is compressible in Z. Then there is a disk $D \subset \operatorname{int} Z$ such that $D \cap F = \partial D$ and such that ∂D is a homotopically non-trivial simple closed curve in F. Since $D \subset \operatorname{int} Z \subset N - V$, we have $D \cap \partial_+ \mathcal{H} = \emptyset$. Furthermore, since $D \subset Z$, we have $D \cap \partial_- \mathcal{H} = D \cap (Z \cap \partial_- \mathcal{H}) = D \cap F = \partial D$. Hence $D \cap \partial \mathcal{H} = \partial D$. It follows that either $D \subset \mathcal{H}$ or $D \cap \mathcal{H} = \partial D$.

If $D \subset \mathcal{H}$, let H_0 denote the component of $\mathcal{H}$ containing D, and let $F_0 \subset \partial_- H_0$ denote the component of F containing ∂D. Since ∂D is homotopically non-trivial, it follows that the inclusion homomorphism $\pi_1(F_0) \to \pi_1(H_0)$ has non-trivial kernel. This contradicts Subsection 3.6.

If $D \cap \mathcal{H} = \partial D$, we fix a regular neighborhood E of D relative to $\overline{Z - \mathcal{H}}$. Then $\mathcal{H}' \doteq \mathcal{H} \cup E$ is obtained from $\mathcal{H}$ by attaching a 2-handle to $\partial_- \mathcal{H}$, and hence $\mathcal{H}' \in \mathcal{C}$ (cf. Subsection 3.3). The surface $\partial \mathcal{H}'$ has the form $((\partial \mathcal{H}) - A) \cup D_1 \cup D_2$, where $A \subset \partial \mathcal{H}$ is a homotopically non-trivial annulus, and D_1 and D_2 are disjoint disks in N such that $(D_1 \cup D_2) \cap \partial \mathcal{H} = \partial A$. It therefore follows from Lemma 3.2 that $\kappa(\partial \mathcal{H}) < \kappa(\partial \mathcal{H}')$. This contradicts the minimality of $\kappa(\mathcal{H})$, and the incompressibility of F in Z is proved.

Since Z is π_1-injective in N by hypothesis, it now follows that F is incompressible in N. Our choice of Z guarantees that $F \neq \emptyset$. Choose any component F_1 of F, and let H_1 denote the component of $\mathcal{H}$ containing F_1. By Subsection 3.4, genus(F_1) is at most the genus of the connected surface $\partial_+\mathcal{H}$. But $\partial_+\mathcal{H}$ is a component of ∂V since $\mathcal{H} \in \mathcal{C}$, and so genus($F_1$) is at most the maximum of the genera of the components of ∂V. Hence conclusion (2) of the proposition holds in this case. $\square$

4. Transporting surfaces downstairs

Lemma 4.1. *Let M be a simple, compact, orientable 3-manifold, let $p : \widetilde{M} \to M$ be a 2-sheeted covering, and let $\tau : \widetilde{M} \to \widetilde{M}$ denote the non-trivial deck transformation. Suppose that $\widetilde{M}$ contains a closed, incompressible surface F_0 of positive genus. Then F_0 is ambiently isotopic to a surface F such that F and $\tau(F)$ meet transversally, and every component of $F \cap \tau(F)$ is a homotopically non-trivial simple closed curve in $\widetilde{M}$.*

Proof. Let $\mathcal{F}$ denote the collection of all surfaces $S \subset \widetilde{M}$ such that S is isotopic to F_0 and S meets $\tau(S)$ transversely. Choose a surface $F \in \mathcal{F}$ so that the number of components of $F \cap \tau(F)$ is minimal. We will show that every component of $F \cap \tau(F)$ is a homotopically non-trivial simple closed curve.

Suppose there exists a homotopically trivial component γ of $F \cap \tau(F)$. Then, since F is incompressible in $\widetilde{M}$, the simple closed curve γ must bound disks $D \subset F$ and $D' \subset \tau(F)$. We assume, without loss of generality, that the disk D' is innermost on $\tau(F)$ in the sense that $D' \cap F = \gamma$. This implies, in particular, that $D \cup D'$ is an embedded 2-sphere in $\widetilde{M}$.

Since $\widetilde{M}$ is irreducible by [10], the 2-sphere $D \cup D'$ bounds a ball B in M. We may observe at this point that the curve γ cannot be invariant under τ. Otherwise, since D' is the unique disk on $\tau(F)$ bounded by γ, it would follow that $\tau(D) = D'$, and hence that the sphere $D \cup D'$ is invariant under τ. Since $\widetilde{M}$ contains an incompressible surface, it is not homeomorphic to S^3, and therefore B is the unique 3-ball bounded by $D \cup D'$. Thus the assumption that γ is invariant implies that the ball B is invariant under the fixed point free map τ, contradicting the Brouwer Fixed Point Theorem. This shows that γ is not invariant under τ. It follows, since D' is innermost, that D' is disjoint from its image under τ.

Now let V be a regular neighborhood of B, chosen so that $V \cap F$ is a regular neighborhood of D and $V \cap F'$ is a regular neighborhood of D'. The disk $F' \cap V$ divides V into two balls, one of which, say U, is disjoint from the interior of D. Since $D' \cap \tau(D') = \emptyset$, we may assume without loss of generality that V has been chosen to be small enough so that $U \cap \tau(U) = \emptyset$. Let E denote the disk in ∂U which is bounded by $F \cap U$ and which is disjoint from $\tau(F)$. We set $A = \overline{F \setminus U}$ and consider the surface $F' = A \cup E$, which is clearly isotopic to F by an isotopy supported in V. We will show that $F' \cap \tau(F') \subset (F \cap \tau(F)) - \gamma$.

We write $F' \cap \tau(F') = (A \cup E) \cap (\tau(A) \cup \tau(F))$ as the union of the four sets $A \cap \tau(A)$, $A \cap \tau(E)$, $E \cap \tau(A)$ and $E \cap \tau(E)$. We have $A \cap \tau(A) \subset F \cap \tau(F) - \gamma$. Since $E \subset U$ and $U \cap \tau(U) = \emptyset$ we have $E \cap \tau(E) = \emptyset$. The sets E and $\tau(F) \supset \tau(A)$ are disjoint by construction, and hence $E \cap \tau(A) = \emptyset$. Finally, $A \cap \tau(E) = \tau(E \cap \tau(A)) = \emptyset$.

We have shown that $F' \cap \tau(F') \subset (F \cap \tau(F)) - \gamma$, and hence that $F' \cap \tau(F')$ has fewer components than $F \cap \tau(F)$. This contradicts the choice of F and completes the proof of the lemma. $\square$

Lemma 4.2. *Let N be a simple, compact, orientable 3-manifold, let $p : \widetilde{N} \to N$ be a 2-sheeted covering, and let $\tau : \widetilde{N} \to \widetilde{N}$ denote the non-trivial deck transformation. Suppose that $F \subset \widetilde{N}$ is a closed, incompressible surface such that F and $\tau(F)$ meet transversally, and every component of $F \cap \tau(F)$ is a homotopically non-trivial simple closed curve in N. Then $N - p(F)$ is π_1-injective in N.*

Proof. Set $F_1 = \tau(F)$, so that F_1 is incompressible in $\widetilde{N}$. Set $C = F \cap F_1$. Let $\widetilde{N}'$ denote the 3-manifold obtained by splitting $\widetilde{N}$ along F, and let F_1' denote the surface obtained by splitting F_1 along C. Then $\widetilde{N}$ and F_1 may be regarded as quotient spaces of $\widetilde{N}'$ and F_1', and F_1' is naturally identified with a properly embedded surface in $\widetilde{N}'$. We have a commutative diagram

$$\begin{array}{ccc} F_1' & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ \widetilde{N}' & \longrightarrow & \widetilde{N} \end{array}$$

where the horizontal maps are quotient maps and the vertical maps are inclusions. The inclusion $F_1 \to \widetilde{N}$ is π_1-injective because F_1 is incompressible in $\widetilde{N}$, and the quotient map $F_1' \to F_1$ is π_1-injective because the components of C are homotopically non-trivial. By commutativity of the diagram it follows that the inclusion $F_1' \to \widetilde{N}'$ is π_1-injective.

Now let $\widetilde{N}''$ denote the 3-manifold obtained by splitting $\widetilde{N}'$ along F_1'. Since the inclusion $F_1' \to \widetilde{N}'$ is π_1-injective, the quotient map $\widetilde{N}'' \to \widetilde{N}'$ is also π_1-injective. On the other hand, the quotient map $\widetilde{N}' \to \widetilde{N}$ is π_1-injective because F is incompressible in $\widetilde{N}$. Hence the composite quotient map $\widetilde{N}'' \to \widetilde{N}$ is π_1-injective. It follows that the inclusion map $\widetilde{N} - (F \cup F_1) \to \widetilde{N}$ is π_1-injective.

Now consider any component Z of $N - p(F)$. Choose a component $\widetilde{Z}$ of $p^{-1}(Z)$. Then $\widetilde{Z}$ is a component of $\widetilde{N} - (F \cup F_1)$, and hence the inclusion $\widetilde{Z} \to \widetilde{N}$ is π_1-injective. Thus in the commutative diagram

$$\begin{array}{ccc} \pi_1(\widetilde{Z}) & \longrightarrow & \pi_1(\widetilde{N}) \\ \downarrow & & \downarrow \\ \pi_1(Z) & \longrightarrow & \pi_1(N) \end{array}$$

the inclusion homomorphism $\pi_1(\widetilde{Z}) \to \pi_1(\widetilde{N})$ is injective, while the vertical homomorphisms are induced by covering maps and are therefore also injective. Since the image of $\pi_1(\widetilde{Z})$ has index at most 2 in $\pi_1(Z)$, the kernel of the inclusion homomorphism $\pi_1(Z) \to \pi_1(N)$ has order at most 2. But $\pi_1(Z)$ is torsion-free because N is simple. Hence $\pi_1(Z) \to \pi_1(N)$ is injective, as asserted by the lemma. □

Lemma 4.3. *Suppose that N is a simple, compact, orientable 3-manifold, that $p : \widetilde{N} \to N$ is a 2-sheeted covering, that $g \geq 2$ is an integer, and that $\widetilde{N}$ contains a closed, incompressible surface of genus g. Then there exist a connected book of I-bundles $\mathcal{V}$ with $V = |\mathcal{V}| \subset N$ and a closed, orientable (possibly disconnected) surface $S \subset \operatorname{int} V$ such that*

(1) $\bar{\chi}(V) = \bar{\chi}(S) = 2g - 2$;
(2) every page of $\mathcal{V}$ has strictly negative Euler characteristic;

(3) $\mathcal{P}_{\mathcal{W}}$ *is* π_1*-injective in* N*;*
(4) $N - V$ *is* π_1*-injective in* N*;*
(5) no component of S *is a sphere; and*
(6) for every page P *of* $\mathcal{V}$*, the set* $S \cap P$ *is a section of the* I*-bundle* P*.*

Proof. According to Lemma 4.1, $\widetilde{N}$ contains a closed, incompressible surface F of genus g such that F and $\tau(F)$ meet transversally, and every component of $F \cap \tau(F)$ is a homotopically non-trivial simple closed curve in N. It follows that $q = p|F : F \to N$ is an immersion with at most double-curve singularities.

The map $q_\sharp : \pi_1(F) \to \pi_1(N)$ is injective because F is incompressible in $\widetilde{N}$ and $p : \widetilde{N} \to N$ is a covering map.

Let us set $X = q(F)$, and let $C \subset X$ denote the union of all double curves of q. Since the components of C are homotopically non-trivial in N and hence in X, the set $\widetilde{C} = q^{-1}(C)$ is a disjoint union of homotopically non-trivial simple closed curves in F. Hence $F - \widetilde{C}$ is π_1-injective in F, and each of its components has non-positive Euler characteristic. Since $q_\sharp : \pi_1(F) \to \pi_1(N)$ is injective, it follows that $q|(F - \widetilde{C}) : (F - \widetilde{C}) \to N$ is π_1-injective.

The set $F - \widetilde{C}$ is mapped homeomorphically onto $X - C$ by q. In particular, each component of $X - C$ has non-positive Euler characteristic. Furthermore, since $q|(F - \widetilde{C}) : (F - \widetilde{C}) \to N$ is π_1-injective, it now follows that $X - C$ is π_1-injective in N.

In the notation of Subsection 1.7 we have $\Pi(X) = X - C$, and the link in X of every point of C is homeomorphic to the suspension of a four-point set. Since every component of $X - C$ has non-positive Euler characteristic, it follows from Definition 2.6 that X is a book of surfaces. Since each component of C is a simple closed curve, we have $\bar{\chi}(X) = \bar{\chi}(F) = 2g - 2$.

Let W denote a regular neighborhood of X in N. According to Lemma 2.9, we may write $W = |\mathcal{W}|$ for some book of I-bundles $\mathcal{W}$ in such a way that conclusions (2)–(4) of Lemma 2.9 hold. Since $X - C$ is π_1-injective in N, it follows from conclusion (4) of Lemma 2.9 that $\mathcal{P}_{\mathcal{W}}$ is π_1-injective in N.

Since $\chi(W) = \chi(X) = 2 - 2g < 0$, and since $\mathcal{P}_{\mathcal{W}}$ is π_1-injective in N, it follows from Lemma 2.5 that there is a connected book of I-bundles $\mathcal{V}$ with $V = |\mathcal{V}| \subset N$, such that conclusions (1)—(7) of Lemma 2.5 hold. Conclusion (2) of Lemma 2.5 gives $\bar{\chi}(V) = \bar{\chi}(W) = \bar{\chi}(X)$, so that

$$\bar{\chi}(V) = 2g - 2. \tag{4.3.1}$$

It follows from conclusions (1) and (6) of Lemma 2.5 that every binding of $\mathcal{W}$ is contained in a binding of $\mathcal{V}$. Since by conclusion (2) of Lemma 2.9 we have $C \subset \operatorname{int} \mathcal{B}_{\mathcal{W}}$, it follows that $C \subset \operatorname{int} \mathcal{B}_{\mathcal{V}}$.

Let $\mathcal{U}$ denote a regular neighborhood of C in $\operatorname{int} \mathcal{B}_{\mathcal{V}}$. We may suppose $\mathcal{U}$ to be chosen so that $\partial \mathcal{U}$ meets $\Pi(X)$ transversally, and each component of $\mathcal{U} \cap X$ is homeomorphic to $+ \times S^1$, where $+$ denotes a cone on a four-point set. Set $X' = \overline{X - (\mathcal{U} \cap X)}$ and $F' = F \cap q^{-1}(X')$. Then F' and X' are (possibly disconnected) compact 2-manifolds with boundary, and $q' = q|F'$ maps F' homeomorphically onto X'. Let us fix an orientation of F so that F' inherits an orientation, and define an orientation of X' by transporting the orientation of F' via q.

Let $U_1, \dots, U_m$ denote the components of $\mathcal{U}$. We set $B_i = X \cap \partial U_i$. Each component β of B_i is a boundary component of X' and hence has an orientation induced from the orientation of X', which determines a generator of $H_1(U_i;\mathbb{Z})$ via the inclusion isomorphism $H_1(\beta;\mathbb{Z}) \to H_1(U_i;\mathbb{Z})$. We shall say that two components of B_i are *similar* if they determine the same generator of $H_1(U_i;\mathbb{Z})$ via this construction.

The set $(\partial U_i) - B_i$ has four components. Their closures are annuli, which we shall call *complementary annuli*. We shall say that two components of B_i are *adjacent* if their union is the boundary of a complementary annulus, and *opposite* otherwise.

If β and β' are opposite components of $X \cap \partial U_i$, then $q^{-1}(\beta)$ and $q^{-1}(\beta')$ form the boundary of an annulus A in F, which is mapped homeomorphically by q to an embedded annulus in U_i. Since the orientation of F' is the restriction of an orientation of F, the induced orientations of $q^{-1}(\beta)$ and $q^{-1}(\beta')$ determine different generators of $H_1(A;\mathbb{Z})$. In view of our definitions it follows that opposite components of B_i are dissimilar.

Let us call a complementary annulus *bad* if its boundary curves are similar, and *good* otherwise. If β is any component of B_i, the two components of B_i adjacent to β are opposite each other; hence exactly one of them is similar to β. This shows that β is contained in the boundary of exactly one bad annulus and one good annulus. We conclude that ∂U_i contains exactly two good annuli, say A_i and A_i', and that $A_i \cap A_i' = \emptyset$.

The set

$$S = (X - (X \cap \mathcal{U})) \cup (A_1 \cup \cdots \cup A_m) \cup (A_1' \cup \cdots \cup A_m')$$

is a (possibly disconnected) compact PL 2-manifold embedded in V. Since A_i and A_i' are good annuli, the orientation of X' extends to an orientation of S. In particular S is orientable.

We shall show that conclusions (1)–(6) of the present lemma hold when $\mathcal{V}$ and S are defined as above.

According to equation (4.3.1) we have $\bar{\chi}(V) = 2g - 2$. It follows from the construction of S that $\bar{\chi}(S) = \bar{\chi}(X) = 2g - 2$. Hence conclusion (1) of the present lemma holds.

Conclusion (2) of the present lemma follows from conclusion (3) of Lemma 2.5.

Since we have seen that $\mathcal{P}_{\mathcal{W}}$ is π_1-injective in N, it follows from conclusion (6) of Lemma 2.5 that $\mathcal{P}_{\mathcal{V}}$ is π_1-injective in N. This is conclusion (3) of the present lemma.

It follows from Lemma 4.2 that $N - X = N - q(F)$ is π_1-injective in N. It follows from conclusions (1) and (4) of Lemma 2.5 that every component of $N - V$ is also a component of $N - W$, and is therefore ambiently isotopic in N to a component of $N - X$. Hence $N - V$ is π_1-injective in N. This is conclusion (4) of the present lemma.

It follows from the construction of S that $S \cap \mathcal{P}_{\mathcal{V}} = X \cap \mathcal{P}_{\mathcal{V}}$. If P is any page of $\mathcal{V}$, then by conclusion (6) of Lemma 2.5, P is a page of $\mathcal{W}$, and hence $S \cap P = X \cap P$ is a section of the I-bundle P according to conclusion (3) of Lemma 2.9. This establishes conclusion (6) of the present lemma.

In particular it follows that for every page P of $\mathcal{V}$ the surface $P \cap S$ is connected and has non-positive Euler characteristic. On the other hand, the construction of S shows that every component of $S \cap \mathcal{B}_{\mathcal{V}}$ is an annulus. Hence every component of S has non-positive Euler characteristic, and conclusion (5) of the present lemma follows. □

Proposition 4.4. *Suppose that N is a simple, compact, orientable 3-manifold, that $p : \widetilde{N} \to N$ is a 2-sheeted covering, that $g \geq 2$ is an integer, and that $\widetilde{N}$ contains a closed, incompressible surface of genus g. Then either*

(1) N contains a closed, connected, incompressible surface of genus at most g, or
(2) N is closed, and there is a connected book of I-bundles $\mathcal{V}$ with $V = |\mathcal{V}| \subset N$ such that $\bar{\chi}(V) = 2g-2$, every page of $\mathcal{V}$ has strictly negative Euler characteristic, and every component of $\overline{N - V}$ is a handlebody. In particular, the rank of $H_1(N;\mathbb{Z}_2)$ is at most $4g-3$.

Proof of Proposition 4.4. Let us fix a connected book of I-bundles $\mathcal{V}$ with $V = |\mathcal{V}| \subset N$, and a closed, orientable surface $S \subset \operatorname{int} V$, such that conclusions (1) to (6) of Lemma 4.3 hold. We distinguish two cases, depending on whether there does or does not exist a page of $\mathcal{V}$ whose horizontal boundary is contained in a single component of ∂V.

Case I. There is a page P_0 of $\mathcal{V}$ such that $\partial_h P_0$ is contained in a single component Y_0 of ∂V.

According to conclusion (6) of Lemma 4.3, the set $S \cap P_0$ is a section of the I-bundle P_0. Hence there is a properly embedded arc α in V, such that $\alpha \subset P_0$, and such that α meets S transversally in a single point. The endpoints of α lie in $\partial_h P_0 \subset Y_0$. Since Y_0 is connected, there is an arc $\beta \subset Y_0$ with $\partial\beta = \partial\alpha$.

Let σ denote the class in $H_2(N;\mathbb{Z}_2)$ represented by S. Since α is properly embedded in V and meets X transversally in a single point of $\pi_0 \subset \Pi(X)$, the class σ has intersection number 1 with the class in $H_1(N;\mathbb{Z}_2)$ represented by the simple closed curve $\alpha \cup \beta$. In particular $\sigma \neq 0$. Hence some component S_0 of S represents a non-zero class in $H_2(N;\mathbb{Z}_2)$. It follows from conclusions (1) and (5) of Lemma 4.3 that $\bar{\chi}(S_0) \leq \bar{\chi}(S) = 2g-2$, and hence that $\operatorname{genus}(S_0) \leq g$.

Among all closed, orientable surfaces in N that represent non-trivial classes in $H_2(N;\mathbb{Z}_2)$, let us choose one, say S_1, of minimal genus. Then $\operatorname{genus}(S_1) \leq \operatorname{genus}(S_0) \leq g$. If S_1 is compressible in N, a compression of S_1 produces a 2-manifold S_2 with one or two components. Each component of S_2 has strictly smaller genus than S_1, and at least one of them represents a non-trivial class in $H_2(N;\mathbb{Z}_2)$. This contradicts minimality. Hence S_1 is incompressible in N. Since $\operatorname{genus}(S_1) \leq g$, conclusion (1) of the present lemma holds in this case.

Case II. There is no page P_0 of $\mathcal{V}$ such that $\partial_h P_0$ is contained in a single component of ∂V.

In this case, every page of $\mathcal{V}$ is a trivial I-bundle. Furthermore, if T is any component of ∂V, then for every page P of $\mathcal{V}$, at most one component of the horizontal boundary of P is contained in T. Hence

$$\bar{\chi}(T \cap P) \leq \bar{\chi}(P)$$

for every page P of $\mathcal{V}$. Letting P range over the pages of $\mathcal{V}$, and using equation (4.3.1), we find that

$$\bar{\chi}(T) = \sum_P \bar{\chi}(T \cap P) \leq \sum_P \bar{\chi}(P) = \bar{\chi}(V) = 2g-2.$$

This shows that

$$\text{genus}(T) \leq g \tag{4.4.1}$$

for every component T of ∂V.

According to conclusion (4) of Lemma 4.3, $N - V$ is π_1-injective in N. Thus $V \subset N$ satisfies the hypotheses of Proposition 3.8. There are three subcases corresponding to the three alternatives (1)—(3) of Proposition 3.8.

First suppose that alternative (1) of Proposition 3.8 holds, i.e. that V is contained in a ball. Then in particular for any page P of $\mathcal{V}$, the inclusion homomorphism $\pi_1(P) \to \pi_1(W)$ is trivial. But according to conclusions (2) and (3) of Lemma 4.3, we have that $\chi(P) < 0$ (so that $\pi_1(P)$ is non-trivial) and $\mathcal{P}_{\mathcal{W}}$ is π_1-injective in N. This contradiction shows that alternative (1) of Proposition 3.8 cannot hold in this situation.

Next suppose that alternative (2) of Proposition 3.8 holds, i.e. that there exists a connected, incompressible closed surface S_1 in N whose genus is at most the maximum of the genera of the components of ∂V. By equation (4.4.1) this maximum is at most g. Thus conclusion (1) of the present lemma holds in this subcase.

Finally, suppose that alternative (3) of Proposition 3.8 holds, i.e. that N is closed and that every component of $\overline{N - V}$ is a handlebody. We have that $V = |\mathcal{V}|$, where V is a book of I-bundles whose pages all have negative Euler characteristic, and conclusion (1) of Lemma 4.3 gives $\bar{\chi}(V) = 2g - 2$. Since the components of $\overline{N - V}$ are handlebodies, the inclusion of V into N induces a surjection from $H_1(V; \mathbb{Z}_2)$ to $H_1(N; \mathbb{Z}_2)$; hence the latter group has rank at most $4g - 3$ by Lemma 2.11. Furthermore, according to conclusion (6) of Lemma 4.3, for every page P of $\mathcal{V}$, the set $S \cap P$ is a section of the I-bundle P. Thus conclusion (2) of the present proposition holds in this subcase. □

5. Singularity of PL maps

If K is a finite simplicial complex, we shall denote the underlying space of K by $|K|$. A simplicial map $\phi : K_1 \to K_2$ between finite simplicial complexes defines a map from $|K_1|$ to $|K_2|$ which we shall denote by $|\phi|$.

Now suppose that X_1 and X_2 are compact topological spaces and that $f : X_1 \to X_2$ is a continuous surjection. We define a *triangulation* of f to be a quintuple $(K_1, J_1, K_2, J_2, \phi)$, where each K_i is a finite simpicial complex, $J_i : |K_i| \to X_i$ is a homeomorphism, and $f \circ J_1 = J_2 \circ \phi$. When it is unnecessary to specify the K_i and J_i, we shall simply say that ϕ is a triangulation of f.

Note that if f is any PL map from a compact PL space X to a PL space Y, then the surjection $f : X \to f(X)$ admits a triangulation.

Definition 5.1. Let K and L be finite simplicial complexes and let $\phi : K \to L$ be a simplicial map. We define the *degree of singularity* of ϕ, denoted $\text{DS}(\phi)$, to be the number of ordered pairs (v, w) of vertices of K such that $v \neq w$ but $\phi(v) = \phi(w)$.

If f is any PL map from a compact PL space X to a PL space Y, we define the *absolute degree of singularity* of f, denoted $\text{ADS}(f)$, by

$$\text{ADS}(f) = \min_{\phi} \text{DS}(\phi),$$

where ϕ ranges over all triangulations of $f : X \to f(X)$.

5.2. We emphasize that the definition of $\mathrm{ADS}(f)$ is based on regarding f as a map from X to $f(X)$. Hence if f is any PL map from a compact PL space X to a PL space Y, and Z is a PL subspace of Y containing $f(X)$, then the absolute degree of singularity of f is unchanged when we regard f as a PL map from X to Z.

An almost equally trivial immediate consequence of the definition of absolute degree of singularity is expressed by the following result.

Lemma 5.3. *Suppose that X, Y and Z are PL spaces, that X is compact, that $f : X \to Y$ is a PL map, and that h is a PL homeomorphism of $f(X)$ onto a PL subspace of Z. Then $h \circ f : X \to Z$ has the same absolute degree of singularity as f.*

Proof. In view of Subsection 5.2 we may assume that f is surjective and that h is a PL homeomorphism of Y onto Z. Now if $(K_1, J_1, K_2, J_2, \phi)$ is a triangulation of f, then $(K_1, J_1, K_2, h \circ J_2, h \circ \phi)$ is a triangulation of $h \circ f$, and $\mathrm{DS}(h \circ \phi) = \mathrm{DS}(\phi)$. It follows that $\mathrm{ADS}(h \circ f) \leq \mathrm{ADS}(f)$. The same argument, with h^{-1} in place of h, shows that $\mathrm{ADS}(f) \leq \mathrm{ADS}(h \circ f)$. □

Proposition 5.4 (Stallings)**.** *Suppose that Y is a connected PL space and that $p : \widetilde{Y} \to Y$ is a connected covering space, which is non-trivial in the sense that p is not a homeomorphism. Suppose that f is a PL map from a compact connected PL space X to Y, such that the inclusion homomorphism $\pi_1(f(X)) \to \pi_1(Y)$ is surjective. Suppose that $\widetilde{f} : X \to \widetilde{Y}$ is a lift of f. Then $\mathrm{ADS}(\widetilde{f}) < \mathrm{ADS}(f)$.*

Proof. We first prove the proposition in the special case where $f : X \to Y$ is a surjection. In this case we set $m = \mathrm{ADS}(f)$, and we fix a triangulation $(K_1, J_1, K_2, J_2, \phi)$ of the PL surjection f such that $\mathrm{DS}(\phi) = m$. Here, by definition, $J_1 : |K_1| \to X$ and $J_2 : |K_2| \to Y$ are homeomorphisms. Let us identify X and Z with $|K_1|$ and $|K_2|$ via these homeomorphisms. The covering space $\widetilde{Y}$ of Y may be identified with $|\widetilde{K}_2|$ for some simplicial covering complex $\widetilde{K}_2$ of K_2; thus $p = |\sigma|$ for some simplicial covering map $\sigma : \widetilde{K}_2 \to K_2$. The lift $\widetilde{f}$ may be written as $|\widetilde{\phi}|$ for some simplicial lift $\widetilde{\phi} : K_1 \to \widetilde{K}_2$. We shall denote by W the subcomplex $\widetilde{\phi}(K_1)$ of $\widetilde{K}_2$.

Since $\sigma \circ \widetilde{\phi} = \phi$, the definition of degree of singularity implies that $\mathrm{DS}(\widetilde{\phi}) \leq \mathrm{DS}(\phi) = m$. If equality holds here, then the restriction of σ to the vertex set of W is one-to-one. This implies that p restricts to a one-to-one map from $|W|$ to Y. But we have $W = \widetilde{f}(X)$, and the surjectivity of f implies that p maps $|W|$ onto Y; thus p restricts to a homeomorphism from $|W|$ to Y. This is impossible since $p : \widetilde{Y} \to Y$ is a non-trivial connected covering space. Hence we must have $\mathrm{DS}(\widetilde{\phi}) < m$. Since by definition we have $\mathrm{ADS}(\widetilde{\phi}) \leq DS(\widetilde{\phi})$, the assertion of the proposition follows in the case where f is surjective.

We now turn to the general case. Let us set $Z = f(X)$ and $\widetilde{Z} = p^{-1}(Z)$. Since $\widetilde{Y}$ is a non-trivial connected covering space of Y, and since the inclusion homomorphism $\pi_1(Z) \to \pi_1(Y)$ is surjective, $\widetilde{Z}$ is a non-trivial connected covering space of Z. According to Subsection 5.2, regarding $\widetilde{f}$ and f as maps into $\widetilde{Z}$ and Z does not affect their absolute degrees of singularity. Since $f : X \to Z$ is surjective, the inequality now follows from the special case that has already been proved. □

Following the terminology used by Simon in [15], we shall say that a 3-manifold M admits a *manifold compactification* if there is a homeomorphism h of M onto an open subset of a compact 3-manifold Q such that $h(\operatorname{int} M) = \operatorname{int} Q$.

Lemma 5.5. *Suppose that N is a compact, orientable, connected, irreducible PL 3-manifold and that D is a separating, properly embedded disk in N. Let X denote the closure of one of the connected components of $N - D$. Let $\nu \in D$ be a base point, and let $p : (\widetilde{N}, \widetilde{\nu}) \to (N, \nu)$ denote the based covering space corresponding to the subgroup $\mathrm{im}(\pi_1(X, \nu) \to \pi_1(N, \nu))$ of $\pi_1(N, \nu)$. Then $\widetilde{N}$ admits a manifold compactification.*

Proof. Let us set $X_1 = \overline{N - X}$. It will also be convenient to write $X_0 = X$. Then the X_i are compact submanifolds of N, and $X_0 \cap X_1 = D$. We set $H_i = \pi_1(X_i, \nu)$ for $i = 0, 1$. We identify $\pi_1(N, \nu)$ with $H_0 \star H_1$, so that the H_i are in particular subgroups of $\pi_1(N, \nu)$. Thus $(\widetilde{N}, \widetilde{\nu})$ is the based covering space corresponding to the subgroup H_0.

According to the general criterion given by Simon in [15, Theorem 3.1], $\widetilde{N}$ will admit a manifold compactification provided that the following conditions hold:

(i) X_0 and X_1 are irreducible,
(ii) D is π_1-injective in X_0 and X_1,
(iii) each conjugate of H_0 in $\pi_1(N, \nu)$ intersects $\mathrm{im}(\pi_1(D, \nu) \to \pi_1(N, \nu))$ in a finitely generated subgroup, and
(iv) for each $i \in \{0, 1\}$, and for each finitely generated subgroup Z of H_i which has the form $H_i \cap g^{-1} H_0 g$ for some $g \in \pi_1(N, \nu)$, the based covering space of (X_i, ν) corresponding to Z admits a manifold compactification.

Here conditions (ii) and (iii) hold trivially because $\pi_1(D)$ is trivial. Condition (i) follows from the irreducibility of N. (A ball bounded by a sphere in $\mathrm{int}\, X_i$ must be contained in X_i because the frontier of X_i is the disk D, and $\partial D \neq \emptyset$.)

To prove (iv), we consider any $i \in \{0, 1\}$ and any subgroup of H_i having the form $Z = H_i \cap g^{-1} H_0 g$, where $g \in \pi_1(N, \nu)$. Since $\pi_1(N, \nu) = H_0 \star H_1$, we have either (a) $Z = \{1\}$ or (b) $i = 0$ and $g \in H_0$. If (a) holds, then the based covering of (X_i, ν) corresponding to Z is equivalent to the universal cover of X_i. But since X_i is irreducible and has a non-empty boundary, it is a Haken manifold. Hence by [18, Theorem 8.1], the universal cover of X_i admits a manifold compactification. If (b) holds, then the covering corresponding to Z is homeomorphic to X_0 and is therefore a manifold compactification of itself. □

Lemma 5.6. *Suppose that N is a compact, connected, orientable, irreducible PL 3-manifold and that D is a separating, properly embedded disk in N. Let X denote the closure of one of the connected components of $N - D$. Let $\nu \in D$ be a base point, and let $p : (\widetilde{N}, \widetilde{\nu}) \to (N, \nu)$ denote the based covering space corresponding to the subgroup $\mathrm{im}(\pi_1(X, \nu) \to \pi_1(N, \nu))$ of $\pi_1(N, \nu)$. Let $\widetilde{X}$ denote the component of $p^{-1}(X)$ containing $\widetilde{\nu}$ (so that p maps $\widetilde{X}$ homeomorphically onto X). Then every compact PL subset of $\mathrm{int}\, \widetilde{N}$ is PL ambient-isotopic to a subset of $\widetilde{X}$.*

Proof. Since N is a compact, orientable, irreducible 3-manifold with non-empty boundary, it is a Haken manifold. Hence by [18, Theorem 8.1], the universal cover of $\mathrm{int}\, N$ is homeomorphic to $\mathbf{R}^3$. Thus $\mathrm{int}\, \widetilde{N}$ is covered by an irreducible manifold and is therefore irreducible.

According to Lemma 5.5, the manifold $\widetilde{N}$ admits a manifold compactification. Thus there is a homeomorphism h of $\widetilde{N}$ onto an open subset of a compact 3-manifold Q such that $h(\mathrm{int}\, \widetilde{N}) = \mathrm{int}\, Q$. Since $\mathrm{int}\, Q$ is homeomorphic to the irreducible manifold $\mathrm{int}\, \widetilde{N}$, the compact manifold Q is itself irreducible.

The definition of $\widetilde{N}$ implies that the inclusion map $\iota : X \to N$ admits a based lift $\widetilde{\iota} : (X,\nu) \to (\widetilde{N},\widetilde{\nu})$, that $\widetilde{\iota}(X) = \widetilde{X}$, and that $\widetilde{\iota}_\sharp : \pi_1(X,\nu) \to \pi_1(\widetilde{N},\widetilde{\nu})$ is an isomorphism. Hence the inclusion $\widetilde{X} \to \widetilde{N}$ induces an isomorphism of fundamental groups, and if we set $X' = h(\widetilde{X})$, the inclusion $X' \to Q$ induces an isomorphism of fundamental groups.

On the other hand, since the frontier of X in N is D, the frontier of X' in Q is $D' = h(\widetilde{\iota}(D))$, a properly embedded disk in the compact 3-manifold Q. Set $Y = \overline{Q - X'}$. Then in terms of a base point in D' we have a canonical identification of $\pi_1(Q)$ with $\pi_1(X') \star \pi_1(Y)$. Since the inclusion $X' \to Q$ induces an isomorphism of fundamental groups, it follows that $\pi_1(Y)$ is trivial. We also know that Y is irreducible because its frontier in the irreducible manifold Q is a disk. Thus Y is a compact, simply connected, irreducible 3-manifold with non-empty boundary, and is therefore PL homeomorphic to a ball.

We have now exhibited Q as the union of the compact 3-dimensional submanifold X and the PL 3-ball Y, and their intersection is the disk D. It follows that any compact PL subset W of $\operatorname{int} Q$ is PL isotopic to a subset of $\operatorname{int} X$. Since h maps $\operatorname{int} \widetilde{N}$ homeomorphically onto $\operatorname{int} Q$, and maps $\operatorname{int} \widetilde{X}$ homeomorphically onto $\operatorname{int} X'$, the conclusion of the lemma follows. □

Lemma 5.7. *Suppose that K is a compact, connected PL space such that $\pi_1(K)$ has rank ≥ 2 and is freely indecomposable. Suppose that N is a compact, connected, orientable PL 3-manifold which is irreducible but boundary-reducible. Suppose that $f : K \to \operatorname{int} N$ is a π_1-injective PL map, and that the inclusion homomorphism $\pi_1(f(K)) \to \pi_1(N)$ is surjective. Then f is homotopic to a map g such that $\mathrm{ADS}(g) < \mathrm{ADS}(f)$.*

Proof. Since N is boundary-reducible it contains an essential properly embedded disk. If N contains a non-separating essential disk D_0, then there is a separating essential disk D_1 in $N - D_0$ such that the closure of the component of $N - D_1$ containing D_0 is a solid torus J. In this case $\pi_1(\overline{N - J})$ is non-trivial, since $\pi_1(N)$ has rank at least 2; hence D_1 is an essential disk as well. Thus in all cases, N contains a separating essential disk D. We may write $N = X_0 \cup X_1$ for some connected submanifolds X_0 and X_1 of N with $X_0 \cap X_1 = D$. We choose a base point in $\nu \in D$ and set $A_i = \operatorname{im}(\pi_1(X_i,\nu) \to \pi_1(N,\nu))$ for $i = 0, 1$. Then $\pi_1(N,\nu) = A_0 \star A_1$.

If one of the A_i were trivial, then one of the X_i would be a ball since N is irreducible, and D would not be an essential disk. Hence the A_i are non-trivial subgroups. It then follows from the free product structure of $\pi_1(N,\nu)$ that the A_i are of infinite index in $\pi_1(N,\nu)$ and, in particular, that they are proper subgroups.

Since the subgroup $H = f_\sharp(\pi_1(K))$, which is defined only up to conjugacy in $\pi_1(N)$, has rank at least 2 and is freely indecomposable, it follows from the Kurosh subgroup theorem that H is conjugate to a subgroup of one of the A_i. By symmetry we may assume that H is conjugate to a subgroup of A_0. Hence after modifying f by a homotopy we may assume that f maps some base point κ of K to ν and that $f_\sharp(\pi_1(K,\kappa)) \subset A_0$. Hence if $(\widetilde{N},\widetilde{\nu})$ denotes the based covering space of (N,ν) corresponding to the subgroup A_0 of $\pi_1(N)$, then f admits a lift $\widetilde{f} : (K,\kappa) \to (\widetilde{N},\widetilde{\nu})$. Since A_0 is a proper subgroup of $\pi_1(N,\nu)$, the covering space $\widetilde{N}$ is non-trivial. Hence, according to Proposition 5.4, we have $\mathrm{ADS}(\widetilde{f}) < \mathrm{ADS}(f)$.

Let $\widetilde{X}_0$ denote the component of $p^{-1}(X_0)$ containing $\widetilde{\nu}$, so that p maps $\widetilde{X}_0$ homeomorphically onto X_0. According to Lemma 5.6, the compact PL subset $\widetilde{f}(K)$ of $\operatorname{int} \widetilde{N}$ is PL ambient-isotopic to a subset of $\widetilde{X}_0$. In particular, there is a PL homeomorphism j of $\widetilde{f}(K)$ onto a subset L of $\widetilde{X}_0$ such that j, regarded as a map of $\widetilde{f}(K)$ into $\widetilde{N}$, is homotopic to the inclusion $\widetilde{f}(K) \to \widetilde{N}$. It now follows that $p \circ j$ maps $\widetilde{f}(K)$ homeomorphically onto the subset $p(L)$ of $X_0 \subset N$. Hence by Lemma 5.3, if we set $g = p \circ j \circ \widetilde{f} : K \to N$, we have $\mathrm{ADS}(g) = \mathrm{ADS}(\widetilde{f}) < \mathrm{ADS}(f)$. But since $j : \widetilde{f}(K) \to \widetilde{N}$ is homotopic to the inclusion $\widetilde{f}(K) \to \widetilde{N}$, the map $g : K \to N$ is homotopic to f. □

Proposition 5.8. *Suppose that K is a compact, connected PL space such that $\pi_1(K)$ has rank at least 2 and is freely indecomposable. Suppose that f is a π_1-injective PL map from K to the interior of a compact, connected, orientable, irreducible PL 3-manifold M. Then there exist a map $g : K \to M$ homotopic to f with $\mathrm{ADS}(g) \leq \mathrm{ADS}(f)$ and a compact, connected 3-dimensional submanifold N of $\operatorname{int} M$ such that (i) $\operatorname{int} N \supset g(K)$, (ii) the inclusion homomorphism $\pi_1(g(K)) \to \pi_1(N)$ is surjective, (iii) ∂N is incompressible in M, and (iv) N is irreducible.*

Proof. Among all maps from K to M that are homotopic to f, we choose one, g, for which $\mathrm{ADS}(g)$ has the smallest possible value. In particular we then have $\mathrm{ADS}(g) \leq \mathrm{ADS}(f)$. Note also that $f_\sharp : \pi_1(K) \to \pi_1(N)$ is injective.

Now let N be a compact, connected 3-submanifold of M satisfying conditions (i) and (ii) of the statement of the proposition, and choose N so as to minimize the quantity $\kappa(\partial N)$ (see Definition 3.1) among all compact, connected 3-submanifolds satisfying (i) and (ii). We shall complete the proof by showing that N satisfies (iii) and (iv).

We first show that (iv) holds, i.e. that N is irreducible. If $S \subset \operatorname{int} N$ is a 2-sphere, then S bounds a ball $B \subset M$. If we set $N' = N \cup B$, then the pair N' satisfies (i) and (ii). (It inherits property (ii) from N because the inclusion homomorphism $\pi_1(N) \to \pi_1(N')$ is surjective.) But if $B \not\subset N$, it is clear from Definition 3.1 that $\kappa(\partial N') < \kappa(\partial N)$, and the minimality of $\kappa(\partial N)$ is contradicted. Hence we must have $B \subset N$, and irreducibility is proved.

It remains to show that (iii) holds, i.e. that ∂N is incompressible. If this is false, then either ∂N has a sphere component or there is a compressing disk D for ∂N. If ∂N has a sphere component S, then the irreducibility of N implies that N is a ball. But then the injectivity of $g_\sharp : \pi_1(K) \to \pi_1(N)$ implies that $\pi_1(K)$ is trivial, a contradiction to the hypothesis that $\pi_1(K)$ has rank at least 2.

If there is a compressing disk D for ∂N, then either $D \cap N = \partial D$ or $D \subset N$. If $D \cap N = \partial D$, and if we set $N' = N \cup Q$, where Q is a regular neighborhood of D relative to $\overline{M - N}$, then the 3-submanifold N satisfies conditions (i) and (ii). (It inherits property (ii) because the inclusion homomorphism $\pi_1(N) \to \pi_1(N')$ is again surjective.) Now $\partial N'$ has the form $((\partial N) - A) \cup D_1 \cup D_2$, where $A \subset \partial N$ is a homotopically non-trivial annulus, and D_1 and D_2 are disjoint disks in M such that $(D_1 \cup D_2) \cap \partial N = \partial A$. It therefore follows from Lemma 3.2 that $\kappa(\partial N) < \kappa(\partial N')$. Again the minimality of $\kappa(\partial N)$ is contradicted.

Finally, if $D \subset N$, then N is boundary-reducible. As we have already shown that N is irreducible, it follows from Lemma 5.7 that f is homotopic in N to a map g' such that $\mathrm{ADS}(g') < \mathrm{ADS}(g)$. In particular, g' is homotopic to g in M; and since, according to Subsection 5.2, the absolute degrees of singularity of g and g'

do not depend on whether they are regarded as maps into N or into M, we now have a contradiction to the minimality of $\mathrm{ADS}(g)$. □

6. Homology of covering spaces

In this short section we shall apply and extend some results from [13] concerning homology of covering spaces of 3-manifolds. In this section all homology groups are understood to be defined with coefficients in $\mathbb{Z}_2$.

6.1. If N is a normal subgroup of a group G, we shall denote by $G\#N$ the subgroup of G generated by all elements of the form $gag^{-1}a^{-1}b^2$ with $g \in G$ and $a, b \in N$. (This is a special case of the notation used in [17] and [13]. Here we are taking the prime p, which was arbitrary in [17] and [13], to be 2.

6.2. As in Section 1 of [13], for any group Γ, we define subgroups Γ_d of Γ recursively for $d \geq 0$, by setting $\Gamma_0 = \Gamma$ and $\Gamma_{d+1} = \Gamma\#\Gamma_d$. We regard Γ_d/Γ_{d+1} as a $\mathbb{Z}_2$-vector space.

Lemma 6.3. *Let M be a closed 3-manifold and set $r = \mathrm{rk}_2\, M$. Suppose that $\widetilde{M}$ is a regular cover of M whose group of deck transformations is isomorphic to $(\mathbb{Z}_2)^m$ for some integer $m \geq 0$. Then*

$$\mathrm{rk}_2(\widetilde{M}) \geq mr - \frac{m(m+1)}{2}.$$

Proof. We set $\Gamma = \pi_1(M)$ and define Γ_d for each $d \geq 0$ as in Subsection 6.1. We have $\mathrm{rk}_2\, \Gamma/\Gamma_1 = \mathrm{rk}_2\, M = r$. It then follows from [13, Lemma 1.3] that $\mathrm{rk}_2(\Gamma_1/\Gamma_2) \geq r(r-1)/2$.

Let N denote the normal subgroup of Γ corresponding to the regular covering space $\widetilde{M}$. Since $\Gamma/N \cong (\mathbb{Z}_2)^m$, we may write $N = E\Gamma_1$ for some $(r-m)$-generator subgroup E of Γ. It now follows from [13, Lemma 1.4] that

$$\begin{aligned} \mathrm{rk}_2\, \widetilde{M} &= \mathrm{rk}_2\, H_1(E\Gamma_1) \\ &\geq \mathrm{rk}_2(\Gamma_1/\Gamma_2) - \frac{(r-m)(r-m-1)}{2} \\ &\geq \frac{r(r-1)}{2} - \frac{(r-m)(r-m-1)}{2} = mr - \frac{m(m+1)}{2}. \end{aligned}$$

□

The case $m = 2$ of Lemma 6.3 will be applied in the proof of Lemma 8.5.

7. An application of a result of Gabai

This section contains the applications of Gabai's results that were mentioned in the Introduction. The main result of the section is Proposition 7.5.

Lemma 7.1. *Let X be a PL space, let K be a closed, connected, orientable surface of genus $g > 0$, and let $f : K \to X$ be a PL map. Suppose that the homomorphism $f_* : H_2(K;\mathbb{Z}_2) \to H_2(X;\mathbb{Z}_2)$ is trivial. Then the image of $f_* : H_1(K;\mathbb{Z}_2) \to H_1(X;\mathbb{Z}_2)$ has dimension at most g.*

Proof. Since $f_* : H_2(K;\mathbb{Z}_2) \to H_2(X;\mathbb{Z}_2)$ is trivial, it follows that the dual homomorphism $f^* : H^2(X;\mathbb{Z}_2) \to H^2(K;\mathbb{Z}_2)$ is also trivial. Hence for any $\alpha, \beta \in H^1(X)$ we have

$$f^*(\alpha) \cup f^*(\beta) = f^*(\alpha \cup \beta) = 0.$$

Thus if we set

$$V = H^1(K;\mathbb{Z}_2)$$

and let $L \subset V$ denote the image of $f^* : H^1(X;\mathbb{Z}_2) \to H^1(K;\mathbb{Z}_2)$, we have $L \cup L = 0$, i.e.

$$L \subset L^\perp = \{v \in V : v \cup L = 0\}.$$

Hence if d denotes the dimension of L, we have

$$d \le \dim L^\perp.$$

But by Poincaré duality, the cup product pairing on V is non-singular, and so

$$\dim L^\perp = \dim V - \dim L = 2g - d.$$

Hence $d \le g$. As the linear map $f_* : H_1(K;\mathbb{Z}_2) \to H_1(X;\mathbb{Z}_2)$ is dual to $f^* : H^1(X;\mathbb{Z}_2) \to H^1(K;\mathbb{Z}_2)$, its rank is the same as that of f^*, namely d. The conclusion follows. □

Notation 7.2. If F is a closed, orientable 2-manifold, we shall denote by $\mathrm{TG}(F)$ the total genus of F, i.e. the sum of the genera of its components.

Lemma 7.3. *For any compact, connected, orientable* 3*-manifold N, we have*

$$\mathrm{TG}(\partial N) \le \mathrm{rk}_2\, N.$$

Proof. In the exact sequence

$$H_2(N,\partial N;\mathbb{Z}_2) \longrightarrow H_1(\partial N;\mathbb{Z}_2) \longrightarrow H_1(N;\mathbb{Z}_2),$$

Poincaré-Lefschetz duality implies that the vector spaces $H_2(N,\partial N;\mathbb{Z}_2)$ and $H_1(N;\mathbb{Z}_2)$ are of the same dimension, $\mathrm{rk}_2\, N$. Hence we have

$$2\ \mathrm{TG}(\partial N) = \mathrm{rk}_2\, \partial N \le 2\,\mathrm{rk}_2\, N,$$

and the conclusion follows. □

Lemma 7.4. *If N is a compact, connected, orientable* 3*-manifold N such that ∂N has at most one connected component, then $H_2(N;\mathbb{Z})$ is torsion-free.*

Proof. In the exact sequence

$$H_2(\partial N;\mathbb{Z}) \longrightarrow H_2(N;\mathbb{Z}) \longrightarrow H_2(N,\partial N;\mathbb{Z})$$

the inclusion map $H_2(\partial N;\mathbb{Z}) \longrightarrow H_2(N;\mathbb{Z})$ is trivial since ∂N has at most one connected component. Hence the map $H_2(N;\mathbb{Z}) \longrightarrow H_2(N,\partial N;\mathbb{Z})$ is injective, so that $H_2(N;\mathbb{Z})$ is isomorphic to a subgroup of $H_2(N,\partial N;\mathbb{Z})$. But by Poincaré-Lefschetz duality, $H_2(N,\partial N;\mathbb{Z})$ is isomorphic to $H^1(N,\mathbb{Z})$ and is therefore torsion-free. The conclusion follows. □

Proposition 7.5. *Suppose that N is a compact (possibly closed) orientable* 3*-manifold which is irreducible and boundary-irreducible. Suppose that K is a closed, connected, orientable surface of genus $g \ge 2$ and that $\phi : K \to N$ is a π_1-injective PL map. Then either*

(1) N contains a connected (non-empty) closed incompressible surface of genus at most g, or

(2) the $\mathbb{Z}_2$-vector subspace $\phi_*(H_1(K;\mathbb{Z}_2))$ of $H_1(N;\mathbb{Z}_2)$ has dimension at most g.

Furthermore, if $\phi_* : H_1(K;\mathbb{Z}_2) \to H_1(N;\mathbb{Z}_2)$ is surjective and $\partial N \ne \emptyset$, then (1) holds.

Proof. We begin with the observation that N is non-simply connected in view of the existence of the map ϕ. Since N is also irreducible, it follows that no component of ∂N is a sphere. On the other hand, since N is boundary-irreducible, every component of ∂N is π_1-injective in N. Thus every component of ∂N is parallel to an incompressible surface in N.

To prove the first assertion of the proposition we distinguish three cases, which are not mutually exclusive but cover all possibilities.

Case A. The homomorphism $\phi_* : H_2(K;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is trivial.

Case B. The surface ∂N has at least two components.

Case C. The surface ∂N has at most one component, and $\phi_* : H_2(K;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is a non-trivial homomorphism.

To prove the assertion in Case A, we first consider the commutative diagram

$$\begin{array}{ccc} H_2(K;\mathbb{Z}) & \longrightarrow & H_2(N;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_2(K;\mathbb{Z}_2) & \longrightarrow & H_2(N;\mathbb{Z}_2) \end{array}$$

in which the vertical maps are natural homomorphisms and the horizontal maps are induced by ϕ. The left-hand vertical arrow is surjective because the surface K is orientable. Since the top horizontal map is trivial, it follows that the bottom horizontal map is trivial. Hence Lemma 7.1 asserts that the image of $\phi_* : H_1(K;\mathbb{Z}_2) \to H_1(N;\mathbb{Z}_2)$ has dimension at most g. Thus alternative (2) of the conclusion holds in Case A.

In Case B, using Lemma 7.3 and the surjectivity of $\phi_* : H_1(K;\mathbb{Z}_2) \to H_1(N;\mathbb{Z}_2)$, we find that

$$\mathrm{TG}(\partial N) \leq \mathrm{rk}_2 N \leq \mathrm{rk}_2 K = 2g.$$

Since ∂N has at least two components in this case, some component F of ∂N must have genus at most g. By the observation at the beginning of the proof, F is parallel to an incompressible surface in N. Thus alternative (1) of the conclusion holds in Case B.

To prove the assertion in Case C, we begin by considering the commutative diagram

$$\begin{array}{ccc} H_2(K;\mathbb{Z}) & \longrightarrow & H_2(N;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_2(K;\mathbb{R}) & \longrightarrow & H_2(N;\mathbb{R}) \end{array}$$

in which the vertical maps are natural homomorphisms and the horizontal maps are induced by ϕ. Since ∂M has at most one component, Lemma 7.4 asserts that $H_2(N;\mathbb{Z})$ is torsion-free. Hence the right-hand vertical arrow in the diagram is injective. Since the top horizontal map is non-trivial, it follows that the bottom horizontal map is non-trivial. In other words, if $[K]$ denotes the fundamental class in $H_2(K;\mathbb{R})$, then the class $\alpha = f_*([K]) \in H_2(N;\mathbb{R})$ is non-zero.

We shall now apply a result from [7]. For any 2-manifold $\mathcal{F}$ we shall denote by $\chi_{\text{-}}(\mathcal{F})$ the quantity

$$\sum_F \max(\bar{\chi}(F), 0),$$

where F ranges over the components of $\mathcal{F}$. As in [7], given a class z in $H_2(M;\mathbb{R})$, we denote by $x_s(z)$ and $x(z)$, respectively, the "norm based on singular surfaces" and the Thurston norm of z. Since α is by definition realized by a map of the surface K into N, and since $\chi\_(K) = 2g-2$, we have $x_s(\alpha) \leq 2g-2$. But it follows from [7, Corollary 6.18] that $x(\alpha) = x_s(\alpha)$. Hence $x(\alpha) \leq 2g-2$. By definition this means that if $\mathcal{F}$ is a closed orientable embedded surface in $\operatorname{int} N$ such that the fundamental class $[\mathcal{F}] \in H_1(\mathcal{F};\mathbb{R})$ is mapped to α under inclusion, and if $\mathcal{F}$ is chosen among all such surfaces so as to minimize $\chi\_(\mathcal{F})$, then $\chi\_(\mathcal{F}) \leq 2g-2$. Since $\alpha \neq 0$ we have $\mathcal{F} \neq \emptyset$.

Since N is irreducible, any sphere component of $\mathcal{F}$ must be homologically trivial in N. We may assume that every torus component of F is compressible, as otherwise alternative (1) of the conclusion holds. Under this assumption, if T is a torus component of $\mathcal{F}$, compressing T yields a sphere which must be homologically trivial; hence T is itself homologically trivial. Thus after discarding homologically trivial components of $\mathcal{F}$ whose Euler characteristics are ≥ 0, we may suppose that no component of $\mathcal{F}$ is a sphere or torus. The minimality of $\chi\_(\mathcal{F})$ now implies that $\mathcal{F}$ is incompressible.

Let F be any component of $\mathcal{F}$. Then F is an incompressible closed surface in N, and we have

$$\chi\_(F) \leq \chi\_(\mathcal{F}) \leq 2g-2.$$

Hence F has genus at most g, and alternative (1) holds. This completes the proof of the first assertion of the proposition.

To prove the second assertion, suppose that $\phi_* : H_1(K;\mathbb{Z}_2) \to H_1(N;\mathbb{Z}_2)$ is surjective, that $\partial N \neq \emptyset$, and that alternative (2) holds. Then $\operatorname{rk}_2 N \leq g$, and it follows from Lemma 7.3 that $\operatorname{TG}(\partial N) \leq g$. In particular, any component F of the non-empty 2-manifold ∂N has genus at most g. By the observation at the beginning of the proof, F is parallel to an incompressible surface in N. Thus alternative (1) of the conclusion holds. □

8. Towers

In this section we prove a result, Proposition 8.10, which summarizes the tower construction described in the Introduction. Our main topological result, Theorem 8.13, will then be proved by combining Proposition 8.10 with results from the earlier sections. We begin by introducing some machinery that will be needed for the statement and proof of Proposition 8.10.

Definition 8.1. Suppose that n is a non-negative integer. We define a *height-n tower of 3-manifolds* to be a $(3n+2)$-tuple

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n),$$

where $M_0, \ldots, M_n$ are compact, connected, orientable PL 3-manifolds, N_j is a compact, connected 3-dimensional PL submanifold of M_j for $j = 0, \ldots, n$, and $p_j : M_j \to N_{j-1}$ is a PL covering map for $j = 1, \ldots, n$. We shall refer to M_0 as the *base* of the tower $\mathcal{T}$ and to N_n as its *top*. We define the *tower map associated to* $\mathcal{T}$ to be the map

$$h = \iota_0 \circ p_1 \circ \iota_1 \circ p_2 \circ \cdots \circ p_n \circ \iota_n : N_n \to M_0,$$

where $\iota_j : N_j \to M_j$ denotes the inclusion map for $j = 0, \ldots, n$.

8.2. Consider any tower of 3-manifolds

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n).$$

Note that for any given j with $0 \leq j < n$, the manifold N_j is closed if and only if its finite-sheeted covering space M_{j+1} is closed. Note also that if, for a given j with $0 \leq j \leq n$, the submanifold N_j of the (connected) manifold M_j is closed, then we must have $N_j = M_j$, so that in particular M_j is closed.

It follows that if M_j is closed for a given j with $0 \leq j \leq n$, then M_i is also closed for every i with $0 \leq i \leq j$. Thus either all the M_j have non-empty boundaries, or there is an index j_0 with $0 \leq j_0 \leq n$ such that M_j is closed when $0 \leq j \leq j_0$ and M_j has non-empty boundary when $j_0 < j \leq n$. Furthermore, in the latter case, for each $j < j_0$ we have $N_j = M_j$.

8.3. In particular, if in a tower of 3-manifolds

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n)$$

the manifold M_j is closed for a given $j \leq n$, then for every i with $0 \leq i < j$ the composition

$$p_{j-1} \circ \cdots \circ p_i : M_j \to M_i$$

is a well-defined covering map, whose degree is the product of the degrees of $p_i, \ldots, p_{j-1}$.

Definition 8.4. A tower of 3-manifolds

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n)$$

will be termed *good* if it has the following properties:

(1) M_j and N_j are irreducible for $j = 0, \ldots, n$;
(2) ∂N_j is a (possibly empty) incompressible surface in M_j for $j = 0, \ldots, n$;
(3) the covering map $p_j : M_j \to N_{j-1}$ has degree 2 for $j = 1, \ldots, n$; and
(4) for each j with $2 \leq j \leq n$ such that M_j is closed, the four-fold covering map (see Subsection 8.3)

$$p_j \circ p_{j-1} : M_j \to M_{j-2}$$

is regular and has covering group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 8.5. *Suppose that*

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n)$$

is a good tower of 3*-manifolds and that* j_0 *is an index with* $0 \leq j_0 \leq n$ *such that* M_{j_0} *is closed. Set* $r = \mathrm{rk}_2\, M_0$ *and assume that* $r \geq 3$*. For any index* j *with* $0 \leq j \leq j_0$*, we have*

$$\mathrm{rk}_2\, M_j \geq 2^{j/2}(r-3) + 3$$

if j *is even, and*

$$\mathrm{rk}_2\, M_j \geq 2^{(j-1)/2}(r-3) + 2$$

if j *is odd.*

In particular, we have $\mathrm{rk}_2\, M_j \geq r - 1$ *for each* j *with* $0 \leq j \leq n$ *such that* M_j *is closed, and we have* $\mathrm{rk}_2\, M_j \geq 2r - 4$ *for each* j *with* $2 \leq j \leq n$ *such that* M_j *is closed.*

Proof. According to Subsection 8.2, M_j is closed for every index j with $0 \leq j \leq j_0$. We shall first show that for every even j with $0 \leq j \leq j_0$ we have $\operatorname{rk}_2 M_j \geq 2^{j/2}(r-3)+3$. For $j = 0$ this is trivial since $r = \operatorname{rk}_2 M_0$. Now, arguing inductively, suppose that j is even, that $0 < j \leq n$, and that $\operatorname{rk}_2 M_{j-2} \geq 2^{(j-2)/2}(r-3)+3$. Since the definition of a good tower implies that M_j is a regular $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-cover of M_{j-2}, we apply Lemma 6.3 with $m = 2$ to deduce that

$$\operatorname{rk}_2 M_j \geq 2(\operatorname{rk}_2 M_{j-2}) - 3 \geq 2(2^{(j-2)/2}(r-3)+3) - 3 = 2^{j/2}(r-3)+3.$$

This completes the induction and shows that $\operatorname{rk}_2 M_j \geq 2^{j/2}(r-3)+3$ for every even index j with $2 \leq j \leq j_0$. Finally, if j is an odd index with $2 < j \leq j_0$, then since $j-1$ is even we have $\operatorname{rk}_2 M_{j-1} \geq 2^{(j-1)/2}(r-3)+3$; and since M_j is a 2-sheeted cover of M_{j-1}, it is clear that $\operatorname{rk}_2 M_j \geq \operatorname{rk}_2 M_{j-1} - 1 \geq 2^{(j-1)/2}(r-3)+2$. □

Definition 8.6. If

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n)$$

is a height-n tower of 3-manifolds, then for any m with $0 \leq m \leq n$, the $(3m+2)$-tuple

$$\mathcal{T}^- = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_m, M_m, N_m)$$

is a height-m tower. We shall refer to the tower $\mathcal{T}^-$ as the height-m *truncation* of $\mathcal{T}$. We shall say that a tower $\mathcal{T}^+$ is an *extension* of a tower $\mathcal{T}$, or that $\mathcal{T}^+$ *extends* $\mathcal{T}$, if $\mathcal{T}$ is a truncation of $\mathcal{T}^+$.

In particular, any tower may be regarded as an extension of itself. This will be called the *degenerate* extension.

Definition 8.7. Let $\mathcal{T}$ be a tower of 3-manifolds with base M and top N, and let $h : N \to M$ denote the associated tower map. Let ϕ be a PL map from a compact PL space K to M. By a *homotopy-lift* of ϕ through the tower $\mathcal{T}$ we mean a PL map $\widetilde{\phi} : K \to N$ such that $h \circ \widetilde{\phi}$ is homotopic to ϕ. A homotopy-lift $\widetilde{\phi}$ of ϕ will be termed *good* if the inclusion homomorphism $\pi_1(\widetilde{\phi}(K)) \to \pi_1(N)$ is surjective.

Lemma 8.8. *Suppose that K is a compact PL space with freely indecomposable fundamental group of rank $k \geq 2$. Suppose that $\mathcal{T} = (M_0, N_0, p_1, \ldots, N_n)$ is a good tower of* 3*-manifolds of height n. Suppose that $\phi : K \to M_0$ is a π_1-injective PL map and that $\widetilde{\phi} : K \to N_n$ is a good homotopy-lift of ϕ through the tower $\mathcal{T}$. Suppose that $p_{n+1} : M_{n+1} \to N_n$ is a two-sheeted covering space of N_n and that the map $\widetilde{\phi} : K \to N_n$ admits a lift to the covering space M_{n+1}. Suppose that either*

- (α) *$n \geq 1$, the manifold N_n is closed (so that M_{n+1} is closed; cf. Subsection* 8.2*), and the covering map*

$$p_n \circ p_{n+1} : M_{n+1} \to M_{n-1}$$

is regular and has covering group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, or
- (β) *$\partial M_{n+1} \neq \emptyset$, or*
- (γ) *$n = 0$.*

Then there exists a compact submanifold N_{n+1} of M_{n+1} with the following properties:

(1) $\mathcal{T}^+ = (M_0, N_0, p_1, \dots, N_n, p_{n+1}, M_{n+1}, N_{n+1})$ *is a good height-*$(n+1)$ *tower extending* $\mathcal{T}$, *and*

(2) *there is a good homotopy-lift* $\widetilde{\phi}^+$ *of* ϕ *through the tower* $\mathcal{T}^+$ *such that*

$$\mathrm{ADS}(\widetilde{\phi}^+) < \mathrm{ADS}(\widetilde{\phi}).$$

Proof. Let $h : N_n \to M_0$ be the tower map associated to $\mathcal{T}$. We fix a lift $f : K \to M_{n+1}$ of the map $\widetilde{\phi} : K \to N_n$ to the covering space M_{n+1}. Since $\widetilde{\phi}$ is a homotopy lift of ϕ, the map $h \circ p_{n+1} \circ f : K \to M_0$ is homotopic to ϕ. Since $\phi_\sharp : \pi_1(K) \to \pi_1(M_0)$ is injective, it now follows that $f_\sharp : \pi_1(K) \to \pi_1(M_{n+1})$ is also injective. We may therefore apply Proposition 5.8 to this map f, taking $M = M_{n+1}$ and $N = N_{n+1}$. We choose a map $g : K \to M_{n+1}$ homotopic to f, with $\mathrm{ADS}(g) \le \mathrm{ADS}(f)$, and a compact 3-dimensional submanifold $N = N_{n+1}$ of $\mathrm{int}\, M_{n+1}$, such that conditions (i)—(iv) of Proposition 5.8 hold with $M = M_{n+1}$.

It is clear from the definition that $\mathcal{T}^+ = (M_0, N_0, p_1, \dots, N_n, p_{n+1}, M_{n+1}, N_{n+1})$ is a tower extending $\mathcal{T}$. To show that the tower $\mathcal{T}^+$ is good, we first observe that conditions (1)—(4) of Definition 8.4 hold whenever $j \le n$ because $\mathcal{T}$ is a good tower. For $j = n+1$, conditions (1) and (2) of Definition 8.4 follow from conditions (iv) and (iii) of Proposition 5.8, while condition (3) of Definition 8.4 follows from the hypothesis that $p_{n+1} : M_{n+1} \to N_n$ is a two-sheeted covering. The case $j = n+1$ of condition (4) of Definition 8.4 is clear if alternative (α) of the hypothesis holds, and is vacuously true if alternatives (β) or (γ) hold. Hence $\mathcal{T}^+$ is a good tower.

Since by condition (i) of Proposition 5.8 we have $\mathrm{int}\, N_{n+1} \supset g(K)$, we may regard $g : K \to M_{n+1}$ as a composition $\iota_{n+1} \circ \widetilde{\phi}^+$, where $\iota_{n+1} : N_{n+1} \to M_{n+1}$ is the inclusion map and $\widetilde{\phi}^+$ is a PL map from K to N_{n+1}. Since g is homotopic to f, the map $h \circ p_{n+1} \circ \iota_{n+1} \circ \widetilde{\phi}^+ = h \circ p_{n+1} \circ g : K \to M_0$ is homotopic to ϕ. It follows that $\widetilde{\phi}^+$ is a homotopy-lift of ϕ through the tower $\mathcal{T}^+$. Condition (ii) of Proposition 5.8 asserts that the inclusion homomorphism $\pi_1(\widetilde{\phi}^+(K)) \to \pi_1(N_{n+1})$ is surjective, which according to Definition 8.7 means that the homotopy-lift $\widetilde{\phi}^+$ of ϕ is good.

Finally, since the homotopy-lift $\widetilde{\phi}$ of ϕ is good by hypothesis, the inclusion homomorphism $\pi_1(\widetilde{\phi}^+(K)) \to \pi_1(N_n)$ is surjective. As f is a lift of $\widetilde{\phi}$ to the non-trivial covering space M_{n+1} of N_n, it follows from Proposition 5.4 that $\mathrm{ADS}(f) < \mathrm{ADS}(\widetilde{\phi})$. But we chose g in such a way that $\mathrm{ADS}(g) \le \mathrm{ADS}(f)$, and according to Subsection 5.2 we have $\mathrm{ADS}(\widetilde{\phi}^+) = \mathrm{ADS}(g)$. Hence $\mathrm{ADS}(\widetilde{\phi}^+) < \mathrm{ADS}(\widetilde{\phi})$. □

Lemma 8.9. *Suppose that K is a closed orientable surface of genus $g \ge 2$. Suppose that $\mathcal{T}$ is a good tower of 3-manifolds of height n. Let M denote the base of $\mathcal{T}$, and assume that $\mathrm{rk}_2\, M \ge g+3$. Suppose that $\phi : K \to M$ is a π_1-injective PL map and that $\widetilde{\phi}$ is a good homotopy-lift of ϕ through the tower $\mathcal{T}$. Then at least one of the following alternatives holds:*

(1) *N_n contains a connected (non-empty) closed incompressible surface of genus at most g;*

(2) *$n \ge 1$ and N_{n-1} contains a connected (non-empty) closed incompressible surface of genus at most g; or*

(3) there exist a height-$(n+1)$ extension $\mathcal{T}^+$ of $\mathcal{T}$ which is a good tower and a good homotopy-lift $\widetilde{\phi}^+$ of ϕ through the tower $\mathcal{T}^+$, such that

$$\mathrm{ADS}(\widetilde{\phi}^+) < \mathrm{ADS}(\widetilde{\phi}).$$

Proof. We write

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \dots, p_n, M_n, N_n),$$

so that $M = M_0$. We distinguish several cases.

Case A. $\partial N_n \neq \emptyset$, and the homomorphism $\widetilde{\phi}_* : H_1(K; \mathbb{Z}_2) \to H_1(N_n; \mathbb{Z}_2)$ is surjective.

Case B. $\partial N_n \neq \emptyset$, and $\widetilde{\phi}_* : H_1(K; \mathbb{Z}_2) \to H_1(N_n; \mathbb{Z}_2)$ is not surjective.

Case C. $n = 0$.

Case D. $n \geq 1$, and N_n is closed.

In Case A, all the hypotheses of the final assertion of Proposition 7.5 hold with $\widetilde{\phi}$ in place of ϕ. It therefore follows from the final assertion of Proposition 7.5 that conclusion (1) of the present lemma holds.

In Case B, the map $\widetilde{\phi} : K \to N_n$ admits a lift to some two-sheeted covering space $p_{n+1} : M_{n+1} \to N_n$ of N_n. Since $\partial N_n \neq \emptyset$, we have $\partial M_{n+1} \neq \emptyset$. This is alternative (β) of the hypothesis of Lemma 8.8. It therefore follows from Lemma 8.8 that conclusion (3) of the present lemma holds.

In Case C the argument is identical to the one used in Case B, except that we have alternative (γ) of Lemma 8.8 in place of alternative (β).

We now turn to Case D. In this case, as was observed in Subsection 8.2, we have $N_n = M_n$ and $N_{n-1} = M_{n-1}$, and p_n is a two-sheeted covering map from M_n to M_{n-1}.

Let us set $r = \mathrm{rk}_2\, M \geq g + 3$. According to Lemma 8.5, for any index j such that $1 \leq j \leq n$ and such that M_j is closed, we have $\mathrm{rk}_2\, M_j \geq r - 1$. In particular, if we set $d = \mathrm{rk}_2\, M_{n-1}$, we have $d \geq r - 1 \geq g + 2$.

Now set $\widetilde{\phi}^- = p_n \circ \widetilde{\phi} : K \to M_{n-1}$. Then $X = \widetilde{\phi}^-_*(H_1(K; \mathbb{Z}_2))$ is a subspace of the d-dimensional $\mathbb{Z}_2$-vector space $V = H_1(M_{n-1}; \mathbb{Z}_2)$.

The hypotheses of Proposition 7.5 hold with N and ϕ replaced by M_{n-1} and $\widetilde{\phi}^-$. Hence either M_{n-1} contains a connected (non-empty) closed incompressible surface of genus at most g, or X has dimension at most g. The first alternative gives conclusion (2) of the present lemma.

There remains the subcase in which X has dimension at most g. Since $d \geq r - 1 \geq g + 2$, the dimension of X is then at most $g \leq d - 2$.

In this subcase we shall show that $\widetilde{\phi} : K \to M_n$ admits a lift to some two-sheeted covering space $p_{n+1} : M_{n+1} \to M_n$ of $M_n = N_n$ for which alternative (α) of the hypothesis of Lemma 8.8 holds. It will will then follow from Lemma 8.8 that conclusion (3) of the present lemma holds.

Let q denote the natural homomorphism from $\pi_1(M_{n-1})$ to $H_1(M_{n-1}; \mathbb{Z}_2)$. The two-sheeted cover M_n of M_{n-1} corresponds to a normal subgroup of $\pi_1(M_{n-1})$ having the form $q^{-1}(Z)$, where Z is some $(d-1)$-dimensional vector subspace of V. Since $\widetilde{\phi}^-$ admits the lift $\widetilde{\phi}$ to M_n, we have $X \subset Z \subset V$. Since in addition we have $\mathrm{rk}_2\, X \leq d - 2 < d - 1 = \mathrm{rank}\, Z$, there exists a $(d-2)$-dimensional vector subspace Y of V with $X \subset Y \subset Z$. The subgroup $q^{-1}(Y)$ determines a regular covering space M_{n+1} of M_{n-1} with covering group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since $q^{-1}(Y) \subset q^{-1}(Z)$, the degree-four covering map $M_{n+1} \to M_{n-1}$ factors as the composition of a degree-two covering map $p_{n+1} : M_{n+1} \to M_n$ with $p_n : M_n \to M_{n+1}$. Thus the covering space $p_{n+1} : M_{n+1} \to M_n$ satisfies alternative (α) of Lemma 8.8. It remains to show that $\widetilde{\phi}$ admits a lift to M_{n+1}.

Since $\widetilde{\phi}^-_\sharp(\pi_1(K)) \subset q^{-1}(X) \subset q^{-1}(Y)$, the map $\widetilde{\phi}^-$ admits a lift to the four-fold cover M_{n+1} of M_{n-1}. Since M_{n+1} is a regular covering space of M_{n-1}, there exist four different lifts of ϕ^- to M_{n+1}. But $\widetilde{\phi}^-$ can have at most two lifts to M_n, and each of these can have at most two lifts to M_{n+1}. Hence each lift of $\widetilde{\phi}^-$ to M_n admits a lift to M_{n+1}. In particular, $\widetilde{\phi}$ admits a lift to M_{n+1}. □

Lemma 8.10. *Suppose that K is a closed, orientable surface of genus $g \geq 2$. Suppose that M is a closed, orientable 3-manifold such that $\operatorname{rk}_2 M \geq g+3$ and that $\phi : K \to M$ is a π_1-injective PL map. Suppose that*

$$\mathcal{T}_0 = (M_0, N_0, p_1, \ldots, N_{n_0})$$

is a good tower with base M such that ϕ admits a good homotopy-lift through $\mathcal{T}_0$. Then either

(1) $n_0 \geq 1$, and N_{n_0-1} contains a connected (non-empty) closed incompressible surface of genus at most g, or
(2) there exists a good tower $\mathcal{T}_1$ which is a (possibly degenerate) extension of $\mathcal{T}_0$, such that the top N of $\mathcal{T}_1$ contains a connected (non-empty) closed incompressible surface of genus at most g, and ϕ admits a good homotopy-lift $\widetilde{\phi}_1$ through the tower $\mathcal{T}_1$.

Proof. Let us fix a good homotopy-lift $\widetilde{\phi}_0$ of ϕ through $\mathcal{T}_0$. Let $\mathcal{S}$ denote the set of all ordered pairs $(\mathcal{T}, \widetilde{\phi})$ such that $\mathcal{T}$ is a good tower which is an extension of $\mathcal{T}_0$ and $\widetilde{\phi}$ is a good homotopy-lift of ϕ through $\mathcal{T}$. Then we have $(\mathcal{T}_0, \phi_0) \in \mathcal{S}$, and so $\mathcal{S} \neq \emptyset$. Hence there is an element $(\mathcal{T}_1, \widetilde{\phi}_1)$ of $\mathcal{S}$ such that $\operatorname{ADS}(\widetilde{\phi}_1) \leq \operatorname{ADS}(\widetilde{\phi})$ for every element $(\mathcal{T}, \widetilde{\phi})$ of $\mathcal{S}$. Let us write

$$\mathcal{T}_1 = (M_0, N_0, p_1, \ldots, N_{n_1}).$$

The hypotheses of Lemma 8.9 now hold with $\mathcal{T}_1$ and $\widetilde{\phi}_1$ in place of $\mathcal{T}$ and $\widetilde{\phi}$. Hence one of the following alternatives must hold:

8.9(1) N_{n_1} contains a connected (non-empty) closed incompressible surface of genus at most g;
8.9(2) $n_1 \geq 1$, and N_{n_1-1} contains a connected (non-empty) closed incompressible surface of genus at most g; or
8.9(3) there exist a height-(n_1+1) extension $\mathcal{T}^+$ of $\mathcal{T}_1$ which is a good tower and a good homotopy-lift $\widetilde{\phi}^+$ of ϕ through the tower $\mathcal{T}^+$, such that

$$\operatorname{ADS}(\widetilde{\phi}^+) < \operatorname{ADS}(\widetilde{\phi}_1).$$

If 8.9(1) holds, then the tower $\mathcal{T}_1$ has the property asserted in conclusion (2) of the present lemma. If 8.9(2) holds, and if $n_1 > n_0$ (i.e. $\mathcal{T}_1$ is a non-degenerate extension of $\mathcal{T}_0$), then the height-(n_1-1) truncation $\mathcal{T}_1'$ of $\mathcal{T}_1$ is an extension of $\mathcal{T}_0$, and conclusion (2) holds with $\mathcal{T}_1'$ in place of $\mathcal{T}_1$. If 8.9(2) holds and $n_1 = n_0$ (i.e. $\mathcal{T}_1$ is a degenerate extension of $\mathcal{T}_0$), conclusion (1) of the present lemma holds. Finally, if 8.9(3) holds, then $(\mathcal{T}^+, \widetilde{\phi}^+) \in \mathcal{S}$, and we have a contradiction to the minimality of $\mathrm{ADS}(\widetilde{\phi}_1)$. □

Proposition 8.11. *Suppose that g is an integer ≥ 2, that M is a closed, orientable 3-manifold with $\mathrm{rk}_2 M \geq g + 3$, and that $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group. Then there exists a good tower*

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n),$$

with base $M = M_0$, such that N_n contains a connected incompressible closed surface of genus $\leq g$.

Proof. Let K denote a closed, orientable surface of genus g. The hypothesis implies that there is a π_1-injective PL map $\phi : K \to M$. According to Proposition 5.8, there exist a PL map $\widetilde{\phi}_0 : K \to M$ homotopic to ϕ and a compact, connected 3-submanifold N_0 of $\mathrm{int}\, M$ such that (i) $\mathrm{int}\, N_0 \supset \widetilde{\phi}_0(K)$, (ii) the inclusion homomorphism $\pi_1(\widetilde{\phi}_0(K)) \to \pi_1(N_0)$ is surjective, (iii) ∂N_0 is incompressible in M, and (iv) N_0 is irreducible. According to the definitions, this means that $\mathcal{T}_0 = (M, N_0)$ is a good tower of height 0 and that $\widetilde{\phi}_0$ is a good homotopy-lift of ϕ through $\mathcal{T}_0$.

We apply Lemma 8.10 with these choices of $\mathcal{T}_0$ and $\widetilde{\phi}_0$. Conclusion (1) of Lemma 8.10 cannot hold since $\mathcal{T}_0$ has height 0. Hence conclusion (2) must hold. The extension $\mathcal{T} = \mathcal{T}_1$ of $\mathcal{T}_0$ given by conclusion (2) is a good tower whose top contains a connected, closed incompressible surface of genus at most g. □

Lemma 8.12. *Suppose that*

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n)$$

is a good tower of 3-manifolds such that M_0 is simple. Then the manifolds M_j and N_j are all simple for $j = 0, \ldots, n$.

Proof. By hypothesis M_0 is simple. If M_j is simple for a given $j \leq n$, then since N_j is a submanifold of M_j bounded by a (possibly disconnected and possibly empty) incompressible surface, it is clear from Definitions 1.10 that N_j is simple. If $j < n$, it then follows from Subsection 1.11 that the two-sheeted covering space M_{j+1} of N_j is also simple. □

The following theorem is the main topological result of this paper.

Theorem 8.13. *Let g be an integer ≥ 2. Let M be a closed simple 3-manifold such that $\mathrm{rk}_2 M \geq 4g - 1$ and $\pi_1(M)$ has a subgroup isomorphic to a genus-g surface group. Then M contains a closed, incompressible surface of genus at most g.*

Proof. Applying Proposition 8.11, we find a good tower

$$\mathcal{T} = (M_0, N_0, p_1, M_1, N_1, p_2, \ldots, p_n, M_n, N_n),$$

with base M_0 homeomorphic to M and with some height $n \geq 0$, such that N_n contains a connected incompressible closed surface F of genus $\leq g$. According to the definition of a good tower, ∂N_n is incompressible (and, *a priori*, possibly empty) in M_n. Hence N_n is π_1-injective in M_n. Since F is incompressible in N_n, it follows that it is also incompressible in M_n.

Since M is simple, it follows from Lemma 8.12 that all the M_j and N_j are simple.

Let m denote the least integer in $\{0,\dots,n\}$ for which M_m contains a closed incompressible surface S_m of genus at most g. To prove the theorem it suffices to show that $m = 0$. Let h denote the genus of S_m.

Suppose to the contrary that $m \geq 1$. Then the hypotheses of Proposition 4.4 hold with N_{m-1} and M_m playing the respective roles of N and $\widetilde{N}$. Suppose that conclusion (1) of Proposition 4.4 holds, i.e. that N_{m-1} contains an incompressible closed surface S_{m-1} with $\mathrm{genus}(S_{m-1}) \leq h \leq g$. According to the definition of a good tower, ∂N_{m-1} is an incompressible (and possibly empty) surface in M_{m-1}. Hence N_{m-1} is π_1-injective in M_{m-1}. Since S_{m-1} is incompressible in N_{m-1}, it follows that it is also incompressible in M_{m-1}. We therefore have a contradiction to the minimality of m.

Hence conclusion (2) of Proposition 4.4 must hold; in particular, N_{m-1} is closed, so that $N_{m-1} = M_{m-1}$, and $\mathrm{rk}_2\, M_{m-1} = \mathrm{rk}_2\, N_{m-1} \leq 4h - 3 \leq 4g - 3$. On the other hand, since by hypothesis we have $\mathrm{rk}_2\, M_0 \geq 4g - 1$, it follows from Lemma 8.5 that for any index j such that $0 \leq j \leq n$ and such that M_j is closed, we have $\mathrm{rk}_2\, M_j \geq 4g - 2$. This is a contradiction, and the proof is complete. □

9. Proof of the geometric theorem

As a preliminary to the proof of Theorem 9.6 we shall point out how the Marden tameness conjecture, recently established by Agol [1] and by Calegari-Gabai [4], strengthens the results proved in [3].

We first recall some definitions from [3, Section 8]. Let Γ be a discrete torsion-free subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$, and let $k \geq 2$ be an integer. We say that λ is a *k-Margulis number* for Γ, or for $M = \mathbb{H}^3/\Gamma$, if for any k elements $\xi_1,\dots,\xi_k \in \Gamma$ and for any $z \in \mathbb{H}^3$, we have that either

- the group $\langle \xi_1,\dots,\xi_k\rangle$ is generated by at most $k-1$ abelian subgroups, or
- $\max_{i=1}^{k} \mathrm{dist}(\xi_i \cdot z, z) \geq \lambda$.

We say that λ is a *strong k-Margulis number* for Γ, or for M, if for any k elements $\xi_1,\dots,\xi_k \in \Gamma$, and for any $z \in \mathbb{H}^3$, we have that either

- the group $\langle \xi_1,\dots,\xi_k\rangle$ is generated by at most $k-1$ abelian subgroups, or
- $\displaystyle\sum_{i=1}^{k} \frac{1}{1+e^{\mathrm{dist}(\xi_i\cdot z,z)}} \leq \frac{k}{1+e^{\lambda}}$.

We note that if λ is a strong k-Margulis number for Γ, then λ is also a k-Margulis number for Γ.

A group Γ is termed *k-free,* where k is a positive integer, if every subgroup of Γ whose rank is at most k is free.

Theorem 9.1. *Let $k \geq 2$ be an integer and let Γ be a discrete subgroup of $\mathrm{Isom}_+(\mathbb{H}^3)$. Suppose that Γ is k-free and purely loxodromic. Then $\log(2k-1)$ is a strong k-Margulis number for Γ.*

Proof. This is the same statement as [3, Proposition 8.1], except that the latter result contains the additional hypothesis that Γ is k-tame, in the sense that every subgroup of Γ whose rank is at most k is topologically tame. (To say that a discrete torsion-free subgroup Δ of $\mathrm{Isom}_+(\mathbb{H}^3)$ is topologically tame means that $\mathbb{H}^3/\Delta$ is diffeomorphic to the interior of a compact 3-manifold.) But the main theorem of [1] or [4] asserts that any finitely generated discrete torsion-free subgroup Δ of $\mathrm{Isom}_+(\mathbb{H}^3)$ is topologically tame. □

By combining this with another result from [3], we shall prove:

Theorem 9.2. *Suppose that M is an orientable hyperbolic 3-manifold without cusps and that $\pi_1(M)$ is 3-free. Then either M contains a hyperbolic ball of radius $(\log 5)/2$ or $\pi_1(M)$ is a free group of rank 2.*

Proof. We may write $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \mathrm{Isom}(\mathbb{H}^3)$ is discrete and purely loxodromic. Since $\Gamma \cong \pi_1(M)$ is 3-free, it follows from Theorem 9.1 that $\log 5$ is a strong 3-Margulis number, and in particular a Margulis number, for Γ (or equivalently for M). According to [3, Theorem 9.1], if M is a hyperbolic 3-manifold without cusps, if $\pi_1(M)$ is 3-free and if λ a 3-Margulis number for M, then either M contains a hyperbolic ball of radius $\lambda/2$, or $\pi_1(M)$ is a free group of rank 2. The assertion follows. □

Corollary 9.3. *Suppose that M is a closed orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 3-free. Then M contains a hyperbolic ball of radius $(\log 5)/2$. Hence the volume of M is greater than 3.08.*

Proof. It follows from Theorem 9.2 that either M contains a hyperbolic ball of radius $(\log 5)/2$ or $\pi_1(M)$ is a free group of rank 2. The latter alternative is impossible, because Γ, as the fundamental group of a closed hyperbolic 3-manifold, must have cohomological dimension 3, whereas a free group has cohomological dimension 1. Thus M must contain a hyperbolic ball of radius $(\log 5)/2$.

The lower bound on the volume now follows by applying Böröczky's density estimate for hyperbolic sphere-packings as in [6]. □

Theorem 9.4 (Agol-Storm-Thurston)**.** *Suppose that M is a closed orientable hyperbolic 3-manifold containing a connected incompressible closed surface S. Then either $\mathrm{Vol}(M) > 3.66$ or the manifold X obtained by splitting M along S has the form $X = |\mathcal{W}|$ for some (possibly disconnected) book of I-bundles $\mathcal{W}$ such that each page of $\mathcal{W}$ has strictly negative Euler characteristic.*

Proof. According to [2, Corollary 2.2], if S is an incompressible closed surface in a closed orientable hyperbolic 3-manifold M, if X denotes the manifold obtained by splitting M along S, and if $K = \overline{X - \Sigma}$ where Σ denotes the relative characteristic submanifold of the simple manifold X, then the volume of M is greater than $\bar{\chi}(K) \cdot 3.66$. Hence either $\mathrm{Vol}(M) > 3.66$ or $\chi(K) = 0$.

In the case where $\chi(K) = 0$ we note that each component of K must have Euler characteristic ≤ 0, because the components of the frontier of K in X are essential annuli in X. Since $\chi(K) = 0$ it follows that each component of K has Euler characteristic 0. Hence if Y denotes the union of all components of Σ with strictly negative Euler characteristic, and if we set $Z = \overline{X - Y}$, then each component of Z has Euler characteristic 0. But Z is π_1-injective in X because its frontier components are essential annuli. Since X is simple, the components of Z are solid tori. Since $Y = \overline{X - Z}$ is an I-bundle with $Y \cap Z = \partial_v Y$ and the components of $\partial_v Y$ are π_1-injective in X and hence in Z, it follows from the definition that X is a book of I-bundles. □

Proposition 9.5. *Suppose that M is a closed orientable hyperbolic 3-manifold such that $\mathrm{rk}_2 M \geq 7$. Suppose that $\pi_1(M)$ has a subgroup isomorphic to a genus-2 surface group. Then $\mathrm{Vol}\, M \geq 3.66$.*

Proof. Since M is simple and $\operatorname{rk}_2 M \geq 7$, it follows from Theorem 8.13 that M contains a closed incompressible surface of genus 2.

Suppose that $\operatorname{Vol} M < 3.66$. Let X denote the manifold obtained by splitting M along S. According to Theorem 9.4, each component of $M - S$ has the form $|\mathcal{W}|$ for some book of I-bundles $\mathcal{W}$.

Consider the subcase in which X is connected. Since S has genus 2, we have $\bar{\chi}(X) = 2$. By Lemma 2.11 it follows that

$$\operatorname{rk}_2(X) \leq 2\bar{\chi}(X) + 1 \leq 5.$$

Hence

$$\operatorname{rk}_2 M \leq \operatorname{rk}_2 X + 1 \leq 6,$$

a contradiction to the hypothesis.

There remains the case in which X has two components, say X_1 and X_2. Since S has genus 2, we have $\bar{\chi}(X_i) = 1$ for $i = 1, 2$. By Lemma 2.11, it follows that

$$\operatorname{rk}_2(X_i) \leq 2\bar{\chi}(X_i) + 1 = 3.$$

Hence

$$\operatorname{rk}_2 M \leq \operatorname{rk}_2 X_1 + \operatorname{rk}_2 X_2 \leq 6,$$

and we have a contradiction. (The bound of 6 in this last inequality could easily be improved to 4, but this is obviously not needed.) □

We can now prove our main geometrical result.

Theorem 9.6. *Let M be a closed orientable hyperbolic 3-manifold such that* $\operatorname{Vol} M \leq 3.08$. *Then* $\operatorname{rk}_2 M \leq 6$.

Proof. Assume that $\operatorname{rk}_2 M \geq 7$. If $\pi_1(M)$ has a subgroup isomorphic to a genus-2 surface group, then it follows from Proposition 9.5 (with $g = 2$) that $\operatorname{Vol} M \geq 3.66 > 3.08$, a contradiction to the hypothesis. There remains the possibility that $\pi_1(M)$ has no subgroup isomorphic to a genus-2 surface group. In this case, since $\operatorname{rk}_2 M \geq 5$, it follows from [3, Proposition 7.4 and Remark 7.5] that $\pi_1(M)$ is 3-free. Hence by Corollary 9.3 we have $\operatorname{Vol} M > 3.08$, and again the hypothesis is contradicted. □

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