GROUP ACTIONS ON $\mathbb{R}$-TREES

MARC CULLER *and* JOHN W. MORGAN

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0. *Introduction*

An $\mathbb{R}$-tree is a non-empty metric space in which any two points are joined by a unique arc, and in which every arc is isometric to a closed interval in the real line. Group actions on $\mathbb{R}$-trees arise naturally from groups of isometries of hyperbolic space, and have had significant application in the study of hyperbolic manifolds. In **[6]**, **[11]** and **[13]** it is shown that the space of conjugacy classes of representations of a finitely generated group G into $\mathrm{SO}(n, 1)$ has a natural compactification whose ideal points are isomorphism classes of actions of G on $\mathbb{R}$-trees. From a different point of view, both Gromov and Thurston **[17]** have constructed $\mathbb{R}$-trees as limits of sequences of hyperbolic spaces, scaled so that the curvature goes to $-\infty$. These theorems suggest that the study of group actions on $\mathbb{R}$-trees can be viewed as a natural extension of representation theory; we take this point of view here.

There are two features of group actions on $\mathbb{R}$-trees which are addressed in this paper. The first is that isometries of $\mathbb{R}$-trees behave in many ways like isometries of hyperbolic space, and that groups of isometries of $\mathbb{R}$-trees resemble subgroups of $\mathrm{SO}(n, 1)$. The second is that, for a fixed finitely generated group G, the space of all actions of G on $\mathbb{R}$-trees has strong compactness properties.

Actions on simplicial trees are useful in combinatorial group theory, as was made clear in Serre's book **[16]**. Actions on $\mathbb{R}$-trees have applications in this area as well. For example, it was shown in **[7]** that the space of all free properly discontinuous actions of a free group on $\mathbb{R}$-trees is a contractible space, the analogue for free groups of the Teichmüller space associated to a closed surface group. It follows from results proved here that this space has a compactification in which the ideal points are actions on $\mathbb{R}$-trees for which stabilizers of non-degenerate arcs are cyclic. Thus the compactification is the strict analogue of Thurston's compactification of Teichmüller space as described in **[13]**. We hope that this compact space will be useful in studying outer automorphisms of free groups in the same way that Thurston's compactification of Teichmüller space was useful in studying automorphisms of surfaces.

The definition of an $\mathbb{R}$-tree was first given by Tits **[18]**, who introduced them as generalizations of local Bruhat–Tits buildings for rank-1 groups and showed that certain groups of higher rank cannot act on $\mathbb{R}$-trees without fixed points. (Tits only considered $\mathbb{R}$-trees which are complete as metric spaces. The assumption of completeness is usually irrelevant. However, it does make a difference in the case of an infinitely generated group acting so that every element has a fixed point.) It was observed in **[18]** that $\mathbb{R}$-trees behave like hyperbolic spaces and that the space of ends of an $\mathbb{R}$-tree is analogous to the sphere at infinity of hyperbolic space.

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In [**5**] Chiswell studied actions of a group on trees in terms of associated *based* length functions on the group. If $X \times G \to X$ is an action of a group G by isometries of a metric space X, then a *based length function* L_p associated to a base point $p \in X$ is defined by $L_p(g) = \operatorname{dist}(p, p \cdot g)$. Chiswell gave an axiomatic characterization of the (integer-valued) based length functions associated to actions of G on simplicial trees in terms of axioms which had been studied earlier by Lyndon [**10**]. For each function L: $G \to \mathbb{R}$ satisfying these axioms, Chiswell constructed an action of G on a contractible metric space T and a point $p \in T$ so that $L = L_p$. He viewed the space T as being a generalized simplicial tree. Later, Alperin and Moss [**2**] and, independently, Imrich [**9**], showed that T is an $\mathbb{R}$-tree in our sense. We will make use of these results in several of our proofs.

We will study the actions of a group on $\mathbb{R}$-trees in terms of their associated translation length functions. These are analogues of characters of representations. If G acts by isometries on hyperbolic space then the length $\|g\|$ of an element g of G is zero if g has no axis (i.e. if g is a parabolic isometry) and otherwise is the distance of translation of g along its axis in hyperbolic space. Notice that if G is the fundamental group of a closed hyperbolic manifold M, then $\|g\|$ is the length of the closed geodesic in M representing the conjugacy class of g. If a group G acts by isometries on an $\mathbb{R}$-tree T then, by analogy, we define the *translation length* of $g \in G$ to be

$$\|g\| = \inf_{x \in T} \operatorname{dist}(x, x \cdot g).$$

Let Ω denote the set of conjugacy classes in G. Since translation length functions are constant on conjugacy classes, we may regard them as points of the (usually infinite-dimensional) Euclidean space $\mathbb{R}^\Omega$ whose coordinates are indexed by the elements of Ω. The subspace $\mathrm{LF}(G) \subset \mathbb{R}^\Omega$ of all translation length functions of actions of G on $\mathbb{R}$-trees is an analogue of the character variety of a group.

Any action of G on an $\mathbb{R}$-tree whose translation length function is non-trivial determines a point in the projective space P^Ω which is the quotient of $\mathbb{R}^\Omega - \{0\}$ by the action of the group of homotheties. Of course, an action of G on hyperbolic space with a non-trivial length function also corresponds to a point in P^Ω. Since scaling the metric has no effect on the image in P^Ω, this is the natural ambient space in which to view actions on hyperbolic space converging to actions on $\mathbb{R}$-trees. In fact, it is in this space that compactification theorems of [**17**], [**13**], and [**11**] take place. The compactness results in this paper concern the image, $\mathrm{PLF}(G)$, of $\mathrm{LF}(G) \cap (\mathbb{R}^\Omega - \{0\})$ in P^Ω. (In [**18**] and [**13**] it is shown that a finitely generated group of isometries of an $\mathbb{R}$-tree has a common fixed point if and only if its translation length function is trivial. Hence, a fixed-point-free action of a finitely generated group determines a point of P^Ω.)

In the first section we list some of the most basic properties of isometries of $\mathbb{R}$-trees; our discussion overlaps substantially with that in [**18**] as well as that in [**13**].

In §§ 2 and 3 we develop the analogy between actions on $\mathbb{R}$-trees and linear representations. We will restrict attention to the case of $\mathrm{SO}(2,1)$ where the analogy is clearest. Recall that a linear representation is *reducible* if it has a non-trivial invariant subspace, and that it is *semi-simple* (or completely reducible) if every invariant subspace has an invariant complement. The connected component of the identity in $\mathrm{SO}(2,1)$ is the group, $\mathrm{SO}_0(2,1)$, of orientation-preserving isometries of the hyperbolic plane. A representation of G into $SO_0(2, 1)$ is reducible if and only if the induced action of G on the hyperbolic plane has

(i) a fixed point in the plane, or

(ii) an invariant line in the plane, or

(iii) a fixed point in the circle at infinity.

Such a representation is semi-simple in Cases (i) and (ii). (Of course, the irreducible representations are also semi-simple.)

The space at infinity for an ℝ-tree T, which is called its space of *ends*, is defined to be the limit of the inverse system $\pi_0(T - B)$ where B ranges over all closed and bounded sets in T. (Clearly, automorphisms of T extend continuously to the space of ends of T.) By analogy with the hyperbolic case, we say that an action of a group G on an ℝ-tree T is *reducible* if one of the following holds:

(i) every element of G fixes a point of T; or

(ii) there is a line in T which is invariant under the action of G; or

(iii) there is an end of T which is fixed by G.

An action is *semi-simple* if it is irreducible, if it has a fixed point, or if it is of Type (ii) above. An action of Type (ii) which does not preserve the orientation on an invariant line will be called *dihedral*. Actions of Type (ii) which preserve the orientation on an invariant line will be called *shifts*. (Note that for actions on hyperbolic space, if every element of the group has a fixed point then the whole group has one as well. This is not necessarily true for actions on ℝ-trees if the group in question is not finitely generated.)

If G is irreducibly represented in $SO(2, 1)$ then it is an easy exercise to show that G contains a free group of rank 2. The analogue for ℝ-trees is proved in § 2 (see Theorem 2.7):

THEOREM. *If G acts irreducibly on an ℝ-tree then G contains a free group of rank* 2.

Every character of a non-semi-simple representation of a group into $SO(2, 1)$ equals the character of some semi-simple (even diagonalizable) representation into $SO(2, 1)$. The same is true for actions on ℝ-trees (see Corollary 2.4).

THEOREM. *The translation length function of a non-semi-simple action on an ℝ-tree equals that of an action on ℝ by translations.*

Recall that a semi-simple representation of G into $SO(2, 1)$ is determined up to conjugacy by its character. In the third section we shall show that semi-simple actions of G on ℝ-trees are essentially determined by their translation length functions. If the tree contains a proper subtree which is invariant under the action of G, then the translation length function associated to the action on the subtree is the same as that associated to the original action. Thus, we consider actions which are *minimal* in the sense that the tree contains no proper invariant subtree. (In Proposition 3.1 we prove that minimal invariant subtrees always exist for semi-simple actions.) Our first main result is the following uniqueness theorem (Theorem 3.7).

THEOREM. *Suppose that $T_1 \times G \to T_1$ and $T_2 \times G \to T_2$ are two minimal semi-simple actions of a group G on $\mathbb{R}$-trees with the same translation length function. Then there exists an equivariant isometry from T_1 to T_2. If either action is not a shift then the equivariant isometry is unique.*

Analogously to the case of representations into SO(2, 1), this result is not true for non-semi-simple actions.

The theory of non-semi-simple actions on $\mathbb{R}$-trees diverges somewhat from that of non-semi-simple representations into SO(2, 1). At the end of § 3 we give some examples to illustrate these differences.

In the fourth section we turn from the study of a single action to that of the space of all actions of a given group G on $\mathbb{R}$-trees. Recall that $\mathrm{PLF}(G) \subset \mathrm{P}^{\Omega}$ is the subspace whose elements are projectivized translation length functions of fixed-point-free actions of G on $\mathbb{R}$-trees. Our second main result (Theorem 4.5) is:

THEOREM. *If G is a finitely generated group then the space* $\mathrm{PLF}(G)$ *is a compact subset of* P^{Ω}.

If G acts on an $\mathbb{R}$-tree, we say that a subgroup $H \subset G$ is an *arc stabilizer* if H is the maximal subgroup of G stabilizing some non-degenerate arc in the $\mathbb{R}$-tree. In the study of free groups or fundamental groups of surfaces, the actions which arise geometrically either are free or have abelian arc stabilizers. In general the free actions do not form a closed set. For example, measured laminations on a surface determine a compact space of actions of its fundamental group on $\mathbb{R}$-trees. A dense subset consists of free actions, but there are some for which the stabilizer of a point contains a non-abelian free subgroup. Nevertheless, for all of these actions the stabilizer of any non-degenerate arc is cyclic (cf. **[15]**). Thus, it is natural to ask in general whether there is a compact space of actions for which the arc stabilizers are 'small'. We define $\mathrm{SLF}(G)$ to be the subspace of $\mathrm{PLF}(G)$ which consists of all projective classes of translation length functions of actions for which no arc stabilizer contains a free subgroup of rank 2. In § 5 we show that this subspace is closed, giving us the following (Theorem 5.3):

THEOREM. *If G is a finitely generated group then the space* $\mathrm{SLF}(G)$ *is a compact subset of* P^{Ω}.

In proving Theorem 4.5 we consider the space $\Psi\mathrm{LF}(G)$ of 'pseudo-length functions' on G. This space is defined by piecewise linear inequalities and contains $\mathrm{PLF}(G)$. The proof of Theorem 4.5 uses a number of properties of pseudo-length functions which follow formally from the defining inequalities; these are proved in § 6.

We learned late in the course of this work that Alperin and Bass were working independently on many of the same questions. There is substantial overlap between their work **[1]** and the work in §§ 1, 2, 3, 4, and 6 of this paper. One difference is that while we restrict ourselves to $\mathbb{R}$-trees they work with Λ-trees for a general ordered abelian group Λ. We thank Ken Brown for pointing out to us Tits's paper **[18]** and for several other helpful remarks.

We end this section by listing some questions that we find interesting.

(1) Is every pseudo-length function a translation length function?† If so, this would give a direct method of constructing the actions on $\mathbb{R}$-trees which compactify the character variety.

(2) Are the simplicial actions dense in PLF(G) or in SLF(G)? If so, this would give a structure theorem for those groups whose space of conjugacy classes of discrete and faithful representations into SO(n, 1) is non-compact. Such groups would decompose non-trivially as free products with amalgamation, or as HNN-extensions, along virtually abelian subgroups.

(3) What is the topology of PLF(G) and SLF(G)? For example, when are these spaces infinite-dimensional?

(4) Which finitely generated groups act freely on $\mathbb{R}$-trees? Peter Shalen asks whether such a group must be a free product of abelian groups and surface groups.

1. *Elements of the geometry of $\mathbb{R}$-trees*

In this section we review a number of properties of isometries of $\mathbb{R}$-trees. At the end of this section we list five 'axioms' which describe how translation lengths behave under the operations in the group of isometries of an $\mathbb{R}$-tree. A natural question (which we cannot answer) is whether these axioms actually characterize translation length functions, or indeed whether it is possible to characterize translation length function in terms of axioms, such as these, which deal only with single elements or pairs of elements.

Recall that in an $\mathbb{R}$-tree any two points are joined by a unique embedded arc; this arc will be called a *geodesic*. It is also part of the definition that, with the induced metric, each geodesic in an $\mathbb{R}$-tree is isometric to an interval in the real line. Whenever convenient we will assume that a geodesic is parametrized by arc-length (i.e. by distance from one of its endpoints). If α is an oriented geodesic then the reverse of α will be denoted $\bar{\alpha}$.

If two geodesics have the same initial endpoint then their intersection is a geodesic. We define a *direction* at a point p of an $\mathbb{R}$-tree to be an equivalence class of geodesics with initial point p under the following relation: $\alpha \sim \alpha'$ if α meets α' in a geodesic of positive length. If α and β are two geodesics in an $\mathbb{R}$-tree and the initial point of α equals the terminal point of β, then the product path $\alpha\beta$ is a geodesic if and only if the direction of $\bar{\alpha}$ is distinct from the direction of β.

1.1. *If T_1 and T_2 are disjoint non-empty closed subtrees of T then there is a unique shortest geodesic having its initial point in T_1 and its terminal point in T_2.*

Proof. It suffices to show that there is a unique geodesic such that its initial point is in T_1, its terminal point is in T_2, and its interior is disjoint from $T_1 \cup T_2$. To see that an arc with these properties exists, consider a geodesic γ joining a point of T_1 to a point of T_2. The closure in γ of $\gamma - (T_1 \cup T_2)$ has the desired properties. Suppose that α and β were two distinct such arcs. Without loss of generality we may assume that the initial points of α and β are distinct. Let γ be a geodesic from the initial point of α to that of β. Then γ is contained in T_1; since α

† *Added in proof*: William Parry has answered this question in the affirmative.

and β meet T_1 in points, the arc $\bar{\alpha}\gamma\beta$ is a geodesic. This is impossible since it has endpoints in T_2 but is not contained in T_2.

The geodesic in 1.1 will be called the *spanning* geodesic from T_1 to T_2. If T_1 and T_2 are closed subtrees which meet in a point then we will also call that point the spanning geodesic from T_1 to T_2.

1.2. *If T_1, T_2, and T_3 are closed subtrees of T with $T_1 \cap T_3$ and $T_2 \cap T_3$ both non-empty but with $T_1 \cap T_2 \cap T_3 = \varnothing$ then $T_1 \cap T_2 = \varnothing$.*

Proof. If there is a non-degenerate arc in T with one end point in T_1 and the other in T_2 and which is otherwise disjoint from $T_1 \cup T_2$ then $T_1 \cap T_2 = \varnothing$. The spanning geodesic between $T_1 \cap T_3$ and $T_2 \cap T_3$ in T_3 is such an arc.

For each isometry g of T we define the *characteristic set* of g to be the set $C_g = \{x \in T \mid \operatorname{dist}(x, x \cdot g) = \|g\|\}$.

1.3. *The characteristic set C_g is a closed non-empty subtree of T, which is invariant under the action of g. In addition,*

(i) *if $\|g\| = 0$ then C_g is the fixed set of g;*

(ii) *if $\|g\| > 0$ then C_g is isometric to the real line and the action of g on C_g is translation by $\|g\|$; and*

(iii) *for any $p \in T$, we have* $\operatorname{dist}(p, p \cdot g) = \|g\| + 2\operatorname{dist}(p, C_g)$.

Proof. It is clear from the definition that C_g is invariant under the action of g.

First assume that g has a fixed point. Clearly, in this case, $\|g\| = 0$ and C_g is the fixed set of g. In particular, it is closed and non-empty. If an isometry of T fixes both endpoints of a geodesic then it must fix the entire geodesic. Thus C_g is connected and hence is a subtree of T. Using 1.1, we can easily see that (iii) holds in this case.

Next suppose that g has no fixed point. Let x be any point of T and let α be the arc from x to $x \cdot g$. An easy geometric argument shows that $\alpha \cap \alpha \cdot g$ and $\alpha \cap \alpha \cdot g^{-1}$ are subarcs of α which do not contain its midpoint. Let β be the spanning arc from $\alpha \cdot g^{-1}$ to $\alpha \cdot g$. Then β is a subarc of α which has positive length and meets each of $\beta \cdot g$ and $\beta \cdot g^{-1}$ exactly in one endpoint. This implies that the union A of the arcs $\beta \cdot g^n$ for all $n \in \mathbb{Z}$ is isometric to $\mathbb{R}$ and that g acts on A by translation with β as fundamental domain. If p is a point of T then it is easily shown that

$$\operatorname{dist}(p, p \cdot g) = \operatorname{length} \beta + 2\operatorname{dist}(p, A),$$

from which it follows that $\|g\| = \operatorname{length} \beta > 0$ and $A = C_g$.

The proof of 1.3 shows that if $\|g\| = 0$ then g has a fixed point. It also follows from this argument that, for any non-zero integer n, $C_g \subset C_{g^n}$, with equality if g has no fixed point.

The arguments in the proof of 1.3 are given in more detail, and in greater generality, in **[13]**.

1.4. DEFINITION. If g is an isometry of T with no fixed point then g is *hyperbolic*; otherwise g is *elliptic*. If g is hyperbolic then the invariant line C_g is called the *axis* of g.

The action of a hyperbolic element g on its axis C_g induces an orientation of C_g with respect to which g translates in the positive direction. A point p is contained in the axis of g if and only if the geodesic α from p to $p . g$ has the property that the image under g of the direction of α at p is distinct from the direction of $\bar{\alpha}$ at $p \cdot g$.

1.5. *Let g and h be isometries of T. If $C_g \cap C_h$ is empty, or if it is a single point and g and h are hyperbolic, then*

$$\|gh\| = \|gh^{-1}\| = \|g\| + \|h\| + 2\,\mathrm{dist}(C_g, C_h).$$

Proof. Let α be the spanning geodesic from C_g to C_h. Let p and q be the respective initial and terminal points of α. Let β and γ be the geodesics from $p \cdot g^{-1}$ to p and from q to $q \cdot h$ respectively. (In the case when g and h have

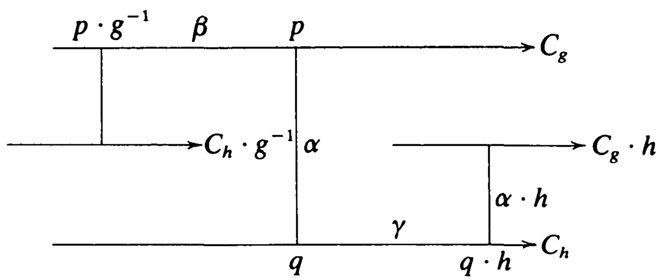

FIG. 1

disjoint axes, these geodesics are shown in Fig. 1.) The path $\omega = \beta\alpha\gamma(\bar{\alpha} \cdot h)$ is a geodesic since $\beta \subset C_g$, α meets $C_g \cup C_h$ only at its endpoints, $\gamma \subset C_h$, and $\bar{\alpha} \cdot h$ meets C_h only at its initial point. Moreover, $\beta \cdot gh \subset C_g \cdot h$ while $\bar{\alpha} \cdot h$ meets $C_g \cdot h$ only at its terminal point. Thus ω meets its translate under gh in a single point. This proves that $\omega \subset C_{gh}$ and that

$$\begin{aligned}\|gh\| &= \mathrm{length}(\omega) = \mathrm{length}(\beta) + \mathrm{length}(\gamma) + 2\,\mathrm{length}(\alpha)\\ &= \|g\| + \|h\| + 2\,\mathrm{dist}(C_g, C_h).\end{aligned}$$

Applying this same argument with h replaced by h^{-1} completes the proof.

An elaboration of the argument in 1.5 also shows:

1.6. *Under the hypothesis of* 1.5, *if g and h are hyperbolic isometries then $C_{gh} \cap C_{hg}$ equals the spanning geodesic from C_g to C_h.*

1.7. *If T_0 is a closed subtree of T disjoint from C_g then $T_0 \cap T_0 \cdot g = \varnothing$.*

Proof. Let α be the spanning arc from T_0 to C_g. Then $\alpha \cdot g$ is the spanning arc from $T_0 \cdot g$ to C_g. If g is hyperbolic then these arcs are disjoint. Otherwise they meet in a single point; namely the endpoint which is contained in C_g. But if there exists $p \in T_0 \cap T_0 \cdot g$, then the spanning arc from p to C_g would contain both α and $\alpha \cdot g$. This would imply that α meets $\alpha \cdot g$ in an arc of positive length, a contradiction.

1.8. *Let g and h be hyperbolic isometries of T. Then $C_g \cap C_h \neq \varnothing$ if and only if*

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|.$$

Moreover, $\|gh\| > \|gh^{-1}\|$ if and only if $C_g \cap C_h$ contains a non-degenerate arc and the orientations induced by g on C_g and by h on C_h agree on $C_g \cap C_h$.

Proof. It follows from 1.5 that if $C_g \cap C_h = \varnothing$ then $\|gh\| = \|gh^{-1}\| \neq \|g\| + \|h\|$. We therefore may assume that $C_g \cap C_h \neq \varnothing$.

First suppose that C_g meets C_h in a segment of positive length and that the induced orientations agree on the overlap. Let p be an interior point of the intersection. Consider the geodesic α from $p \cdot g^{-1}$ to $p \cdot h$. The direction of α equals that of the positive ray on C_g. It is mapped by gh to the direction of the positive ray on C_h emanating from $p \cdot h$. However, the direction of $\bar{\alpha}$ equals that of the negative ray on C_h emanating from $p \cdot h$. This shows that α is a fundamental domain for the action of gh on its axis. Therefore, $\|gh\|$ equals the length of α, which is $\|g\| + \|h\|$. Also, we have

$$\|gh^{-1}\| \leq \operatorname{dist}(p \cdot g^{-1}, p \cdot h) < \|g\| + \|h\|.$$

This proves the result in the case when $C_g \cap C_h$ contains a non-degenerate arc.

Now suppose that $C_g \cap C_h$ is a single point. Let p be this point. Let α be defined as above. The direction of $\alpha . gh$ points out of C_h, but that of $\bar{\alpha}$ points along C_h. Thus α is a fundamental domain for the action of gh on its axis in this case as well. This shows that $\|gh\| = \|g\| + \|h\|$. Similarly, $\|gh^{-1}\| = \|g\| + \|h\|$.

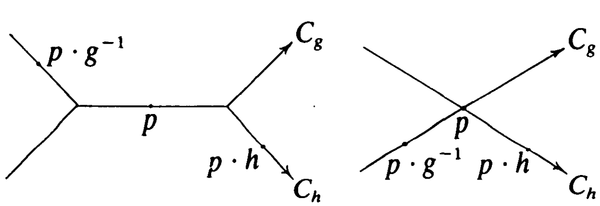

FIG. 2

1.9. *Let g and h be isometries of T with $C_g \cap C_h \neq \varnothing$. Then* $\max(\|gh\|, \|gh^{-1}\|) \leq \|g\| + \|h\|$.

Proof. Let p be a point of $C_g \cap C_h$. Then

$$\|gh\| \leq \operatorname{dist}(p \cdot g^{-1}, p \cdot h) \leq \operatorname{dist}(p \cdot g, p) + \operatorname{dist}(p, p \cdot h) = \|g\| + \|h\|.$$

A similar calculation applies to $\|gh^{-1}\|$.

1.10. *Suppose that g and h are isometries of T and that $C_g \cap C_h$ contains an arc J of length $\Delta \geq \|g\| + \|h\|$. If g and h are both hyperbolic then make the further assumption that the orientations induced by g and h on their axes agree on the overlap. Then $ghg^{-1}h^{-1}$ fixes a subarc of J of length $\Delta - \|g\| - \|h\|$. (If*

$\Delta = \|g\| + \|h\|$ then this statement is interpreted to mean that $ghg^{-1}h^{-1}$ fixes a point of J.)

Proof. The result is immediate from the description in 1.3 of the action of an isometry on its characteristic set.

1.11. We summarize this section by listing some fundamental properties of translation length functions which are all easy consequences of the facts proved above. We assume here that G is a group of isometries of an $\mathbb{R}$-tree T and that $\| \ \|$ denotes its translation length function.

I. $\|1\| = 0$.

II. $\|g\| = \|g^{-1}\|$ for all $g \in G$.

III. $\|g\| = \|hgh^{-1}\|$ for all $g, h \in G$.

IV. For all $g, h \in G$, either $\|gh\| = \|gh^{-1}\|$ or $\max(\|gh\|, \|gh^{-1}\|) \leq \|g\| + \|h\|$.

V. For all $g, h \in G$ such that $\|h\| > 0$ and $\|g\| > 0$, either

$$\|gh\| = \|gh^{-1}\| > \|g\| + \|h\|$$

or

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|.$$

2. *Actions on $\mathbb{R}$-trees with non-trivial translation length function*

Throughout this section we shall be considering actions on $\mathbb{R}$-trees whose associated translation length functions are not identically zero. These actions naturally fall into three classes: irreducible, dihedral, and those with a fixed end. Of course, all non-semi-simple actions are of the last type, and the semi-simple actions of the last type are shifts. In this section we characterize the three types of actions, both geometrically and in terms of their translation length functions.

By a *ray* in an $\mathbb{R}$-tree we mean the image of an isometric embedding of the ray $[0, \infty)$. If p is the image of 0 under such an embedding then the ray will be said to *emanate* from p. Recall that the space of ends of an $\mathbb{R}$-tree T is the limit of the inverse system $\pi_0(T - B)$ where B ranges over the closed and bounded subsets of T. Let p be a fixed point of T. For every point of the limit there exist a ray emanating from p, and a cofinal sequence of components of complements of closed and bounded sets representing the limit point, such that every set in the sequence meets the ray. There is only one such ray for each end. Thus the ends of T correspond one-to-one with the rays in T which emanate from the point p. Given two distinct ends of T, the corresponding rays will meet in an interval with p as an endpoint. The union of these rays minus the interior of their intersection will be a line in the tree. Conversely, given any line in an $\mathbb{R}$-tree, the two oppositely oriented rays emanating from a point in the line determine two ends of the tree. This pair of ends does not depend on the choice of the point; we will say in this situation that the line *joins* the pair of ends of the tree. It is easy to see that there is only one line joining each pair of ends.

We begin by giving a geometric criterion for reducibility.

2.1. LEMMA. *Let $T \times G \to T$ be an action. Suppose that g is a hyperbolic* *element of G, and suppose that for all hyperbolic elements $h \in G$ the intersection $C_g \cap C_h$ is non-empty. Then the action is reducible.*

Proof. Each hyperbolic element h fixes exactly two ends of T, namely those determined by the two ends of its axis C_h. Let S be the subset of all ends which are fixed by every hyperbolic element of G. Clearly, S is invariant under G and the cardinality of S is at most 2. If the cardinality of S is 2, then the line between the two ends contained in S is invariant under G; hence the action is reducible. By definition, if S consists of a single point, then the action is reducible. It therefore suffices to prove that $S \neq \emptyset$.

We first show that if $h \in G$ is hyperbolic, then g and h fix a common end. Suppose not; i.e. suppose that there is a hyperbolic element h of G such that $C_g \cap C_h$ is bounded. Then by 1.2, for all sufficiently large n we have $C_g \cap C_g \cdot h^n = \emptyset$. This is equivalent to $C_g \cap C_{h^{-n}gh^n} = \emptyset$, which contradicts our hypothesis.

Since each hyperbolic element $h \in G$ fixes an end in common with g, either $S \neq \emptyset$ or there are two hyperbolic elements h and k of G which fix opposite ends of C_g. Thus $C_g \cap C_h \cap C_k$ is bounded. Using 1.2 again, for some n we would have $C_k \cap C_{g^{-n}hg^n} = \emptyset$. We may assume, without loss of generality, that the natural orientations of C_h and C_k agree on the overlaps with the natural orientation of C_g. Then the axis of the product $k^{-1}g^{-n}h^{-1}g^n$ may be constructed as in the proof of 1.5; this axis will meet C_g in the (bounded) spanning arc from C_k to $C_{g^{-n}hg^n}$, contradicting our hypothesis. Therefore $S \neq \emptyset$.

2.2. THEOREM. *Let $T \times G \to T$ be an action of a group G on an $\mathbb{R}$-tree T whose translation length function is non-trivial. Then the following are equivalent:*

(i) *the action has a fixed end;*

(ii) *for all elements g and h of G we have $C_g \cap C_h \neq \emptyset$;*

(iii) *for all elements g and h of G the intersection $C_g \cap C_h$ is unbounded;*

(iv) *there is a ray R in T such that for each $g \in G$ the intersection $R \cap C_g$ contains a subray of R.*

Any semi-simple action with these properties is a shift.

Proof. Clearly, (iv) implies (iii) which implies (ii). Let us assume that (ii) holds and prove (i). By Lemma 2.1, statement (ii) implies that the action is reducible. Since the action has a non-trivial translation length function, it either has a fixed end or is dihedral. Were it dihedral, there would be elements r and s in G acting as reflections with distinct fixed points on the invariant line. By 1.2, we would have $C_r \cap C_s = \emptyset$, contradicting (ii).

Now let us show that (i) implies (iv). Suppose that the action has a fixed end. Let R be a ray going to a fixed end of T. Since the end is fixed, for each $g \in G$ the intersection $R \cap R \cdot g$ contains a subray of R. This means, by 1.7, that C_g meets every subray of R, and hence that $R \cap C_g$ is a ray.

Clearly, the only semi-simple actions satisfying (i) are shifts.

2.3. COROLLARY. *For an action of G on an $\mathbb{R}$-tree with non-trivial translation length function the following are equivalent:*

(i) *the action has a fixed end;*

(ii) *the translation length function is given by* $|\rho(g)|$ *where* ρ: $G \to \mathbb{R}$ *is a homomorphism;*

(iii) *for all* g *and* h *in* G, $\|[g, h]\| = 0$.

Proof. We first prove that (i) implies (ii). If the action has a fixed end, then there is a ray R such that for each $g \in G$ the intersection $C_g \cap R$ is unbounded. Each $g \in G$ translates $R \cap R \cdot g^{-1}$ along R, and $\|g\| = \mathrm{dist}(x, x \cdot g)$ for any $x \in R \cap R \cdot g^{-1}$. We define $\rho(g) = \pm\mathrm{dist}(x, x \cdot g)$ for any $x \in R \cap R \cdot g^{-1}$ where the sign is '$+$' if g moves x towards the end of R, and is '$-$' in the opposite case. Clearly, $|\rho(g)| = \|g\|$. We claim that ρ is a homomorphism. Obviously, $\rho(1) = 0$ and $\rho(g^{-1}) = -\rho(g)$. Hence we need only show that $\rho(gh) = \rho(g) + \rho(h)$. This is immediate from a consideration of the action of gh on any

$$x \in R \cap R \cdot g^{-1} \cap R \cdot h^{-1}g^{-1}.$$

Clearly (ii) implies (iii). For the proof that (iii) implies (i), assume that $\|[g, h]\| = 0$ for all g and h in G. This means, by 1.5, that

$$\emptyset \neq C_g \cap C_{hg^{-1}h^{-1}} = C_g \cap (C_{g^{-1}}) \cdot h^{-1},$$

or equivalently, that $C_g \cap C_g \cdot h \neq \emptyset$. By 1.7 this implies that $C_g \cap C_h \neq \emptyset$. Thus, by 2.2 the action has a fixed end.

2.4. COROLLARY. *The translation length function of a non-semi-simple action of* G *equals that of a shift.*

Now let us turn to dihedral actions.

2.5. THEOREM. *For an action of* G *on an* $\mathbb{R}$*-tree with non-trivial translation length function the following are equivalent:*

(i) *the action is dihedral;*

(ii) *the translation length function is given by* $\|g\| = N(\rho(g))$ *where* ρ: $G \to \mathrm{Iso}(\mathbb{R})$ *is a homomorphism whose image contains a reflection, and* N *denotes the translation length function for the natural action of* $\mathrm{Iso}(\mathbb{R})$ *on* $\mathbb{R}$;

(iii) *the translation length function is non-trivial on some simple commutator in* G, *yet it vanishes on all simple commutators of hyperbolic elements.*

Proof. Suppose the action is dihedral. Then the translation length function of the action is equal to that of the action restricted to an invariant line. Hence (ii) holds.

Now suppose that (ii) holds; say that $\|g\| = N(\rho(g))$ for some homomorphism ρ: $G \to \mathrm{Iso}(\mathbb{R})$. Set $G_0 = \rho^{-1}(\mathbb{R})$ where $\mathbb{R} \subset \mathrm{Iso}(\mathbb{R})$ has been identified with the subgroup of translations. Clearly all hyperbolic elements of G lie in G_0. Since $\|\ \|$ restricted to G_0 is given by the absolute value of a homomorphism to $\mathbb{R}$, it follows from Corollary 2.3 that $\|[g, h]\| = 0$ for all $g, h \in G$ hyperbolic. There exists $g \in G - G_0$; necessarily $\rho(g)$ is a reflection. Since the translation length function is not identically zero, there is an element $h \in G_0$ with $\|h\| \neq 0$. Then $\|[g, h]\| = 2\,\|h\| \neq 0$.

Now suppose that (iii) holds. Since the commutator of any two hyperbolic elements has a fixed point, we can argue as in 2.3 to show that if g and h are hyperbolic elements then $C_g \cap C_h \neq \varnothing$. Thus, by Lemma 2.1, the action is reducible. Since the translation length function does not vanish on the commutator subgroup, by Corollary 2.3 the action does not have a fixed end. Since it has a non-trivial translation length function, it must then be dihedral.

Now we turn to the study of irreducible actions.

2.6. LEMMA. *Suppose that g and h are hyperbolic isometries of an $\mathbb{R}$-tree T such that $C_g \cap C_h$ is either empty or has length less than* $\min(\|g\|, \|h\|)$. *Then g and h generate a free group of rank 2 which acts freely and properly discontinuously on T.*

Proof. If $C_g \cap C_h \neq \varnothing$, let H be the union of two intervals which are fundamental domains for the actions of g and h on their respective axes and which each contain $C_g \cap C_h$ in their interiors. If $C_g \cap C_h = \varnothing$, choose intervals on C_g and C_h which are fundamental domains for the actions of g and h and so that the union of their interiors with the spanning arc is connected, and set H equal to this union.

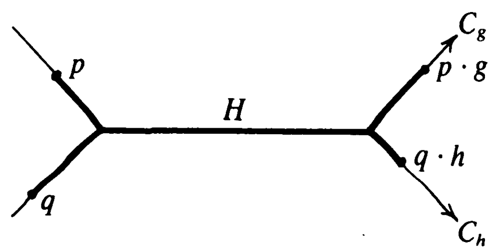

FIG. 3

We claim that H meets $H \cdot g^{\pm 1}$ and $H \cdot h^{\pm 1}$ at its endpoints, and is disjoint from $H \cdot w$ where w is any reduced word in g and h other than 1, $g^{\pm 1}$, or $h^{\pm 1}$. The proof is by induction on the length of w; it is easy to check in the case where w has length 1. For the inductive hypothesis we consider the four sets A, B, C, and D which are the interiors of the components of $T - \operatorname{Int}(H)$ which respectively contain the points p, q, $p \cdot g$, and $q \cdot h$. We show that if w is a reduced word in g and h of length at least 1 then $H \cdot w$ is contained in one of A, B, C, or D according to whether the last letter of w is g^{-1}, h^{-1}, g, or h. This clearly implies our claim. Observe that $B \cdot g \subset C$, $C \cdot g \subset C$, and $D \cdot g \subset C$. Thus if the last letter of w is g then $w = vg$ where the last letter of v is not g^{-1}. By induction $H \cdot v \subset B \cup C \cup D$. Thus $H \cdot w \subset C$. The cases where w ends in one of the other three letters are handled in the same way.

Let S be the subtree of T which is the union of the images of H under the action of $\langle g, h \rangle$. It is easily checked that $\langle g, h \rangle$ acts freely and properly discontinuously on S with fundamental domain H. It follows immediately from the previous paragraph that g and h generate a free group of rank 2.

2.7. THEOREM. *Let $T \times G \to T$ be an action with non-trivial translation length function. The following are equivalent:*

(i) *the action is irreducible;*

(ii) *there are hyperbolic elements* g *and* h *in* G *with* $\|[g, h]\| \neq 0$;

(iii) *there are hyperbolic elements* g *and* h *in* G *such that* $C_g \cap C_h$ *is an arc of finite positive length;*

(iv) G *contains a free group of rank* 2 *which acts freely and properly discontinuously on* T.

Proof. That (i) implies (ii) is immediate from 2.3 and 2.5.

Suppose that there are hyperbolic elements g and h of G with $\|[g, h]\| \neq 0$. This means by 1.10 that the intersection $C_g \cap C_h$ is a bounded subset of T. *If* C_g and C_h are disjoint axes then C_{gh} meets C_g in a finite arc. A fundamental domain for the action of gh on its axis is shown in Fig. 4 (cf. 1.5).

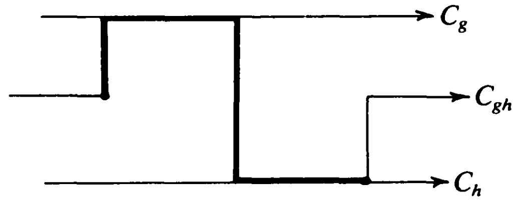

FIG. 4

If C_g and C_h are axes which meet in a single point then we again have that C_{gh} meets C_g in an arc of finite positive length. The axis of C_{gh} is shown in Fig. 5.

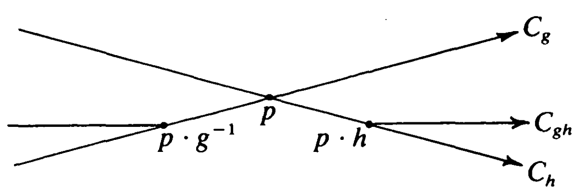

FIG. 5

Thus in all cases we can find hyperbolic elements whose axes intersect in a non-degenerate interval of finite length. This shows that (ii) implies (iii).

Now let us suppose that g and h are hyperbolic elements whose axes intersect in a non-degenerate interval of finite length. By taking powers we can assume that $\|g\|$ and $\|h\|$ are both longer than the length of the intersection $C_g \cap C_h$. By Lemma 2.6, the subgroup $\langle g, h\rangle$ is a free group of rank 2 acting freely and properly discontinuously on T.

Lastly we show that (iv) implies (i). Clearly if G contains a free group of rank 2 acting freely and properly discontinuously then there are hyperbolic elements g and h in G with $\|[g, h]\| \neq 0$. Thus by 2.3 the action has no fixed end, and by 2.5 the action is not dihedral. The only remaining possibility is that it is irreducible.

One consequence of these results is that the type of an action of G on an ℝ-tree with non-trivial translation length function is determined by its translation length function according to the following scheme:

FIXED END	$\Leftrightarrow$	$\| \ \|$ is trivial on $[G, G]$,
DIHEDRAL	$\Leftrightarrow$	$\| \ \|$ is non-trivial on $[G, G]$ but $\|[g, h]\| = 0$ for all hyperbolic g and h,
IRREDUCIBLE	$\Leftrightarrow$	$\|[g, h]\| \neq 0$ for some hyperbolic elements g and h.

3. *Uniqueness results for semi-simple actions*

In the previous section we showed that the three types of semi-simple actions are distinguished by their translation length functions. Here we will extend these results by showing that minimal semi-simple actions are completely determined by their translation length functions. At the end of the section we discuss some examples of non-semi-simple actions.

We begin with a result on the existence of minimal invariant subtrees. Recall that if $T \times G \to T$ is an action and if $T_0 \subset T$ is a subtree invariant under G, then it is a *minimal invariant subtree* provided that it contains no proper invariant subtree.

3.1. PROPOSITION. *If G acts on an $\mathbb{R}$-tree with a non-zero translation length function then there exists a unique minimal invariant subtree.*

Proof. Since the translation length function is non-zero, there exists a hyperbolic element of G. Consider the union of the axes of all the hyperbolic elements of G. Clearly this set is invariant, and, by the argument given in the proof of 1.3, it is contained in any invariant non-empty subset of the tree. To show that this set is the minimal invariant subtree we must show that it is connected. But this follows immediately from the fact that if C_g and C_h are disjoint axes then they both have non-empty intersection with the axis C_{gh} (cf. the proof of 2.7).

It is shown in [18] and [13] that if G is finitely generated and acts on an $\mathbb{R}$-tree so that every element of G has length 0, then G has a fixed point and hence has a minimal invariant subtree. However, there are fixed-point-free actions of non-finitely generated groups without minimal invariant subtrees. An example is given at the end of the section (see Remark 3.10).

In proving that the translation length function of a minimal semi-simple action determines the action, the reducible case is easily handled. In the irreducible case the proofs depend on the existence of certain special pairs of hyperbolic isometries.

3.2. DEFINITION. Let g and h be isometries of an $\mathbb{R}$-tree T. We will say that g and h are a *good pair* of isometries if the following conditions are satisfied:

(i) the elements g and h are hyperbolic;
(ii) the axes C_g and C_h meet in an arc of positive length;
(iii) the orientations on the axes C_g and C_h induced respectively by g and h agree on the arc $C_g \cap C_h$;
(iv) the length of $C_g \cap C_h$ is strictly less than $\min(\|g\|, \|h\|)$.

3.3. REMARK. Note that if g and h are hyperbolic isometries of an $\mathbb{R}$-tree such that C_g meets C_h in an arc of positive but finite length, then there exist integers $a > 0$ and b such that g^a and h^b form a good pair of isometries. The sign of b is chosen to ensure that the induced orientations agree. The magnitudes of a and b are chosen so that $\|g^a\|$ and $\|h^b\|$ are greater than the length of $C_g \cap C_h = C_{g^a} \cap C_{h^b}$.

3.4. LEMMA. *Suppose that g and h are a good pair of isometries. Let* Δ *be the length of the geodesic* $C_g \cap C_h$. *Then*

(i) $\|gh\| = \|g\| + \|h\|$,

(ii) $\|gh^{-1}\| = \|g\| + \|h\| - 2\Delta > 0$,

(iii) $\|ghg^{-1}h^{-1}\| = 2\,\|g\| + 2\,\|h\| - 2\Delta > 0$, *and*

(iv) *the three axes* C_g, C_h, *and* $C_{gh^{-1}}$ *meet in a single point.*

Proof. Part (i) follows immediately from 1.8. For parts (ii), (iii), and (iv) we will use the facts that fundamental domains for the actions of gh, gh^{-1}, and hg on their respective axes are as shown in Fig. 6. These fundamental domains can be constructed as in 1.8.

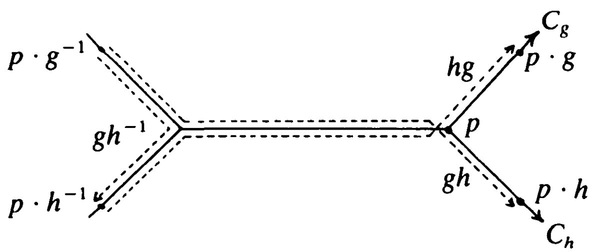

FIG. 6

Parts (ii) and (iv) are immediate from the figure. For part (iii) note that $C_g \cap C_h = C_{gh} \cap C_{hg}$. Thus gh and hg are also a good pair and their axes intersect in an arc of length Δ. So by (ii) and (i),

$$\|ghg^{-1}h^{-1}\| = \|gh\| + \|hg\| - 2\Delta = 2\,\|g\| + 2\,\|h\| - 2\Delta.$$

3.5. REMARK. Suppose that g and h are hyperbolic elements with disjoint axes or axes which meet in a single point. By inspection of the proof of 1.5, one checks that the elements gh and $g^{-1}h^{-1}$ either are a good pair or have axes which meet in a single point. Furthermore, we have $\text{length}(C_{gh} \cap C_{g^{-1}h^{-1}}) = \text{dist}(C_g, C_h)$. Thus, by 3.4 part (i) or 1.5, in these cases we have the following analogue of the formula in (iii):

$$\|ghg^{-1}h^{-1}\| = 2\,\|g\| + 2\,\|h\| + 4\,\text{dist}(C_g, C_h).$$

Good pairs of isometries can be characterized in terms of lengths as follows.

3.6. LEMMA. *Two isometries g and h of an* $\mathbb{R}$*-tree are a good pair if and only if*

$$0 < \|g\| + \|h\| - \|gh^{-1}\| < 2\min(\|g\|, \|h\|).$$

Proof. The two inequalities in the statement hold for a good pair of isometries by Lemma 3.4 part (ii), since by the definition of a good pair $0 < \Delta < \min(\|g\|, \|h\|)$. To prove the converse, note first that the inequalities imply that both g and h are hyperbolic. It then follows from 1.5 that the axes C_g and C_h have non-empty intersection. By 1.8 they meet in a geodesic of positive length and the orientations must agree on the overlap. Assume that $\min(\|g\|, \|h\|)$ is less than or equal to the length of $C_g \cap C_h$. Let J be an arc in $C_g \cap C_h$ of length at least $\min(\|g\|, \|h\|)$, and let p be the positive endpoint of the segment J. Consider the geodesic arcs from p to $p \cdot g^{-1}$ and $p \cdot h^{-1}$. One of these geodesics must be contained in the other. Thus, we have

$$\|gh^{-1}\| \leqslant \operatorname{dist}(p \cdot g^{-1}, p \cdot h^{-1}) = | \, \|g\| - \|h\| \, |.$$

This implies that

$$\|g\| + \|h\| - \|gh^{-1}\| \geqslant \|g\| + \|h\| - | \, \|g\| - \|h\| \, | = 2 \min(\|g\|, \|h\|),$$

which contradicts the second inequality in the hypothesis.

Both the uniqueness theorem below and the compactness theorems of §§ 4 and 5 depend heavily on work of Chiswell [**5**] and Alperin and Moss [**2**] or Imrich [**9**]. Chiswell considers real-valued functions L on a group G which satisfy the following three axioms of Lyndon [**10**]:

(i) $L(1) = 0$;

(ii) $L(g^{-1}) = L(g)$ for all $g \in G$;

(iii) $C(g, h) > C(h, k)$ implies $C(g, k) = C(h, k)$ for all $g, h, k \in G$, where by definition $C(g, h) = \frac{1}{2}[L(g) + L(h) - L(hg^{-1})]$.

It is easy to see that the based length function for any action of G on a based $\mathbb{R}$-tree satisfies these axioms. Conversely, given a function $L: G \to \mathbb{R}$ satisfying the axioms, Chiswell constructs an action of G on a metric space T and a point $p \in T$ such that the based length function L_p equals L. It is implicit from the construction of T that if $T' \times G \to T'$ is an action of G on an $\mathbb{R}$-tree and if there exists $p' \in T'$ such that the based length function $L_{p'}$ equals L then there is a unique G-equivariant embedding of (T, p) into (T', p'). Moreover, by results of Alperin and Moss [**2**] or Imrich [**9**], T itself is an $\mathbb{R}$-tree. Thus if the action on T' is minimal then (T, p) and (T', p') are uniquely G-equivariantly isometric.

3.7. THEOREM. *Suppose that $T_1 \times G \to T_1$ and $T_2 \times G \to T_2$ are two minimal semi-simple actions of a group G on $\mathbb{R}$-trees with the same translation length function. Then there exists an equivariant isometry from T_1 to T_2. If either action is not a shift then the equivariant isometry is unique.*

Proof. If the translation length function of a semi-simple action is identically zero, then by definition the action has a fixed point. If such an action is minimal then the $\mathbb{R}$-tree must be a point. The theorem is obvious in this case. Thus we can assume that both of the length functions in question are non-zero.

We first consider the case where one of the actions is reducible. Since reducibility can be detected from the translation length function, in this case both actions are reducible. If a minimal semi-simple action is reducible and has a non-trivial translation length function, then the tree is isometric to $\mathbb{R}$. It is an easy exercise to show that effective actions on $\mathbb{R}$ are determined by their translation length function. Thus we must only show that the kernel of the action is determined by the translation length function. To see this, observe that an element g of G fixes the entire line if and only if $\|g\| = 0$ and $\|[g, h]\| = 0$ for all hyperbolic elements $h \in G$. (There must exist a hyperbolic element since the action is minimal.) An equivariant isometry must take fixed points to fixed points; hence if either action is not a shift then there is a unique equivariant isometry from T_1 to T_2.

We now assume that both actions are irreducible. We will show that there exist points $p_i \in T_i$ for $i = 1, 2$ which determine the same based length function on G. The result will then follow from the work of Chiswell, Alperin and Moss, and Imrich described above.

Since both actions are irreducible, by Theorem 2.7 and Remark 3.3 there exist elements g and h of G which form a good pair of isometries of T_1. By Lemma 3.6 these elements are also a good pair of isometries of T_2. By applying part (iv) of Lemma 3.4, we see that for $i = 1, 2$ there is a unique point p_i of intersection of the axes of g, h, and gh^{-1} in T_i. To complete the argument we need only exhibit a common formula for the based length functions L_{p_i} in terms of the translation length function. We claim that for $k \in G$ and $i = 1, 2$,

$$\begin{aligned} L_{p_i}(k) &= \operatorname{dist}(p_i, p_i \cdot g) \\ &= \operatorname{dist}(C_g \cap C_h \cap C_{gh^{-1}}, C_g \cdot k \cap C_h \cdot k \cap C_{gh^{-1}} \cdot k) \\ &= \max(\operatorname{dist}(C, D)), \end{aligned}$$

where the maximum is taken as C ranges over the three lines C_g, C_h, $C_{gh^{-1}}$, and as D ranges over the three lines $C_g \cdot k$, $C_h \cdot k$, and $C_{gh^{-1}} \cdot k$. This is because no geodesic starting at p_i can meet all three of the axes C_g, C_h, and $C_{gh^{-1}}$ in geodesics of positive length. Thus the geodesic from p_i to $p_i \cdot k$ must in fact be the spanning geodesic from one of these three lines to one of the three images under k. To obtain the formula note that by 1.5 and 1.8, if g and h are hyperbolic then the distance between their axes is given by

$$\operatorname{dist}(C_g, C_h) = \tfrac{1}{2} \max(0, \|gh\| - \|g\| - \|h\|).$$

We have shown that the based length functions L_{p_i} are completely determined by the translation length function and hence are equal. Thus the trees T_i are equivariantly isometric.

Since any equivariant isometry from T_1 to T_2 must send the axes to axes, it must send p_1 to p_2. The uniqueness statement now follows immediately from the results of Chiswell, Alperin and Moss, and Imrich.

Polycyclic groups, i.e. those having a normal series with cyclic quotients, can act on hyperbolic space with no fixed point and no invariant line. For example, it is easy to construct unipotent subgroups of $SL_2(\mathbb{C})$ which are free abelian of rank 2 and act in this way. The next proposition shows that this is not true for actions of such groups on ℝ-trees.

3.8. PROPOSITION. *If G has a subgroup of finite index which is polycyclic, then any action of G on an ℝ-tree has either a fixed point or a unique invariant line.*

Proof. We prove this result first for polycyclic groups by induction on the length of the normal series. If the series has length 1, then the group is cyclic, and the result follows from 1.3 and 1.7.

Now let us turn to the inductive step. Suppose that we have an extension

$$1 \to N \to G \to A \to 1,$$

where the result holds for N, and A is a cyclic group. Let $T \times G \to T$ be an action of G. The action restricted to N has either a fixed point or a unique invariant line.

Let C_N denote the fixed subtree or the unique invariant line for N. Since N is normal in G, we know that G acts on C_N. If C_N is a line, then it is the unique invariant line for the action of G. If C_N is the fixed subtree of N, then the action of G on C_N factors through an action of A on C_N. Since the result holds for cyclic groups, it holds for this action of G.

This completes the proof for polycyclic groups. Next suppose that G has a subgroup of finite index which is of this type. There is a normal subgroup $H \subset G$ which is polycyclic and has finite index. For any action $T \times G \to T$ we know that H has either a fixed point or a unique invariant line. Let C_H be the fixed subtree or the unique invariant line for H. It is invariant under G. Thus, if C_H is the unique invariant line for H, it is also an invariant line for the action of G. Since hyperbolic isometries have infinite order, any finite group of isometries must consist only of elliptic elements. It is shown in **[13]** and **[18]** that such a group has a fixed point. Thus, if C_H is the fixed subtree for H, then, since G/H is a finite group, its action on C_H has a fixed point. Consequently, G fixes a point of C_H.

3.9. EXAMPLE. We give an explicit construction to illustrate the failure of Theorem 3.7 for non-semi-simple actions. Also, this example contains a subaction with trivial translation length function which has no minimal invariant subtree.

Let T be the rooted infinite tree in which each edge has two descendants. We label the directions at any vertex of T, except the root, by 'a' (above), 'l' (left), and 'r' (right). The two directions at the root are labelled 'l' and 'r'.

Consider $\mathbb{R}$ as a simplicial tree with vertices at the integer points. For each integer n let T_n be a copy of T. Form a homogeneous simplicial tree B with vertices of order 3 by joining the root of T_n to the integer point n by an edge. Extend the labelling of the directions at vertices as indicated in Fig. 7.

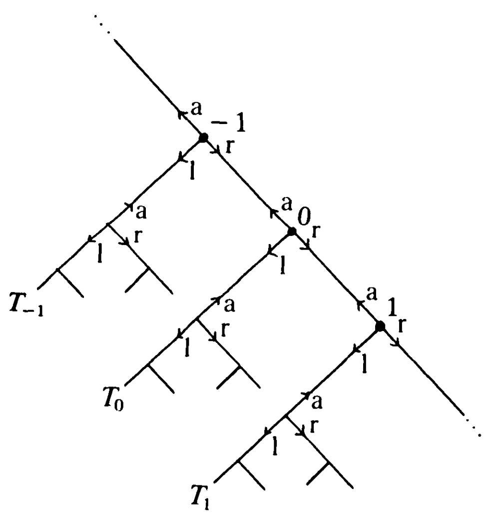

FIG. 7

Let g be the automorphism of B which shifts the line by one unit and maps T_n to T_{n+1} so as to preserve the labels. Let r be the automorphism of order 2 which fixes $(-\infty, 0] \cup \bigcup_{n<0} T_n$ and interchanges the two isomorphic subtrees T_0 and $[1, \infty) \cup \bigcup_{n>0} T_n$, preserving the labelling of directions at each vertex other than 0. Consider the group G of isometries of B generated by g and r. It is clear that G

fixes the end of T corresponding to the ray $(-\infty, 0]$, and hence the action is reducible. This action is easily seen to be transitive on edges and hence to be minimal.

The translation length function for the action of G on B is, by 2.3, the absolute value of a homomorphism of G to $\mathbb{Z}$ which sends g to 1 and r to 0 (the exponent sum in g). Let K be the kernel of this homomorphism. We will show that K is an infinite 2-group. This implies by 2.7, or by results in [**18**], that any action of G on an $\mathbb{R}$-tree is reducible.

Of course, the translation length function for the action of G on B is also a translation length function for an action of G on $\mathbb{R}$. Thus this example shows that the semi-simplicity assumption is essential in 3.7.

CLAIM. *Every element of K has order a power of 2, but K has infinite exponent.*

Proof. Let $k \in K$. It acts on B fixing some vertex $v_0 \in B$. If v is any vertex of B then k sends the a-direction at v to the a-direction at $v \cdot k$, and either preserves or interchanges the l- and r-labelling of directions at v. Let $B_n(v_0)$ denote the ball of radius n centred at v_0. We will show that for all $n \geqslant 0$ the element k^{2^n} fixes $B_n(v_0)$ point-wise. This is true for $n = 0$. Suppose by induction that $h = k^{2^{n-1}}$ fixes $B_{n-1}(v_0)$ point-wise. Then h^2 fixes $B_{n-1}(v_0)$ point-wise and leaves invariant the labelling at all vertices of $B_{n-1}(v_0)$. Hence it fixes $B_n(v_0)$. This completes the induction.

Since the generators of G preserve the labelling at all but at most one vertex of B, there are only finitely many vertices of B where k interchanges the l- and r-labels. Let us choose N sufficiently large so that all of these vertices are contained in $B_{N-1}(v_0)$. Clearly, k and all its powers leave $B_{N-1}(v_0)$ invariant, and preserve the labelling of directions at any vertex of B outside of $B_{N-1}(v_0)$. Thus, k^{2^N} fixes the labellings of all directions at all vertices of $B - B_{N-1}(v_0)$. It also fixes $B_N(v_0)$ point-wise and hence leaves invariant all directions at all vertices of $B_{N-1}(v_0)$. Since k^{2^N} fixes a vertex of B and leaves invariant labellings at all vertices of B, it is the identity. This proves that every $k \in K$ has order a power of 2.

We prove by induction on n that there is an element h_n of K fixing $0 \in B$ which preserves the labelling of directions outside the ball B_{n-1} centred at 0 and which acts as a cycle of order 2^n on the descendents of 0 at distance n. Clearly, the order of such an h_n is 2^n. We take h_1 to be r. Suppose we have already defined h_{n-1}. Let r_{n-1} be $g^{-n+1} r g^{n-1}$. It interchanges the descendents of exactly one vertex at distance $n-1$ from 0. Thus the element $r_{n-1} h_{n-1}$ satisfies the properties required of h_n.

3.10. REMARK. For the above action the quotient B/G is a graph with one vertex and one edge. By the Bass–Serre theory [**16**], such actions correspond one-to-one with descriptions of G as an HNN-extension. In this case, conjugation by the stable letter g gives an isomorphism from the base group to a proper subgroup of itself. Suppose that we are given any HNN-decomposition of G of this type, and consider the associated action of G on a simplicial tree T. Examination of the Bass–Serre construction shows that when the extension is of this type, the tree is not a line (since the inclusion is proper), and that the action has a fixed end (namely the positive end of the axis of the stable letter). The action is also transitive on edges and hence gives a minimal reduction action on an $\mathbb{R}$-tree other than $\mathbb{R}$. Moreover, we claim that the kernel H of the natural homomorphism $G \to \mathbb{Z}$ acts on T with no minimal invariant subtree.

To see this note that the action restricted to H has the following properties:

(1) each $h \in H$ has a fixed point in T;

(2) H fixes an end of T;

(3) $\bigcap_{h \in H} C_h = \varnothing$.

For any action satisfying these three properties there is no minimal invariant subtree. To see this choose a ray $R = [0, \infty)$ in T going to an invariant end of T. For each $h \in H$ and for each $n \in \mathbb{N}$ we know, by the argument in 2.2, that $C_h \cap [n, \infty)$ is non-empty. This means that $E_n = \bigcup_{h \in H} [n, \infty) \cdot h$ is a subtree of T. Clearly each E_n is invariant under H, and E_{n+1} is a proper subtree of E_n. Let $T' \subset T$ be an invariant subtree, and let $x \in T'$. Let $a \in R$ be the closest point of R to x. We claim that $[a, \infty) \subset T'$ and hence that $E_n \subset T'$ for all $n \geqslant a$. If so, then T' cannot be minimal.

Let b be a point of R farther from 0 than a. We shall show that $b \in T'$. Since $\bigcap_{h \in H} C_h = \varnothing$, there is an $h \in H$ such that $b \cdot h \neq b$. Let s be such that $R \cap R \cdot h = [s, \infty)$. Clearly, $R \cap C_h = [s, \infty)$. Along R we have $a < b < s$, so that the interval $[s, x]$ contains b and thus the direction of $[s, x]$ at s agrees with the negative direction on R. This means that $[s, x] \cap C_h = s$. Hence $[s, x]$ is contained in the geodesic from x to $x \cdot h$, as is b. Thus, b is contained in the invariant subtree T'.

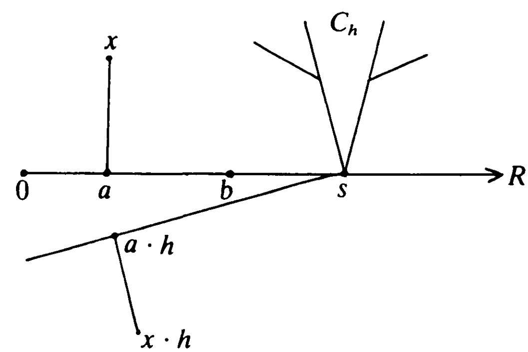

FIG. 8

3.11. REMARK. In Example 3.9, the group G is generated by two elements. We do not know if it is finitely presented, though we suspect that it is not. It was pointed out to us by Steve Gersten that there are finitely presented groups which are HNN-extensions as above. One well-studied example is R. J. Thompson's group, which has the following presentation (due to Freyd):

$$\langle x_i, i \geqslant 0 \mid x_i^{-1} x_j x_i = x_{j+1}, 0 \leqslant i < j \rangle.$$

It is shown in [4] that this realizes the group as a HNN-extension: the subgroup is $\langle x_j \mid j \geqslant 1 \rangle$ and the two inclusions are the identity and conjugation by x_0. This gives a minimal, fixed-point-free action of the group on a simplicial tree which is not isometric to $\mathbb{R}$. In fact, this group has recently been shown to have no free subgroups of rank 2 (see [3]). Thus it also admits no irreducible action on $\mathbb{R}$-trees. A finite presentation of this group is given in [4] and is

$$\langle x, y \mid y^{-1} x^{-1} y x y = x^{-2} y x^2, \ y^{-1} x^{-2} y x^2 y = x^{-3} y x^3 \rangle.$$

4. *Compactness results*

Throughout this section we fix a finitely generated group G. Since translation length functions are constant on conjugacy classes in G, we can consider them as points in $\mathbb{R}^{\Omega}$, where Ω is the set of conjugacy classes in G. We may then define the subset PLF(G) of the projective space P^{Ω} consisting of all projective classes of non-zero translation length functions. In order to study this space, we need to introduce a closely related subspace of P^{Ω}. Define a *pseudo-length function* on a group G to be a function $\| \ \|: G \to \mathbb{R}^{\geq 0}$ satisfying Axioms I–V of 1.11. Since pseudo-length functions are also constant on conjugacy classes, they too may be regarded as points in $\mathbb{R}^{\Omega}$. The space of projective classes of non-trivial pseudo-length functions will be denoted ΨLF(G). Clearly, PLF(G) $\subset \Psi$LF(G). We begin by showing that ΨLF(G) is a compact subset of P^{Ω}. Thus, if all pseudo-length functions were translation length functions, the compactness result, Theorem 4.5, would be immediate. Unfortunately, we do not know if this is the case. We are, however, able to show that pseudo-length functions enjoy many of the same properties that translation length functions do. A consequence of this is the result that the space PLF(G) is a closed subset of ΨLF(G), and hence compact. The proof of the closure of PLF(G) relies on a basic dichotomy, Proposition 4.4, for pseudo-length functions. One consequence of this dichotomy is that any reducible pseudo-length function is actually the translation length function of a reducible action. (See 4.3 for the definition of reducible.) The proof of Proposition 4.4 is rather technical. Out of consideration for the reader, we have postponed it until § 6.

In order to prove the compactness of ΨLF(G) let us establish the following notation. Fix a finite generating set $\{x_1, x_2, \ldots, x_n\}$ of G and let F be the free group on $\{x_1, x_2, \ldots, x_n\}$. We denote by ϕ the natural surjection $F \to G$. Let D denote the set of all elements of F which have the form

$$x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \ldots x_{i_k}^{\pm 1},$$

where $i_1, i_2, \ldots, i_k$ are distinct integers chosen from $\{1, 2, \ldots, n\}$. If w is an arbitrary element of F then we will denote by $|w|$ the length of a cyclically reduced word in $\{x_1, x_2, \ldots, x_n\}$ which represents a conjugate of w. We will abuse notation in the usual way by identifying elements of F with reduced words in the generators. If $\gamma \in G$ then $|\gamma|$ will denote the minimum of $|w|$ over all $w \in F$ with $\phi(w) = \gamma$.

We first show that ΨLF(G) is the image of a bounded subset of $\mathbb{R}^{\Omega}$. The proof is based on the following.

4.1. Proposition. *In the above notation, if* $\| \ \|: G \to \mathbb{R}$ *is any pseudo-length function then, for all* $w \in F$, $\|\phi(w)\| \leq M\,|w|$ *where* $M = \max_{d \in D}(\|\phi(d)\|)$. *In particular,* $\|\gamma\| \leq M\,|\gamma|$ *for all* $\gamma \in G$.

Proof. The proof is by induction on $|w|$. The conclusion of the proposition clearly holds for any element w of D, and hence for any word of length 1 in our generators. Assume that $\|\phi(w)\| \leq M\,|w|$ whenever $|w| < k$. Let w be a reduced word such that $|w| = k$. Since $\| \ \|$ and $| \ |$ are constant on conjugacy classes and on orbits of the inversion operator, we may assume that w is cyclically reduced, and we may replace it by any cyclic permutation of itself or by its inverse. If w is not in D, then some letter must appear at least twice in w. Thus we can write w in one of the following forms:

$$(*) \qquad w = Ax_iBx_i$$

or

$$(**) \qquad w = Ax_iBx_i^{-1},$$

where in either case w is cyclically reduced and A and B are subwords of w. In Case (*), A or B may be empty.

We first observe that Case (**) can be reduced to Case (*). By Axiom IV and the induction hypothesis we have that either $\|\phi(w)\| = \|\phi(Ax_i^2B^{-1})\|$ or

$$\begin{aligned}\|\phi(w)\| \leq \max(\|\phi(w)\|, \|\phi(Ax_i^2B^{-1})\|) &\leq \|\phi(Ax_i)\| + \|\phi(x_iB^{-1})\| \\ &\leq M(|Ax_i| + |x_iB^{-1}|) \\ &\leq M\,|w|.\end{aligned}$$

Thus we may assume that $\|\phi(w)\| = \|\phi(Ax_i^2B^{-1})\|$; since $|w| \geq |Ax_i^2B^{-1}|$, it suffices to prove that the conclusion holds for $Ax_i^2B^{-1}$. But

$$\|\phi(Ax_i^2B^{-1})\| = \|\phi(B^{-1}Ax_i^2)\|$$

and either $|B^{-1}Ax_i^2| < |w|$ or else $B^{-1}Ax_i^2$ is in Case (*).

To prove the result in Case (*) we again apply Axiom IV and the induction hypothesis to conclude that either

$$\|\phi(w)\| = \|\phi(Ax_iBx_i)\| = \|\phi(AB^{-1})\| \leq M(|w| - 2) < M\,|w|$$

or

$$\begin{aligned}\|\phi(w)\| = \|\phi(Ax_iBx_i)\| &\leq \max(\|\phi(Ax_iBx_i)\|, \|\phi(AB^{-1})\|) \\ &\leq \|\phi(Ax_i)\| + \|\phi(Bx_i)\| \\ &\leq M(|Ax_i| + |Bx_i|) \\ &= M\,|w|.\end{aligned}$$

4.2. THEOREM. *If G is a finitely generated group then the space $\Psi\mathrm{LF}(G)$ is a compact subset of P^Ω which is defined by an infinite set of weak linear inequalities.*

Proof. We continue to use the notation defined above. Let

$$I = \prod_{\gamma \in \Omega} [0, |\gamma|] \subset \mathbb{R}^\Omega.$$

Let $\Sigma \subset \Omega$ be the (finite) set of conjugacy classes represented by $\phi(w)$ for $w \in D$. Let $J \subset I$ be the subset consisting of all points $p \in I$ for which there exists $\sigma \in \Sigma$ such that the σth coordinate of p is 1. Clearly J is a compact subset of $I - \{0\}$.

We claim that the image of J in P^Ω contains $\Psi\mathrm{LF}(G)$. Suppose that $\alpha \in \Psi\mathrm{LF}(G)$ and let $\|\ \|$ be a non-trivial pseudo-length function in the projective class α. Since $\|\ \|$ is non-trivial, Proposition 4.1 implies that $\|\sigma\| > 0$ for some $\sigma \in \Sigma$. We may rescale $\|\ \|$ so that $M = \max_{\sigma \in \Sigma}(\|\sigma\|) = 1$. For any $\gamma \in \Omega$ and $g \in \gamma$ we have $\|g\| \leq |\gamma|$ and $\|\sigma\| = 1$ for some element σ of Σ. Thus α is contained in the image of J.

Now, since the image of J in P^Ω is compact, it follows that $\Psi\mathrm{LF}(G)$ has compact closure; we need only show that it is closed. For a given choice of elements g and h, each of the Axioms I–V is equivalent to a conjunction of finitely many linear equations or weak linear inequalities. Thus the set of pseudo-length functions is the intersection of an infinite family of closed subsets of $\mathbb{R}^{\Omega}$ and hence is closed. It is also invariant under the action of the group of homotheties, so ΨLF(G), being its image in the quotient P^{Ω}, is closed.

Before proving the compactness result for PLF(G) let us extend some of the notions we introduced for translation length functions to pseudo-length functions.

4.3. Let $\| \ \|$ be a pseudo-length function. An element $g \in G$ is *hyperbolic* if $\|g\| > 0$; otherwise it is *elliptic*. A non-trivial pseudo-length function $\| \ \|$ is *reducible* provided $\|ghg^{-1}h^{-1}\| = 0$ for all hyperbolic $g, h \in G$. Two elements g and h of G form a *good pair* (cf. 3.6) if

$$0 < \|g\| + \|h\| - \|gh^{-1}\| < 2\min(\|g\|, \|h\|).$$

The following theorem, which is analogous to several results in § 2, is used to prove that PLF(G) is compact.

4.4. PROPOSITION. *Let $\| \ \|$: $G \to \mathbb{R}$ be a non-trivial pseudo-length function. Either there exists a good pair of elements of G or $\| \ \|$ is the translation length function of an action of G on $\mathbb{R}$.*

This result is a formal consequence of the defining axioms for pseudo-length functions. We postpone the proof of this result until § 6, although the statement will be used to prove the following:

4.5. THEOREM. *If G is a finitely generated group then the space* PLF(G) *is a compact subset of* P^{Ω}.

Proof. Consider a sequence (α_n) of points of PLF(G). Since ΨLF(G) is compact, there is a subsequence of (α_n) which converges to the projective class of a pseudo-length function $\| \ \|$. We may choose representatives for the classes in this subsequence to obtain a sequence $\| \ \|_m$ of translation length functions which converges to the pseudo-length function $\| \ \|$. We must show that $\| \ \|$ is a translation length function. By Proposition 4.4, if there is no good pair of elements for $\| \ \|$ then it is a translation length function. Thus we can assume that there exists a good pair of elements g and h of G. The condition on pseudo-length functions that g and h be a good pair is clearly open. Thus, if we let T_m denote a tree upon which G acts minimally with translation length function $\| \ \|_m$, then, for m sufficiently large, g and h form a good pair for the action on T_m. By taking a further subsequence we may assume that g and h form a good pair for the action on all T_m.

We now appeal again to the results of Chiswell and Alperin and Moss or Imrich. Let p_m be the point in T_m which is the intersection of the axes of g, h, and gh^{-1}. Let L_m be the based length function for the action of G on T_m associated to the point p_m. As in the proof of Theorem 3.7, L_m can be expressed in terms of $\| \ \|_m$. The formula for L_m is a maximum of linear combinations of lengths of elements of G, and hence is continuous as a function on $\mathbb{R}^G$. Thus the based length functions L_m converge to a function L: $G \to \mathbb{R}$. We claim that the function L satisfies Chiswell's axioms for based length functions. In fact, the set of functions from G to $\mathbb{R}$ which satisfy Chiswell's axioms is closed since, for a given choice of elements g, h, and k, each axiom is equivalent to a conjunction of disjunctions of finitely many linear weak inequalities. Using the results of Chiswell and Alperin and Moss or Imrich we construct an action $T_\infty \times G \to T_\infty$ and a point $p \in T_\infty$ such that $L_p = L$. Let $\|\ \|_\infty$ be the translation length function of this action.

It follows easily from 1.3 that for any action $T \times G \to T$, for any point $p \in T$, and for any $g \in G$, we have

$$\|g\| = \max(0, L_p(g^2) - L_p(g)).$$

Applying this to the actions on T_m and the points $p_m \in T_m$, and using the fact that the L_m converge to L we see that, for all $g \in G$,

$$\|g\|_\infty = \lim_{m\to\infty} \|g\|_m.$$

Thus, $\|\ \|_\infty = \|\ \|$. Therefore PLF(G) is closed in ΨLF(G) and hence is compact.

4.6. Remark. The proof of Theorem 4.5 shows that, away from the set of translation length functions of actions on $\mathbb{R}$, PLF(G) is locally defined by (infinitely many) weak linear inequalities.

5. *Actions without free arc stabilizers*

In this section we consider minimal actions on $\mathbb{R}$-trees such that no stabilizer of a non-degenerate arc in the tree contains a free subgroup of rank 2. Recall that the space of projective classes of translation length functions of such actions is denoted SLF(G). The main result of this section is that SLF(G) is a closed subset of PLF(G) and hence is compact. First we treat the case when there is a non-trivial reducible translation length function whose projective class is contained in the closure of SLF(G).

5.1. Lemma. *Let G be a finitely generated group. If there is a reducible action whose projective class is in the closure of* SLF(G), *then G contains no free subgroup of rank* 2, *and hence* SLF(G) = PLF(G).

Proof. We begin by considering the case when SLF(G) itself contains the projective class of a reducible action. Then the translation length function is induced from a homomorphism of G to Iso($\mathbb{R}$). If the action has an invariant line, then the kernel, K, of this homomorphism fixes the line and is thus contained in an arc stabilizer. If the action has a fixed end then, since each element of K has a fixed point, any finitely generated subgroup of K has a common fixed point. Such a subgroup must therefore fix the ray from its fixed point to the end fixed by G, and hence is contained in an arc stabilizer. Since we are assuming that the projective class of this reducible action belongs to SLF(G), it follows that K contains no free group of rank 2. Consequently, the same is true of G.

We now consider limit points of SLF(G). Note that the compactness of PLF(G) implies that the closure of SLF(G) consists of projective classes of translation length functions. Let $\|\ \|$ be a translation length function whose projective class is a limit of classes in SLF(G). There exists a sequence $\|\ \|_m$ of translation length functions converging to $\| \ \|$, where each $\| \ \|_m$ corresponds to a minimal action on an $\mathbb{R}$-tree T_m for which stabilizers of non-degenerate arcs contain no free subgroups of rank 2. By the above we can assume that for all m the action of G on T_m is irreducible. Choose an element $g \in G$ such that $\|g\| > 0$. Fix M with $\|g\|_m > 0$ for all $m \geqslant M$. By Lemma 2.1, there is a hyperbolic element h of G such that in T_M we have $C_g \cap C_h = \emptyset$. Let $k = hgh^{-1}$. Since g and k are conjugate in G, we have $\|k\|_m = \|g\|_m > 0$ for all $m > M$. By 1.7, $C_g \cap C_k = \emptyset$ in T_m. By Lemma 2.6, k and g generate a free subgroup of rank 2 in G. Thus for any $m > M$ the length Δ_m of $C_g \cap C_k \subset T_M$ is less than $4\,\|g\|_m = 4\,\|k\|_m$. For otherwise by 1.10 one of the two free subgroups of rank 2, $\langle [g, k], [g^2, k] \rangle$ or $\langle [g, k^{-1}], [g^2, k^{-1}] \rangle$, would fix a non-degenerate arc in T_m. This means that, for each $m > M$, either g^4 and one of $k^{\pm 4}$ form a good pair for the action of G on T_m, or else g^4 and h^4 have axes in T_m which meet in at most a point.

Thus by 3.4 or 3.5 we have

$$\|g^4k^4g^{-4}k^{-4}\|_m \geqslant 2\,\|g^4\|_m + 2\,\|k^4\|_m - 2\Delta_m \geqslant 8\,\|g\|_m.$$

Taking the limit yields $\|[g^4, k^4]\| \geqslant 8\,\|g\| > 0$. Clearly then $\| \ \|$ is not the translation length function of an action of G on $\mathbb{R}$.

The compactness of SLF(G) is a consequence of the following geometric fact.

5.2. Lemma. *Let p_1 and p_2 be points of an $\mathbb{R}$-tree T. Let g_1 and g_2 be isometries of T which move both p_1 and p_2 a distance less than ε. If* $\operatorname{dist}(p_1, p_2) > 2\varepsilon$ *then $C_{g_1} \cap C_{g_2}$ contains a geodesic of length at least* $\operatorname{dist}(p_1, p_2) - 2\varepsilon$.

Proof. Let N_1 and N_2 be the closed ε-neighbourhoods of p_1 and p_2 respectively. These are disjoint subtrees of T. Let α be the unique shortest geodesic joining a point of N_1 to a point of N_2. We will show that α is contained in the characteristic set of both g_1 and g_2. By symmetry it suffices to consider $g = g_1$. By 1.3, an isometry moves each point more than the distance from the point to the characteristic set of the isometry. Thus the distance from p_i to C_g is at most ε, so C_g passes through both N_1 and N_2. Thus C_g must contain a geodesic joining a point of N_1 to a point of N_2. By the proof of 1.1, any such arc contains α.

5.3. Theorem. *If G is a finitely generated group then the space* SLF(G) *is a compact subset of* P^{Ω}.

Proof. By Theorem 4.5, we need only show that SLF(G) is closed in PLF(G). Consider a sequence of points of SLF(G) which converges to a point of PLF(G). By choosing appropriate representatives of these projective classes we obtain a sequence $(\| \ \|_n)$ of translation length functions for actions with small arc stabilizers which converges to a translation length function $\| \ \|_\infty$. For $n \leqslant \infty$, let $G \times T_n \to T_n$ be a minimal action whose translation length function is $\| \ \|_n$. For $n < \infty$, we take the action on T_n to have small arc stabilizers. By Lemma 5.1, we can assume that the action on T_∞ is irreducible. We assume that there exists a non-degenerate arc α in T_∞ which is stabilized by a rank-2 free subgroup of G.

This will lead to a contradiction. A subgroup of index at most 2 in the stabilizer of α fixes the endpoints of α. Thus there is a free group $\langle a, b\rangle$ of rank 2 in G which fixes the endpoints of α. The main step in the proof is to show that the trees T_n converge to the tree T_∞ as metric spaces. This implies that the endpoints of α can be approximated in the T_n, for n large, by points which are moved a very small distance by a and b. On the other hand, the distance between these two points is approximately equal to the length of α. We will then apply Lemma 5.2 to conclude that there is an arc in T_n which is stabilized by a free subgroup of rank 2 of $\langle a, b\rangle$.

Since the action of G on T_∞ is irreducible there is a good pair (t, u) for the action of G on T_∞. Since this is an open condition, t and u are a good pair of isometries of T_n for all sufficiently large n. Thus, for almost all positive n and for $n=\infty$, there is a unique point p_n in T_n which is common to C_t, C_u, and $C_{tu^{-1}}$. By the results of Chiswell and Alperin and Moss or Imrich, as in the proof of Theorems 3.7 and 4.5, the based length functions $L_n = L_{p_n}$, for $n<\infty$, converge to $L_\infty = L_{p_\infty}$. Since the actions that we are considering are minimal, each point of T_n lies on the geodesic from p_n to $p_n \cdot g$ for some $g \in G$. (The union of all such geodesics is an invariant subtree.) For any $g \in G$ and $\rho \in [0, 1]$ we let the pair $[g, \rho]_n$ designate the point q_n of T_n which lies on the geodesic from p_n to $p_n \cdot g$ and satisfies $\operatorname{dist}(p_n, q_n) = \rho \operatorname{dist}(p_n, p_n \cdot g)$.

For n an integer or ∞ we will denote by d_n the metric on T_n. We claim that, for all ρ and σ in $[0,1]$,

$$\lim_{n\to\infty} d_n([g, \rho]_n, [h, \sigma]_n) = d_\infty([g, \rho]_\infty, [h, \sigma]_\infty).$$

To see this consider the smallest subtree of T_n containing the three points p_n, $p_n \cdot g$, and $p_n \cdot h$. In general this is a triad as shown in Fig. 9, although it may degenerate to an arc if one of the edges of the triad has length zero. The lengths of the edges of the triad are determined by the distances between pairs of endpoints; they satisfy a non-singular system of linear equations whose

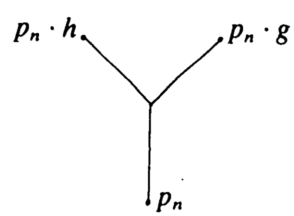

FIG. 9

coefficients are determined by these distances. The distances are given in terms of L_n as follows: $\operatorname{dist}(p_n, p_n \cdot g) = L_n(g)$, $\operatorname{dist}(p_n, p_n \cdot h) = L_n(h)$, and $\operatorname{dist}(p_n \cdot g, p_n \cdot h) = L_n(hg^{-1})$. Let C_n denote the length of the segment joining p_n to the centre point of the triad. Then $C_n = \frac{1}{2}[L_n(g) + L_n(h) - L_n(hg^{-1})]$, and, for n an integer or ∞,

$$d_n([g, \rho]_n, [h, \sigma]_n) = \begin{cases} \rho + \sigma - 2C_n & \text{if } \rho > C_n \text{ and } \sigma > C_n, \\ |\rho - \sigma| & \text{otherwise.} \end{cases}$$

Since this distance depends continuously on L_n, the claim follows.

Next we claim that

$$\lim_{n\to\infty} d_n([g, \rho]_n, [g, \rho]_n \cdot h) = d_\infty([g, \rho]_\infty, [g, \rho]_\infty \cdot h).$$

Here we consider the smallest subtree of T_n containing the four points p_n, $p_n \cdot g$, $p_n \cdot h$, and $p_n \cdot gh$. Again, the lengths of the edges of this subtree are determined by the distances between pairs of its endpoints, and these are given as linear expressions in L_n. The distance from $[g, \rho]_n$ to $[g, \rho]_n \cdot h$ is given by a piecewise linear expression in ρ and the lengths of edges of the subtree. Thus it varies continuously with L_n, which proves the claim.

Let e and f be the endpoints of α. Write $e = [g_1, \rho_1]_\infty$ and $f = [g_2, \rho_2]_\infty$. Set $e_n = [g_1, \rho_1]_n$ and $f_n = [g_2, \rho_2]_n$, and let α_n be the geodesic in T_n with endpoints e_n and f_n. By the discussion above,

$$\lim_{n\to\infty} \text{length}(\alpha_n) = \text{length}(\alpha) > 0.$$

Also,

$$\lim_{n\to\infty} d_n(e_n, e_n \cdot a) = 0 = \lim_{n\to\infty} (f_n, f_n \cdot a),$$

and similarly for b. It follows immediately that

$$\lim_{n\to\infty} \|a\|_n = 0 = \lim_{n\to\infty} \|b\|_n.$$

By 5.2, for n sufficiently large, $C_a \cap C_b$ in T_n contains the middle third of α_n. Now, by 1.10, for n sufficiently large, one of the free groups of rank 2, $\langle [a, b], [a^2, b] \rangle$ or $\langle [a, b^{-1}], [a^2, b^{-1}] \rangle$, fixes a non-degenerate sub-interval of α_n in T_n. This is impossible. This contradiction establishes Theorem 5.3.

6. *Pseudo-length functions*

In this section we will establish the dichotomy (Proposition 4.4) for pseudo-length functions which was used in the proofs of the compactness theorems:

Given a pseudo-length function $\| \;\|$ on G, either there exists a good pair of elements of G or else $\| \;\|$ is the translation length function for an action of G on $\mathbb{R}$.

This section consists of a series of lemmas. They establish various properties of pseudo-length functions directly from the axioms. (These properties are easily seen to hold for translation length functions by direct geometric arguments.) We view these results as progress towards deciding whether or not every pseudo-length function is actually a translation length function.

We assume throughout this section that $\| \;\|: G \to \mathbb{R}$ is a pseudo-length function.

6.1. LEMMA. *For all $g \in G$, $\|g^n\| = |n| \, \|g\|$.*

Proof. The proof is by induction on n. By Axiom II, it suffices to consider $n \geq 0$. The case where $n = 0$ follows from Axiom I, and the case where $n = 1$ is trivial. Assume that $n > 1$ and that $\|g^k\| = |k| \, \|g\|$ for all $0 \leq k < n$. If $\|g\| = 0$

then, by Axiom IV applied to g^{n-1} and g, either $\|g^n\| = \|g^{n-2}\| = 0$ or

$$\|g^n\| = \max(\|g^n\|, \|g^{n-2}\|) \leqslant \|g^{n-1}\| + \|g\| = 0.$$

If $\|g\| > 0$ then, by induction $\|g^{n-2}\| = \|g^{n-1} \cdot g^{-1}\| < \|g^{n-1}\| + \|g\|$. Thus by Axiom V applied to g^{n-1} and g,

$$\|g^n\| = \|g^{n-1} \cdot g\| = \|g^{n-1}\| + \|g\| = |n| \, \|g\|.$$

6.2. LEMMA. *Suppose that g and h are elements of G with $\|g\| > 0$, $\|h\| > 0$, and $\|gh\| = \|g\| + \|h\|$. Then for all positive integers m and n,*

$$\|g^m h^n\| = m \, \|g\| + n \, \|h\|.$$

Proof. Suppose that we have established the result in the special case when $m = 1$. Then for any $m \geqslant 0$ we have

$$\|g^m h\| = \|hg^m\| = \|h\| + m \, \|g\|.$$

Now apply the result in this special case to the elements g^m and h. We conclude that, for any m, $n \geqslant 0$,

$$\|g^m h^n\| = \|g^m\| + n \, \|h\| = m \, \|g\| + n \, \|h\|.$$

Thus, it suffices to prove the result for $m = 1$ and for all $n \geqslant 0$.

Now let $m = 1$. The result is trivial if $n = 0$ or $n = 1$. Assume that $n \geqslant 2$ and $\|gh^k\| = \|g\| + k \, \|h\|$ for all $0 \leqslant k < n$. Since $\|gh^{n-2}\| < \|gh^{n-1}\| + \|h\|$, Axiom V implies that

$$\|gh^n\| = \|gh^{n-1}\| + \|h\| = \|g\| + n \, \|h\|.$$

Lemmas 6.1 and 6.2 are sufficient for the following first step toward the proof of Proposition 4.4:

6.3. PROPOSITION. *Let $\| \ \|$: $G \to \mathbb{R}$ be a non-trivial pseudo-length function. Either there exists a good pair of elements of G or $\| \ \|$ is reducible.*

Proof. Suppose that $\| \ \|$ is not reducible.

CLAIM. *There are elements g and h of G such that $\|g\| > 0$, $\|h\| > 0$, $\|ghg^{-1}h^{-1}\| > 0$, and $\|gh\| = \|g\| + \|h\|$.*

Since $\| \ \|$ is not reducible, there are elements g_1 and h_1 with $\|g_1\| > 0$, $\|h_1\| > 0$, and $\|g_1 h_1 g_1^{-1} h_1^{-1}\| > 0$. By Axiom V,

$$\max(\|g_1 h_1\|, \|g_1 h_1^{-1}\|) > \|g_1\| + \|h_1\|.$$

Replacing h_1 by h_1^{-1}, if necessary, allows us to assume that $\|g_1 h_1\| > \|g_1\| + \|h_1\|$. If equality holds, then we take $g = g_1$ and $h = h_1$. Clearly in this case the elements are as required by the claim. If equality does not hold, then we take $g = g_1 h_1$ and $h = h_1$. With these choices, $\|g\| > 0$, $\|h\| > 0$, and $\|[g, h]\| = \|[g_1, h_1]\| > 0$. Furthermore, since

$$\|gh^{-1}\| = \|g_1\| < \|g\| + \|h\|,$$

Axiom V guarantees that $\|gh\| = \|g\| + \|h\|$. This completes the proof of the claim.

Now set $a = gh$ and $b = hg$. Note that $\|a\| = \|b\| + \|g\| + \|h\|$. By Lemma 6.2 we have

$$\|ab\| = \|ghhg\| = \|g^2h^2\| = 2\,\|g\| + 2\,\|h\| = \|a\| + \|b\|.$$

Thus by Axiom V there are two cases.

Case 1: $\|ab\| = \|a\| + \|b\| > \|ab^{-1}\|$. In this case we show that a and b are a good pair. Clearly, we have $0 < \|a\| + \|b\| - \|ab^{-1}\|$. Also, since $\|ab^{-1}\| = \|ghg^{-1}h^{-1}\| > 0$, we have

$$2\min(\|a\|, \|b\|) = 2\,\|g\| + 2\,\|h\| = \|a\| + \|b\| > \|a\| + \|b\| - \|ab^{-1}\|.$$

Case 2: $\|ab\| = \|a\| + \|b\| = \|ab^{-1}\|$. In this case we show that ab and b^2 are a good pair. First,

$$\|(ab)(b^2)^{-1}\| = \|ab^{-1}\| = \|a\| + \|b\|.$$

Thus,

$$\|ab\| + \|b^2\| - \|(ab)(b^2)^{-1}\| = 2\,\|b\| > 0,$$

and

$$2\min(\|ab\|, \|b^2\|) = 2\min(\|a\| + \|b\|, 2\,\|b\|) > 2\,\|b\|.$$

This completes the proof.

The rest of this section is devoted to the proof that any reducible pseudo-length function is the translation length function of a reducible action.

For the rest of this section $\|\ \|$ *denotes a reducible pseudo-length function.*

6.4. Lemma. *Suppose that* $\|g\| > 0$ *and* $\|h\| > 0$. *Then*

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|$$

and

$$\min(\|gh\|, \|gh^{-1}\|) = |\,\|g\| - \|h\|\,|.$$

Proof. This will follow once we show that

$$\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|$$

and that

$$2\max(\|g\|, \|h\|) = \|gh\| + \|gh^{-1}\|.$$

To do this apply Axiom V to g and hgh^{-1} to conclude, since $\|g(hg^{-1}h^{-1})\| = 0$, that $\|g(hgh^{-1})\| = \|g\| + \|hgh^{-1}\| = 2\,\|g\|$. Next, by Axiom V applied to g and h we have two cases to consider.

Case 1: $\|gh\| = \|gh^{-1}\| > \|g\| + \|h\|$. In this case we have

$$\|gh\| + \|gh^{-1}\| > 2\,\|g\| + 2\,\|h\| = \|ghgh^{-1}\| + \|gh^2g^{-1}\|.$$

According to Axiom V applied to gh and gh^{-1} we have

$$\begin{aligned}\|gh\| + \|gh^{-1}\| &= \max(\|ghgh^{-1}\|, \|gh^2g^{-1}\|)\\ &= 2\max(\|g\|, \|h\|)\\ &\leqslant 2\,\|g\| + 2\,\|h\|.\end{aligned}$$

This is a contradiction, leaving us to consider:

Case 2: $\max(\|gh\|, \|gh^{-1}\|) = \|g\| + \|h\|$. By Axiom IV applied to gh and gh^{-1} we have either

$$\text{(2A)} \qquad \|ghgh^{-1}\| = \|gh^2g^{-1}\|$$

or

$$\text{(2B)} \qquad \max(\|ghgh^{-1}\|, \|gh^2g^{-1}\|) \leqslant \|gh\| + \|gh^{-1}\|.$$

First assume that the equality (2A) holds. Then $\|g\| = \|h\|$. Thus

$$\max(\|gh\|, \|gh^{-1}\|) = 2\,\|g\| = 2\,\|h\|.$$

If $\min(\|gh\|, \|gh^{-1}\|) > 0$ then Axiom V applied to gh and gh^{-1} gives

$$2\,\|g\| = \|ghgh^{-1}\| > \|gh\| + \|gh^{-1}\| > 2\,\|g\|,$$

a contradiction. Thus $\min(\|gh\|, \|gh^{-1}\|) = 0$, so

$$\|gh\| + \|gh^{-1}\| = \max(\|gh\|, \|gh^{-1}\|) = 2\,\|g\| = 2\,\|h\| = 2\max(\|g\|, \|h\|).$$

Now we assume that the equality (2A) does not hold. Since $\|ghgh^{-1}\| = 2\,\|g\|$ and $\|gh^2g^{-1}\| = 2\,\|h\|$, the weak inequality (2B) is equivalent to

$$2\max(\|g\|, \|h\|) \leqslant \|gh\| + \|gh^{-1}\|.$$

We complete the proof in this case by showing that we have equality. This is immediate from Axiom V if $\min(\|gh\|, \|gh^{-1}\|) > 0$. On the other hand, if this minimum is zero then

$$\begin{aligned} 2\max(\|g\|, \|h\|) &\leqslant \|gh\| + \|gh^{-1}\| \\ &= \max(\|g\|, \|h\|) \\ &= \|g\| + \|h\| \leqslant 2\max(\|g\|, \|h\|). \end{aligned}$$

6.5. LEMMA. *If* $\|g\| = 0$ *then either* $\|gh\| = \|h\|$ *for all* h *with* $\|h\| > 0$, *or else* $\|gh\| = 0$ *for all* h *with* $\|h\| > 0$.

Proof. Suppose that h is an element of G with $\|h\| > 0$ such that $\|gh\| > 0$. Since $\|(gh)h^{-1}\| = \|g\| = 0$, applying Lemma 6.4 to gh and h gives

$$|\,\|gh\| - \|h\|\,| = 0.$$

Thus $\|gh\| = \|h\|$. We claim that $\|gh^n\| = |n|\,\|h\|$ for all n. by Lemma 6.4,

$$\max(\|gh\|, \|gh^{-1}\|) = \|h\| = \min(\|gh\|, \|gh^{-1}\|).$$

Thus, $\|gh\| = \|gh^{-1}\|$. Hence, it suffices to prove the result for all $n > 0$. This we do by induction. Assuming the result for all $0 \leqslant m < n$ and applying Lemma 6.4 to gh^{n-1} and h, we have

$$\max(\|gh^n\|, \|gh^{n-2}\|) = \|gh^{n-1}\| + \|h\| > \|gh^{n-2}\|.$$

Thus, $\|gh^n\| = \|gh^{n-1}\| + \|h\| = n\,\|h\|$.

Suppose h and k are elements of G with $\|h\| > 0$ and $\|k\| > 0$ and $\|gh\| = \|h\|$. Replacing h by h^{-1} if necessary, we may assume, by 6.4, that $\|hk\| = \|h\| + \|k\|$.

By Lemma 6.2 we have $\|h^n k\| = n\,\|h\| + \|k\|$ for all $n > 0$. Applying Lemma 6.4 to gh^{-n} and $h^n k$ we conclude that for each positive integer n either $\|gk\| = \|h^n\| + \|h^n\| + \|k\|$ or $\|gk\| = \|k\|$. Clearly we must have $\|gk\| = \|k\|$.

One can now see how to attempt to define an action of G on $\mathbb{R}$ which realizes $\|\ \|$ as a length function. Lemma 6.5 divides the elements g of G with $\|g\| = 0$ into two classes which correspond to those which should act by reflection and those which should act trivially. (If $\|g\| = 0$ and $\|gh\| = 0$ for all h with $\|h\| > 0$ then g should be a reflection.) The elements $g \in G$ such that $\|g\| > 0$ will act on $\mathbb{R}$ by translation. Clearly if $\|g\| > 0$ and $\|h\| > 0$ then g and h should translate in the same direction if and only if $\|gh\| = \|g\| + \|h\|$. Thus the translation directions can be determined by choosing a direction for one translation and comparing the others to it. Since the translations form an index-2 subgroup of Iso($\mathbb{R}$), once the action of the translations has been defined it is not difficult to extend to an action of G.

The next two lemmas show that this procedure for determining the translation directions is consistent.

6.6. LEMMA. *Suppose that s, $u \in G$ satisfy $\|u\| > 0$ and $\|su^n\| \neq 0$ for some n. Then for t equal to either s or s^{-1} we have $\|tu^m\| = m\,\|u\| + \|t\|$ and $\|t^{-1}u^m\| = |\, m\|u\| - \|t\|\,|$ for all $m \geq 0$.*

Proof. If $\|s\| = 0$, then this is immediate from Lemma 6.5 and Lemma 6.1. If $\|s\| > 0$, then by Lemma 6.4 we can choose t equal to s or s^{-1} so that $\|tu\| = \|t\| + \|u\|$. Then, by Lemma 6.2, $\|tu^m\| = \|t\| + m\,\|u\|$ for all $m \geq 0$. Consequently, by Lemma 6.4 again, we have $\|t^{-1}u^m\| = |\, m\,\|u\| - \|t\|\,|$ for all $m \geq 0$.

6.7. COROLLARY. *If s, $u \in G$ satisfy $\|u\| > 0$ and $\|su^n\| \neq 0$ for some n, then $\|su^m\| - \|u^m\|$ is independent of m for m sufficiently large, and is equal to $\pm\|s\|$.*

If $u \in G$ satisfies $\|u\| > 0$, then we define $S_u \subset G$ to be the set of $s \in G$ with $\|su^n\| \neq 0$ for some n, and we define τ_u: $S_u \to \mathbb{R}$ by sending each $s \in S$ to the stable value of $\|su^m\| - \|u^m\|$. We define $V \subset G$ to be the subgroup generated by all $g \in G$ with $\|g\| > 0$. Clearly V is a normal subgroup of G.

6.8. LEMMA. *Suppose that u is a hyperbolic element of G. Then S_u contains V, and τ_u restricted to V defines a homomorphism of V to $\mathbb{R}$ with $\tau_u(v) = \pm\,\|v\|$ for all $v \in V$.*

Proof. Clearly $1 \in S_u$ and $\tau_u(1) = 0$. If g is hyperbolic then by Lemma 6.4, $g \in S_u$. Since V is generated by hyperbolic elements, to prove that $V \subset S_u$ and that τ_u restricted to V is a homomorphism, it suffices to show that if $a, b \in S_u$ then $ab \in S_u$ and $\tau_u(ab) = \tau_u(a) + \tau_u(b)$. We consider $\|abu^k\| = \|au^m \,.\, u^{-m}bu^k\|$. If m is sufficiently large, and if k is sufficiently large given m, then we have

$$\|au^m\| = m\,\|u\| + \tau_u(a)$$

and

$$\|u^{-m}bu^k\| = (k - m)\,\|u\| + \tau_u(b)$$

with both au^m and $u^{-m}bu^k$ hyperbolic. By Lemma 6.4,

$$\|abu^k\| = \|au^m \,.\, u^{-m}bu^k\| = |\, m\|u\| + \tau_u(a) \pm ((k-m)\,\|u\| + \tau_u(b))\,|.$$

The left-hand side of this equation is independent of m, and for large enough k it holds for arbitrarily many values of m. Since the equation cannot hold if the ambiguous sign is '$-$' for three consecutive values of m, there must exist m for which the sign is '$+$'. Hence

$$\|abu^k\| = k\,\|u\| + \tau_u(a) + \tau_u(b).$$

Clearly then, $ab \in S_u$ and $\tau_u(ab) = \tau_u(a) + \tau_u(b)$. This proves that τ_u is defined on all of V and is a homomorphism on V. By Corollary 6.7, $\tau_u(v) = \pm\,\|v\|$ for all $v \in V$.

At this point we have proved the desired properties for $\|\ \|$ restricted to the normal subgroup V of G:

6.9. COROLLARY. *Let $u \in G$ be hyperbolic. The restriction of $\|\ \|$ to the normal subgroup V of G is the translation length function of the action of V by translations on $\mathbb{R}$ determined by the homomorphism τ_u: $V \to \mathbb{R}$.*

In order to complete the proof we need to understand the relationship between V and G. To do this we need to identify the kernel of the homomorphism τ_u and relate it to V. The next two lemmas accomplish this.

6.10. LEMMA. *Suppose that $\|\ \|$ is non-trivial, and choose $u \in G$ hyperbolic. Let $K \subset G$ be defined by*

$$K = \{g \in G \mid \ \|g\| = 0 \textit{ and } \|gh\| = \|h\| \textit{ whenever } \|h\| > 0\}.$$

Then K is a normal subgroup of G, $K \subset V$, and $K = \mathrm{kernel}(\tau_u)$.

Proof. To show that $K \subset V$, observe that for any $k \in K$ we have $\|ku\| = \|u\| = \|u^{-1}\| > 0$. Thus, $k = (ku)u^{-1} \in V$.

Next we show that $K = \mathrm{kernel}(\tau_u)$. It is immediate from the definitions that $K \subset \mathrm{kernel}(\tau_u)$. Conversely, suppose that $\tau_u(v) = 0$ for some $v \in V$. Then $\|vu^n\| = \|u^n\|$ for all n sufficiently large. By Lemma 6.4 it follows that $\|v\| = 0$. By Lemma 6.5 we have $\|vh\| = \|h\|$ for all $h \in G$ with $\|h\| > 0$. This proves that $v \in K$.

Finally, we show that K is normal in G. To see this assume $k \in K$ and $g \in G$. Then $\|gkg^{-1}\| = \|k\| = 0$, and for any $h \in G$ with $\|h\| > 0$ we have

$$\|gkg^{-1} \,.\, h\| = \|kg^{-1}hg\| = \|g^{-1}hg\| = \|h\|.$$

6.11. LEMMA. *Suppose that $\|\ \|$ is reducible and not identically zero. Then*

(i) *for all $r \in G - V$ and all $v \in V$ we have $rvr^{-1}v \in K$, and*

(ii) $[G:V] \leq 2$.

Proof. (i) Since V is generated by hyperbolic elements, we can assume for the proof of (i) that $\|v\| > 0$. First we show that $\|rvr^{-1}v\| = 0$. We know that $\|r\| = 0$ and $\|rv\| = 0$. Lemma 6.4 applied to rvr^{-1} and v shows that $\|rvr^{-1}v\|$ equals either $2\,\|v\|$ or 0. Suppose that it equals $2\,\|v\|$. By Axiom IV applied to rv and

rv^{-1}, since

$$\|rvr^{-1}v\| > 0 = \|rv\| + \|r^{-1}v\|,$$

we must have

$$0 < 2\|v\| = \|rvr^{-1}v\| = \|rvv^{-1}r\| = \|r^2\| = 0.$$

This is a contradiction so we are left with $\|rvr^{-1}v\| = 0$.

Notice that

$$\|(rvr^{-1}v)v^{-1}\| = \|rvr^{-1}\| = \|v\| = \|v^{-1}\| \neq 0.$$

Thus by Lemma 6.5, $rvr^{-1}v \in K$.

(ii) Let $G' = G/K$ and $V' = V/K$. It suffices to show that $[G':V'] \leqslant 2$. According to part (i), under the conjugation action of G' on V', every element of $G' - V'$ acts by sending v to v^{-1} for all $v \in V$. Since any hyperbolic element of G becomes an element of infinite order in V', it follows that every element of $G' - V'$ acts by the same non-trivial homomorphism on V'. The only way this can happen is if $[G': V'] \leqslant 2$.

Now we are in a position to prove the main result of this section.

6.12. THEOREM. *Every reducible pseudo-length function on G is the translation length function of an action of G on $\mathbb{R}$.*

Proof. Let $\|\ \|$: $G \to \mathbb{R}$ be a reducible pseudo-length function and let K and V be as in Lemmas 6.8 and 6.10. In Corollary 6.9 we constructed an action of V on $\mathbb{R}$ so that an element v of V translates by an amount equal to $\|v\|$. The kernel of this action is K. If $V = G$, then the proof is complete. Suppose that $V \neq G$. We extend the action to an action of all of G on $\mathbb{R}$ by choosing an element $r \in G - V$ and letting it act by reflection fixing any point of $\mathbb{R}$. This defines an action by the full group G because V has index 2 in G and because the relation $rvr = v^{-1}$ holds modulo K for all $v \in V$.

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Department of Mathematics
University of Illinois at Chicago
Box 4348
Chicago
Illinois 60680
U.S.A.

Department of Mathematics
Columbia University
New York
New York 10027
U.S.A.