Topology Vol. 20, pp. 133–145
0040-9383/81/0601-0133/$02.00/0
© Pergamon Press Ltd., 1981. Printed in Great Britain

USING SURFACES TO SOLVE EQUATIONS IN FREE GROUPS

MARC CULLER

(*Received* 11 *July* 1979)

INTRODUCTION

BECAUSE THERE exist groups like those described in [1], it is futile to attempt to study the solution of equations in an arbitrary group. It is reasonable, however, to do so for free groups. Results along these lines have been obtained by Edmonds, Lyndon, Malcev, Schupp, Wicks and others. With few exceptions the equations which have been successfully handled are equivalent to equations involving products of commutators or squares. The defining relations for fundamental groups of 2-manifolds have this form, which suggests that one should be able to use the theory of surfaces to study these equations. This is what will be done here.

If g is an element of the commutator subgroup $[G,G]$ of a G, we define *Genus* (g) to be the least integer n such that there exist elements $a_1, b_1, \ldots, a_n, b_n$ of G with $g = [a_1, b_1]\ldots[a_n, b_n]$. (Here $[a, b]$ denotes $aba^{-1}b^{-1}$). Similarly, if g can be written as a product of squares, we define *Sq* (g) to be the least integer n such that there exist elements $a_1, \ldots, a_n$ of G with $g = a_1^2\ldots a_n^2$.

We begin by describing a homotopy classification of maps from a bounded surface to a 1-complex when the surface satisfies certain minimality conditions. This is applied to give short proofs that there exist effective procedures for computing *Genus* (g) and *Sq* (g) when g is an element of a free group. The existence of such procedures was first shown by Edmunds [2, 3] using cancellation arguments. These algorithms are used in some non-trivial (and somewhat surprising) examples. We then give an exact description of a set of standard forms for words of a given genus. The sets of all solutions to the equations $g = [\alpha_1, \beta_1]\ldots[\alpha_n, \beta_n]$ and $g = \alpha_1^2\ldots\alpha_n^2$ are described under the conditions *Genus* $(g) = n$ and *Sq* $(g) = n$ respectively. Finally, under suitable conditions on groups U and V, we show that there are effective procedures for computing *Genus* (g) and *Sq* (g) when g is an element of the free product $U * V$.

§1. SURFACES AND PRODUCTS OF COMMUTATORS OR SQUARES

We will denote by T_n an orientable surface of genus n with one boundary component. Let P_n be a non-orientable surface of euler characteristic $1 - n$ having one boundary component.

If X is an arc-connected topological space with $\pi_1(X) = G$, then there is a well known one-to-one correspondence between the set of conjugacy classes in G and the set of homotopy classes of maps from the circle S^1 to X. We will denote the conjugacy class of g in G by $[g]$, and if $f: S^1 \to X$ is in the homotopy class corresponding to $[g]$ we will say that f *represents* $[g]$.

1.1 *An element* $g \in \pi_1(X)$ *is a product of n commutators* (resp. *squares*) *if and only if there exists* $f: T_n \to X$ ($f: P_n \to X$) *such that* $f|_{\partial T_n}$ ($f|_{\partial P_n}$) *represents* $[g]$.

Given any expression for g as a product of commutators, a map $f: T_n \to X$ can be constructed from the usual description of T_n as a $(4n + 1)$-gon with identifications on its boundary (see [4]). Conversely, any map $f: T_n \to X$, such that $f|_{\partial T_n}$ represents $[g]$, gives rise to an expression for g as a product of n commutators. This is because $\pi_1(T_n)$ contains elements $a_1, b_1, \ldots, a_n, b_n$ such that the inclusion of ∂T represents the conjugacy class of $[a_1, b_1]\ldots[a_n, b_n]$ in $\pi_1(T)$. Similar arguments apply to representations of g as a product of squares.

1.2 Let Γ_r denote an "r-leafed rose" or wedge product of r circles. The fundamental group of Γ_r is free on r generators, so questions about products of commutators or squares in a free group translate into questions about maps from T_n or P_n to Γ_r.

Let F_r be the free group with generators $X_1,\ldots,X_r$. Then F_r is isomorphic to $\pi_1(\Gamma_r)$. To make this isomorphism explicit we will assume that each "petal" of Γ_r is oriented. Let p_i be a point on the ith petal having a neighborhood homeomorphic to an interval, and let the base point $*$ be disjoint from p_i, $i = 1,\ldots,r$.

Any oriented path in Γ_r, transverse to $p_1,\ldots,p_r$, corresponds to a word in the letters $\{X_1^{\mp 1},\ldots,X_r^{\mp 1}\}$. The word is obtained by listing the intersections of the path with the points $p_1,\ldots,p_r$. If the path crosses p_i in the same direction as the orientation of the ith petal, then we write X_i. A crossing of p_i in the opposite direction corresponds to X_i^{-1}. Thus any map from S^1 to Γ_r, transverse to the points $p_1,\ldots,p_r$, corresponds to a cyclic word.

1.3 *Tight maps.* If S is a surface with boundary we will say that a map $f:S\to\Gamma_r$ is *tight* if: (1) The map f is transverse to $p_1,\ldots,p_r$. (2) The restriction of f to each component of ∂S corresponds to a reduced cyclic word. (3) The set $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ is a union of disjoint properly embedded arcs which cut S into disks.

1.4 THEOREM. *Let $w \in [F_r, F_r]$ be an element of genus n. If $f:T_n\to\Gamma_r$ is any map such that $f|_{\partial T_r}$ represents $[w]$, then f is homotopic to a tight map.*

Proof. We will modify f by a series of operations which preserve its homotopy class and which will eventually produce a tight map. To conserve symbols the new map obtained at each step will also be called f.

First we put f in general position with respect to $p_1,\ldots,p_r$. Then, since $f|_{\partial T_n}$ is homotopic to a map $:S^1\to\Gamma_r$ which corresponds to a reduced cyclic word, we may change f on an annular neighborhood of ∂T_n so that $f|_{\partial T_n}$ corresponds to a reduced cyclic word. This can be done so that the resulting map is still transverse to $p_1,\ldots,p_r$. (See [5])

Now $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ consists of arcs and simple closed curves. We claim that each of these simple closed curves bounds a disk in T_n. For if σ is a simple closed curve in $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ then we may define a new surface S by cutting T_n along σ and capping off the two new boundary circles. By sending the caps to the point $f(\sigma)$, S can be mapped into Γ_r so that the restriction to ∂S represents $[w]$. However, unless σ bounds a disk in T_n, a computation of euler characteristics will show that the bounded component of S has genus less than n. This would contradict *Genus* $(w) = n$, so σ must bound a disk in T_n.

Next we eliminate the simple closed curves in $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$. Let σ be a simple closed curve with $f(\sigma) = p_i$. Let B be the disk bounded by σ in T_n. Redefine f to have the value p_i at every point of B, and then perturb it in a small neighborhood of B to reduce the number of simple closed curves in $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$. This does not change the homotopy class of f because $\pi_2(\Gamma_r)=0$. Thus by induction we can assume that $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ consists only of arcs.

If we cut T_n along the arcs in $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ we obtain a number of components. We must verify that each of these is a disk. But the boundary circles of each component are mapped by f to contractible curves in Γ_r. Thus if any component were not a disk we could replace it by disks (one for each boundary circle) and obtain a contradiction to *Genus* $(w)=n$ as above. $\square$

The proof of Theorem 1.4 is valid without the hypothesis that the surface be orientable. However, to state the non-orientable version we need a definition. If $w\in F_r$ is either a product of commutators or a product of squares, define $\chi(w)$ to be the largest integer n such that there exists a surface S with one boundary component and $\chi(S)=n$, and a map $f:S\to\Gamma_r$ such that $f|_{\partial S}$ represents [w].

1.5 THEOREM. *Let $w\in F_r$, let S be a surface with one boundary component, and suppose that $f:S\to\Gamma_r$ is a map such that $f|_{\partial S}$ represents $[w]$. If $\chi(S)=\chi(w)$ then f is homotopic to a tight map.*

Proof. The proof is similar to that of Theorem 1.5.

1.6 *The pairing of a tight map.* If S is a surface with one boundary component and $f:S\to\Gamma_r$ is a tight map, then $f|_{\partial S}:S^1\to\Gamma_r$ corresponds to a reduced cyclic word W. Let $|W|$ be the length of W. By a *letter* we mean one of the $|W|$ occurences of a factor $X_i^{\pm 1}$ in W. A pairing of the letters of W is induced by f; two letters being paired if they correspond to the endpoints of an arc in $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$. If S is orientable then each letter X_i in W is paired with an X_i^{-1}. Otherwise an X_i may be paired with either an X_i or an X_i^{-1}.

Throughout this paper a *pairing* of the letters of W will be assumed to have the property that each X_i^ϵ occurring in W is paired with either an X_i^ϵ or an $X_i^{-\epsilon}$. If each X_i^ϵ is paired with an $X_i^{-\epsilon}$ then the pairing will be called an *orientable pairing.* (Here $\epsilon=\pm 1$).

1.7 THEOREM. *Let S and T be surfaces with one boundary component, and let $f:S\to\Gamma_r$ and $g:T\to\Gamma_r$ be tight maps. If $f|_{\partial S}$ and $g|_{\partial T}$ both correspond to the same word W and if f and g induce the same pairing of the letters of W, then there is a homeomorphism $h:S\to T$ such that f is homotopic to $g\circ h$.*

Proof. There are handle decompositions of S and T induced respectively by f and g. Let A be an annular neighborhood of ∂S. For each arc α in $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ there is a 1-handle with core $\alpha\cap\overline{(S-A)}$ attached to A. The 1-handle is attached with a half-twist if and only if the end-points of α both correspond to an X_i or both correspond to an X_i^{-1} (Fig. 1). Let A_1 be the union of A and these 1-handles. Since f is tight, S is obtained by attaching a 2-handle to each component of ∂A_1 except ∂S_1. There is a similar handle decomposition induced on T by g.

The attachment of the 1-handles in each of these handle structures is completely specified by the pairing of the letters of W. It follows that the handle decompositions are isomorphic, and that S and T are homeomorphic. Moreover, we can assume that the homeomorphism $h:S\to T$ maps $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ to $g^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ so that each point of ∂S

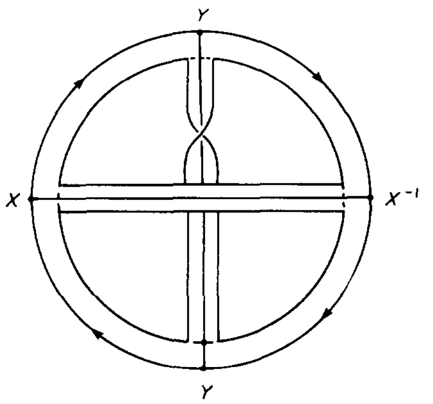

Fig. 1. Attaching 1-handles.

$\cap\ f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ is mapped to the point of $\partial T\ \cap\ g^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ which corresponds to the same letter of W.

To construct a homotopy between f and $g \circ h$ we note that the two maps agree on $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$. We can then construct a homotopy between the restrictions of f and $g \circ h$ to $\partial S\ \cup\ f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ because $\Gamma_r - \{p_1, \ldots, p_r\}$ is contractible. Finally, since $S - \left(\partial S\ \cup\ f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)\right)$ consists of open 2-cells, and $\pi_2(\Gamma_r) = 0$, we can extend the homotopy over all of S. □

1.8 Suppose we are given a reduced cyclic word W and a pairing ρ of the letters of W. We can then construct a surface S and a tight map $f{:}S \to \Gamma_r$ so that $f|_{\partial S}$ corresponds to W, and so that f induces the pairing ρ.

We construct S to have the handle structure described in the proof of theorem 1.7. The map f is easily defined in terms of this handle structure. By 1.7, S is unique up to homeomorphism. Thus we are justified in calling S the surface *associated* to the pairing ρ.

If there are d 2-handles in the handle decomposition of S then $\chi(S) = d - |W|/2$. The number d can be computed directly from the pairing—we will do this in the examples of §2.

§2. COMPUTATION OF *GENUS* (w) *AND* $Sq(w)$

2.1 Theorem. *There is an effective procedure for computing Genus* (w) *for any element* $w \in [F_r, F_r]$.

Proof. Let n = *Genus* (w) and let W be the reduced cyclic word which represents $[w]$. By 1.1 and 1.2, T_n is associated with some orientable pairing of the letters of W. Thus *Genus* (w) can be computed as the minimum of the genera of the surfaces associated with orientable pairings of the letters of W. □

2.2 Theorem. *There is an effective procedure for computing* $Sq(w)$ *for any element* $w \in F_r$ *such that* w *is a product of squares.*

Proof. Let W be the reduced cyclic word which represents $[w]$. As in Theorem 2.1 we can compute $\chi(w)$ as the maximum of the euler characteristics of the surfaces associated with pairings of the letters of W. It may happen that one of these surfaces with euler characteristic equal to $\chi(w)$ is not orientable (i.e. not associated with an orientable pairing). In this case $Sq(w) = -2-\chi(w)$. Otherwise, it is claimed, $Sq(w) = 1-\chi(w)$.

It is clear that $Sq(w) \geq 2-\chi(w)$. Thus the claim will be proved by exhibiting a map $f:P_{2-\chi(\omega)} \to \Gamma_r$ such that the restriction of f to the boundary represents $[w]$. Let S be an orientable surface with $\chi(S) = \chi(w)$ and let $g:S \to \Gamma_r$ be such that $g|_{\partial S}$ represents $[w]$. Since S is the quotient of $P_{2-\chi(w)}$ obtained by identifying an appropriate moebius strip to a point, we may take f to be the composition of g with the quotient map. □

2.3 *Remarks.* (1) It follows from §1 that an element $w \in F_r$ can be written as a product of squares if and only if the word for w has even exponent sum in each letter.

(2) Theorems 2.1 and 2.2 generalize work of Wicks[6] who showed that $w \in F_r$ is a commutator iff $[w]$ is represented by a reduced cyclic word of the form $XYZX^{-1}Y^{-1}Z^{-1}$. This implies that there is an effective procedure for deciding if w is a commutator. In §3 we will see that there are "standard forms" analogous to $XYZX^{-1}Y^{-1}Z^{-1}$ for elements of higher genus.

Algorithms for deciding if an element of a free group is a product of n commutators or a product of n squares were discovered independently by Edmunds [2, 3] Goldstein and Turner [7] and Culler [8].

2.4 *Star graphs.* Let S be a surface with one boundary component and let $f:S \to \Gamma_r$ be a tight map. Suppose that W is the cyclic reduced word corresponding to $f|_{\partial S}$, and that ρ is the pairing of the letters of W induced by f. Number the letters of W consecutively using the integers mod $|W|$ so that $W = L_0L_1 \ldots L_{|W|-1}$. We can then describe ρ as an involution of the set $\{0, 1, \ldots, |W|-1\}$.

If we cut S along the arcs $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ then we obtain a surface S', each component of which is a disk. The boundary of each disk consists alternately of arcs which come from ∂S and arcs which come from $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$. Let $\Delta(W, \rho)$ be the 1-complex obtained by collapsing each arc of $\partial S'$ which comes from $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ to a point.

We can describe $\Delta(W, \rho)$ as the directed graph defined as follows. Consider the set Ω of ordered pairs $(k, \rho(k))$. The involution ρ induces an involution on Ω which carries $V_k = (k, \rho(k))$ to $V_k^{-1} = (k, \rho(k))^{-1} = (\rho(k), k) = V_{\rho(k)}$. We take Ω as the set of vertices of $\Delta(W, \rho)$. Each $L_k = X_{i_k}^{\epsilon_k}$ for some i_k and some $\epsilon_k = \pm 1$. For each k, $\Delta(W, \rho)$ has an edge e_k running from V_{k-1} to V_k.

The star graph of W, which was invented by Whitehead [9], is the directed graph $\Sigma(W)$ defined as follows. The vertices of $\Sigma(W)$ are the letters $X_1^{\pm1}, \ldots, X_r^{\pm1}$, and there is an edge from Y to Z^{-1} for each occurrence of YZ as a subword of W. The star graph of W can be obtained from $\Delta(W, \rho)$ by identifying the vertices by mapping $V_k^{\pm1}$ to $X_{i_k}^{\pm1}$.

Thus $\chi(S) = d - |W|/2$, where d is the number of components of $\Delta(W, \rho)$. Each component of $\Delta(W, \rho)$ corresponds to a cycle in $\Sigma(W)$. Each edge of $\Sigma(W)$ occurs in exactly one such cycle. Also, if S is orientable then the cycles can be oriented consistently with the orientation of the edges of $\Sigma(W)$.

2.5 *Example.* Certain words admit only one pairing of their letters, making our algorithms especially easy to apply. Situations of this type give immediate proofs that if $a_1, b_1, \ldots, a_n, b_n$ are elements of a basis of a free group, then

$$\textit{Genus}\ ([a_1, b_1] \cdots [a_n, b_n]) = n \quad [10]$$

and

$$Sq([a_1, b_1] \cdots [a_n, b_n]) = 2n + 1 \quad [10, 11].$$

2.6 *Example.* More surprising examples are provided by the words $[X, Y]^n$ where X and Y are basis elements in a free group. We will show that *Genus* $([X, Y]^n) = [n/2] + 1$ where [ ] denotes the greatest integer function.

Suppose first that n is odd. Consider the pairing ρ defined by

$$\rho(k) = \begin{cases} -k \pmod{4n} \text{ if } k \text{ is odd} \\ 2n - k \pmod{4n} \text{ if } k \text{ is even.} \end{cases}$$

Since n is odd, ρ defines an orientable pairing, and it is easily checked that $\Delta([X, Y]^n, \rho)$ has n components. No more than n components could be obtained with any orientable pairing because any cycle in the star graph of $[X, Y]^n$ which can be oriented consistently with the orientation of $\Sigma([X, Y]^n)$ must involve at least 4 edges. It follows that *Genus* $([X, Y]^n) = (n + 1)/2$ if n is odd.

If n is even one can still show by appealing to the star graph that *Genus* $([X, Y]^n) \geq (n + 1/2)$. Since genus must be an integer, *Genus* $([X, Y]^n) \geq (n/2) + 1$. On the other hand,

$$\begin{aligned} Genus\ ([X, Y]^n) &\leq Genus\ ([X, Y]^{n-1}) + Genus\ ([X, Y]) \\ &= \frac{n}{2} + 1. \end{aligned}$$

Formulas for the commutators involved in a minimal expression for $[X, Y]^n$ as a product of commutators can be determined. One constructs the surface associated to ρ and carries out the algorithm for classifying surfaces while keeping track of the arcs in the inverse image of $\bigcup_{i=1}^{r} p_i$. This produces some peculiar commutator identities. For example,

$$[X, Y]^3 = [XYX^{-1}, Y^{-1}XYX^{-2}]\,[Y^{-1}XY, Y^2].$$

(See 4.2 and Fig. 3)

§3. STANDARD FORMS

Suppose that U is a reduced cyclic word in the generators $\{Y_1, \ldots, Y_n\}$. Let W be a cyclic word which is obtained from U by substituting a reduced word $\phi(Y_i)^\epsilon$ for each letter Y_i^ϵ, $\epsilon = \pm 1$. We will say that W is obtained from U by a *non-cancelling substitution* if $\phi(Y_i) \neq 1$ and there is no cancellation between $\phi(Y_i)^\epsilon$ and $\phi(Y_j)^\delta$ whenever $Y_i^\epsilon Y_j^\delta$ is a subword of W.

We will call U a *quadratic* word if each generator Y_i which appears in U appears exactly twice, each time as either Y_i or Y_i^{-1}. If each Y_i which appears in U appears exactly once as Y_i and once as Y_i^{-1}, then U will be called an *alternating* word. We will call U a *simple* word provided that whenever $Y_i^\epsilon Y_j^\delta$ is a subword of U then $Y_j^{-\delta} Y_i^{-\epsilon}$ is not a subword of U, *and* $Y_i^\epsilon Y_j^\delta$ appears only once in U.

Wicks showed in [4] that if W is a commutator then W is obtained by a non-cancelling substitution from $ABA^{-1}B^{-1}$ or $ABCA^{-1}B^{-1}C^{-1}$. We will show that there are analogous "standard forms" for elements of higher genus and for products of squares. Our theorem is related to a theorem of Edmunds [4, 5].

3.1 THEOREM. *Let W be a reduced cyclic word. If Genus $(W) = n$, then W is obtained by a non-cancelling substutution from a simple alternating word U with*

Genus $(U)=n$ *and* $|U|\leq 12n-6$. *If* $Sq(W)=n>1$, *then* W *is obtained by a non-cancelling substitution from a simple quadratic word* U *with* $Sq(U)=n$ *and* $|U|\leq 6n-6$.

Proof. We will prove the theorem in the case where *Genus* $(W)=n$. The same method applies in the non-orientable case, but T_n must be replaced with a surface S such that $\chi(S)=\chi(W)$.

Let $f:T_n\to\Gamma_r$ be a tight map such that $f|_{\partial T_n}$ corresponds to W. Then $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ cuts T_n into disks, and the boundary of each of these disks consists alternately of arcs in ∂S and arcs in $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$. We distinguish two types of these disks—disks of type 1 being those whose boundary meets ∂T_n in exactly 2 arcs, and disks of type 2 being those whose boundary meets ∂T_n in 3 or more arcs. Let d_1 and d_2 denote respectively the number of disks of type 1 and type 2.

Let A be the subset of T_n which is the union of all of the type 1 disks together with all of the arcs of $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ which are on the boundary of two disks of type 2. Let N be a closed regular neighborhood of A. Then each component of N is a disk which meets ∂T_n in two arcs. These arcs correspond to subwords of W and every letter of W corresponds to a point on the boundary of some component of N. Also, the two arcs of ∂T_n which are contained in the same component of N correspond to inverse words. If we number the components of N and label the two arcs of ∂T_n in the ith component as Y_i and Y_i^{-1}, then we can define U to be the word obtained by listing the arcs of $N\cap\partial T_n$ in order as we go around ∂T_n. Clearly W is obtained by a non-cancelling substitution from U and *Genus* $(U)=n$. It is also clear that U is a simple alternating word.

To estimate the length of U we note that the number of components of N is at most $3d_2/2$, so $|U|\leq 3d_2$. Also, since $|W|\geq 2d_1+3d_1$,

$$\begin{aligned} n = \textit{Genus}\ (W) &= \frac{1}{2}\left(1-d_1-d_2+\frac{1}{2}|W|\right) \\ &\geq \frac{1}{2}\left(1+\frac{1}{2}d_2\right). \end{aligned}$$

Thus $d_2\leq 4n-2$, and

$$|U|\leq 3d_2\leq 3(4n-2)=12n-6. \square$$

3.2 *Remarks.* (1) It follows from Theorem 3.1 that any simple alternating word of genus n has length at most $12n-6$. Similarly, every simple quadratic word U with Sq $(U)=N>1$ has length at most $6n-6$.

(2) If *Genus* $(W)=1$ or $Sq(W)=2$ the number of standard forms is small enough so that we can list all of the possibilities. If *Genus* $(W)=1$, we obtain the two forms found by Wicks. If Sq $(W)=2$ then W is obtained by a non-cancelling substitution from one of the four cyclic words $AABB$, $ABA^{-1}B$, $ABCA^{-1}CB$, or $AABCCB^{-1}$. In both of these cases the surfaces involved have euler characteristic -1, and hence contain at most two disks of type 2. (See Fig. 2.)

3.3 COROLLARY. *If* $w\in[F_r,F_r]$, *then Genus* $(w^p)\to\infty$ *as* $p\to\infty$.

Proof. Let W be a cyclically reduced word which is conjugate to w. Then W^p is cyclically reduced and conjugate to w^p. It is not hard to show that if V and V^{-1} are

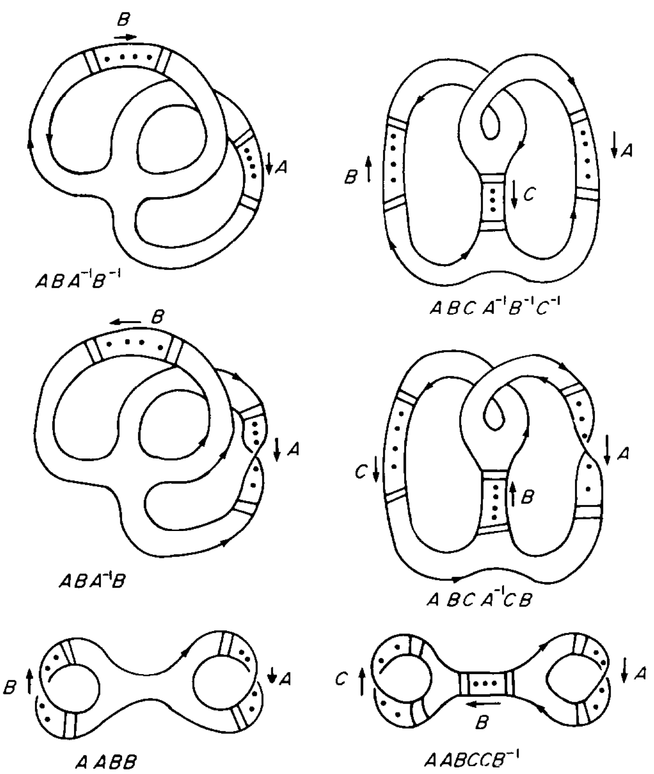

Fig. 2. Possibilities for the inverse image of $\bigcup_{i=1}^{r} p_i$ under a tight map : $T_1 \to F_r$ or : $P_2 \to \Gamma_r$.

both subwords of W^p, then $|V| < |W|/2$. This implies that if W^p is obtained by a non-cancelling substitution from an alternating word U, then $|U| > 2p$. Therefore

$$\text{Genus } (w^p) > \frac{p}{6} + \frac{1}{2}. \square$$

3.4 If ϕ is any automorphism of F_r then *Genus* $(\phi(w)) = Genus(w)$ for any $w \in [F_r, F_r]$. In view of this fact, Corollary 3.3 provides information about automorphisms of F_r. For example, we can see that if u and v are elements of $[F_r, F_r]$ and $v \neq 1$, then there is no automorphism of F_r which sends u to uv and fixes v. For if $\phi \in \mathrm{Aut}(F_r)$ is such that $\phi(u) = uv$ and $\phi(v) = v$, then $\phi^p(u) = uv^p$. This is impossible since an obvious modification of the argument used in the proof of 3.3 shows that *Genus* $(uv^p) \to \infty$ as $p \to \infty$. These ideas will be applied to the study of $\mathrm{Aut}(F_r)$ in a future paper.

§4. THE EQUATIONS $w = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$ AND $w = \alpha_1^2 \cdots \alpha_n^2$

Let g be an element of a group G and let $\Omega(\xi_1, \ldots, \xi_m)$ be a word in the free group Φ with generators $\xi_1, \ldots, \xi_m$. Define a *solution* to the equation $g = \Omega$ to be a homomorphism $\phi : \Phi \to G$ such that $\phi(\Omega) = g$. An m-tuple $(u_1, \ldots, u_m)$ of elements of G satisfies $g = \Omega(u_1, \ldots, u_m)$ if and only if $(u_1, \ldots, u_m) = (\phi(\xi_1), \ldots, \phi(\xi_m))$ for some solution ϕ of the equation $g = \Omega$.

If ϕ is a solution to $g = \Omega$ and if $\alpha \in \mathrm{Aut}(\Phi)$ fixes Ω, then $\phi \circ \alpha$ is also a solution. Thus the stabilizer of Ω in $\mathrm{Aut}(\Phi)$ acts on the set of solutions to $g = \Omega$. We will call the orbits under this action the *stabilizer orbits* of solutions to the equation $g = \Omega$.

A theorem of Hmelevskii [12] which was sharpened by Burns *et al.* [13], states that if w is an element of a free group then there are only a finite number of stabilizer orbits of solutions to $w = [\alpha, \beta]$. We will prove an analogous theorem for the equations $w = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$ and $w = \alpha_1{}^2 \cdots \alpha_n{}^2$.

An example of Lyndon and Wicks [14] shows that, even under the action of a larger group than the stabilizer of W, there is generally more than one orbit of solutions.

4.1 THEOREM. *Let w be an element of F_r. If Genus $(w) = n$ then there are only a finite number of stabilizer orbits of solutions to $w = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$. If Sq $(w) = n$ then there are only a finite number of stabilizer orbits of solutions to $w = \alpha_1{}^2 \cdots \alpha_n{}^2$.*

Proof. We will prove the theorem in the case where *Genus* $(w) = n$. As usual, the non-orientable case is proved by the same method, but T_n must be replaced with the appropriate surface S with $\chi(S) = \chi(w)$.

Let Σ be the set of solutions to $w = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$. Let Φ be the free group with generators $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$. Define an equivalence relation on Σ by $\phi \sim \psi$ iff $\psi = \eta \circ \phi \circ \alpha$, where α is an automorphism of Φ which fixes $[\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$ and η is an inner automorphism of F_r. We claim that each equivalence class is the union of a finite number of stabilizer orbits. For if η is an inner automorphism of F_r such that ϕ and $\eta \circ \phi$ are both in Σ, then η must be conjugation by an element of the centralizer of w. The centralizer of w is a cyclic group which acts transitively (by conjugation) on the stabilizer orbits in each equivalence class. The claim then follows from the observation that conjugation by w fixes each stabilizer orbit.

We must now show that Σ contains only a finite number of equivalence classes. Let W be a reduced cyclic word which represents $[w]$. We can associate to each solution of $w = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$ an orientable pairing of the letters of W. This is done by choosing a base point $*$ on ∂T_n and identifying $\pi_1(T_n, *)$ with Φ so that a curve going around ∂T_n corresponds to $[\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$. Then for any solution $\phi: \Phi \to F_r$ there is a mpa $f: T_n \to \Gamma_r$ so that $f_*: \pi_1(T_n) \equiv \Phi \to \pi_1(\Gamma_r) \equiv F_r$ is equal to ϕ. The map f is homotopic to a tight map which in turn induces an orientable pairing of the letters of W. It follows from Theorem 1.7 that if ϕ and ψ are solutions of $w = \Omega$ which are associated to the same pairing of W, then $\phi \sim \psi$. This implies that Σ contains only a finite number of equivalence classes. □

4.2 There is a procedure for constructing a solution to $w = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$ in each equivalence class—i.e. associated to each appropriate pairing of the letters of the reduced cyclic word W representing $[w]$. This amounts to constructing a surface from the pairing and then carefully carrying out the surface classification algorithm.

It suffices to find a solution, associated to the given pairing ρ, of an equation $w' = [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n]$ where w' is conjugate to w. Suppose that $Genus(w) = n$, and let $f: T_n \to \Gamma_r$ be a tight map such that $f|_{\partial T_n}$ represents $[w]$ and such that f induces ρ. To find the solution we will construct loops $\sigma_1, \tau_1, \ldots, \sigma_n, \tau_n$ in T_n based at a point $* \in T_n$, so that the loop $\sigma_1 \tau_1 \sigma_1{}^{-1} \tau_1{}^{-1} \ldots \sigma_n \tau_n \sigma_n{}^{-1} \tau_n{}^{-1}$ is freely homotopic to ∂T_n. We then will take $\alpha_i = f_*(s_i)$ and $\beta_i = f_*(t_i)$ where s_i and t_i are the homotopy classes of σ_i and τ_i respectively.

Since the arcs of $f^{-1}\left(\bigcup_{i=1}^{r} p_i \right)$ cut T_n into disks, we may view T_n as being obtained by identifying various edges of a family of polygons. These polygons and the identifications can be determined from W and ρ. Each of the edges along which identifications are to be made is labeled with one of the letters $X_1, \ldots, X_n$ and has a specified normal direction which is pulled back from the orientation of the edges of Γ_r. These labels and normal directions are also determinable from W and ρ.

We can make some of the identifications to obtain a single $8n$-gon D. Every second side of D corresponds to an arc of ∂T_n, while the remaining $4n$ sides are identified in pairs according to a certain word. The arcs of $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ corresponds to the arcs of ∂D which are to be identified and to those arcs which have already been identified.

Next we perform cut and paste operations, as in the surface classification algorithm, to obtain an $8n$-gon D' such that T_n is obtained by identifying alternate sides of D' according to the word $[A_1, B_1] \ldots [A_n, B_n]$. At each step we must keep track of the arcs of $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$ and of their normal directions.

Choose a basepoint $*$ in D'. The loops $\sigma_1, \tau_1, \ldots, \sigma_n, \tau_n$ are the images in T_n of arcs joining $*$ to the midpoints of the edges of D'. The words $\alpha_i = f_*(s_i)$ and $\beta_i = f_*(t_i)$ are obtained by listing the intersections of σ_i and τ_i with the arcs of $f^{-1}\left(\bigcup_{i=1}^{r} p_i\right)$. This procedure is carried out in Fig. 3 for the word $[X, Y]^3$ and the pairing ρ given in 2.6.

A similar procedure exists for constructing a solution to $w = \alpha_1^2 \ldots \alpha_n^2$.

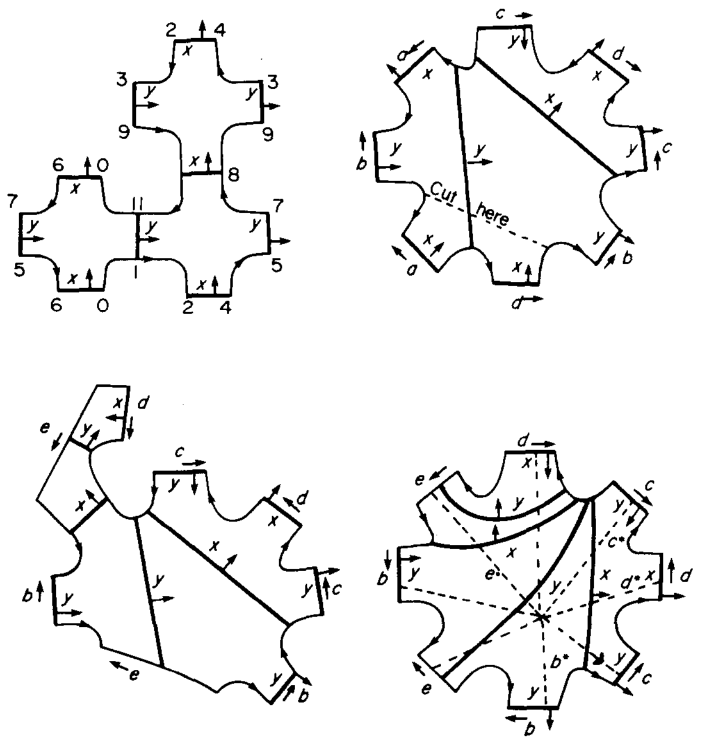

Fig. 3. Cutting and pasting to derive $[X, Y]^3 = [XYX^{-1}, Y^{-1}XYX^{-2}][Y^{-1}XY, Y^2]$. Bold lines correspond to arcs of $f^{-1}(P_1 \cup P_2)$; Straight segments are cut edges. W corresponds to $[c^*, d^*][e^*, b^*]$.

§5. FREE PRODUCTS

Let U and V be groups and let g be an element of the free product $U * V$. Under suitable conditions on U and V our topological methods can be used to give procedures for computing *Genus* (g) and *Sq* (g). We will sketch briefly how this is done.

If $g_1, \ldots, g_k$ are elements of a group G we define *Genus* $(g_1, \ldots, g_k)$ to be the least integer n such that there exist elements $x_2, \ldots, x_k$ and $a_1, b_1, \ldots, a_n, b_n$ of G with $g_1 x_2 g_2 x_2^{-1} \cdots x_k g_k x_k^{-1} = [a_1, b_1] \cdots [a_n, b_n]$. We define *Sq* $(g_1, \ldots, g_k)$ to be the least integer n such that there exist elements $x_2, \ldots, x_k$ and $a_1, \ldots, a_n$ of G such that $g_1 x_2 g_2^{\pm 1} x_2^{-1} \cdots x_k g_k^{\pm 1} x_k^{-1} = a_1^2 \cdots a_n^2$. In order to be consistent with the topology, we will say that *Genus* $(g_1, \ldots, g_k) = 0$ if there exist elements $x_2, \ldots, x_k$ of G such that

$g_1x_2g_2x_2^{-1}\cdots x_ng_nx_n^{-1}=1$. On the other hand, if there exist $x_2,\ldots,x_n$ in G such that $g_1x_2g_2^{\pm1}x_2^{-1}\cdots x_kg_k^{\pm1}x_k^{-1}=1$, then $Sq\ (g_1,\ldots,g_k)=1$.

We will say that *k-Genus* (or *k-Sq*) is *computable* in G if there is an effective procedure for determining whether *Genus* $(g_1,\ldots,g_k)$ (or *Sq* $(g_1,\ldots,g_k)$) is defined, and if so computing it, for all k-tuples of elements of G and all k. We remark that *k-Genus* and *k-Sq* are computable in a free group. This can be proved by applying the techniques of this paper to surfaces with k boundary components.

By a *cyclically reduced word* in $U*V$ will be meant an element of the form $u_1v_1\cdots u_nv_n$ or $v_1u_1\cdots v_nu_n$ where $1\neq u_i\in U$ and $1\neq v_i\in V$. Every nontrivial element of $U*V$ is conjugate to a cyclically reduced word which is unique up to a cyclic permutation of the "letters." The set of all cyclically reduced words conjugate to an element $g\in U*V$ will be called the *reduced cyclic word* representing $[g]$. It is meaningful to talk of the letters of a reduced cyclic word, and the letters have a well-defined order. The *nodes* of a reduced cyclic word W are the ordered pairs (X, Y) where X and Y are consecutive letters of W with X preceding Y. A node (X, Y) has *positive sign* if $X\in U$ and $Y\in V$. If $X\in V$ and $Y\in U$ then (X, Y) has *negative sign.*

5.1 THEOREM. *Let U and V be groups. If k-Genus is computable in U and V, then it is computable in $U*V$. If both k-Genus and k-Sq are computable in U and V, then both are computable in $U*V$.*

Sketch of proof. We sketch a proof of the computability of *k-Genus*. The second statement is proved by an analogous method, using surfaces which need not be orientable.

Let K_u and K_v be spaces with fundamental groups U and V respectively. Form a space K with $\pi_1(K)=U*V$ by identifying the base points of K_u and K_v to a point p. We may assume that p has a neighborhood homeomorphic to an interval.

Let $g_1,\ldots,g_k$ be elements of $U*V$ with *Genus* $(g_1,\ldots,g_k)=n$. If S is an orientable surface of genus n with k boundary components, then there is a map $f:S\to K$ such that the restriction of f to the ith boundary component represents $[g_i]$. As in §1 we can make f transverse to p, arrange that the arcs of ∂S between points of $f^{-1}(p)$ are mapped to loops in K which are not null-homotopic, and eliminate the simple closed curves in $f^{-1}(p)$. Then $f^{-1}(p)$ consists of arcs which cut S into surfaces, although these need not be disks.

Let $W_1,\ldots,W_k$ be reduced cyclic words representing $[g_1],\ldots,[g_k]$ respectively. The ith component of ∂S is divided into arcs by the points of $f^{-1}(p)$, and these arcs are mapped to loops in K_u or K_v which represent the letters of W_i. The points of $F^{-1}(p)$ on the ith component of ∂S correspond to the nodes of W_i.

We can construct S by taking an annulus for each component of ∂S, attaching 1-handles to these annuli, and then attaching surfaces to the boundary of the resulting space. The core of each 1-handle is an arc of $f^{-1}(p)$, and its attachment is determined by the signs of the nodes corresponding to the endpoints of the arc. The surfaces are mapped by f into either K_u or K_v. The genus of each of them must equal the genus of the tuple of elements of U or V corresponding to the images of its boundary components.

Consider all the surfaces obtained by copying this construction—using any pairing of the nodes of $W_1,\ldots,W_k$ to specify the attachment of the 1-handles, and any appropriate way of attaching orientable surfaces of minimal genus. There are only a finite number of surfaces obtained this way. We can determine which of them are orientable from the pairings and we can compute the genera of the orientable ones since *k-Genus* is computable in U and V. Since S has this structure, we can compute *Genus* $(g_1, \ldots, g_k)$ as the minimum of the genera of such surfaces. $\square$

5.2 *Remark.* It can be seen from the proof of Theorem 5.1 that in order to decide if $g \in U*V$ is a product of n commutators we need only be able to decide if

$$\textit{Genus}\ (g_1, \ldots, g_k) \leq n - \frac{k-1}{2}$$

for $g_1, \ldots, g_k \in U$ or $g_1, \ldots, g_k \in V$. Thus to decide if g is a commutator we need to be able to solve the conjugacy problem in U and V and to decide if an element of U or V is a commutator.

5.3 One can obtain standard forms for cyclically reduced words $g \in U*V$ with *Genus* $(g) = n$ or *Sq* $(g) = n$. Wicks has listed the standard forms for commutators and for products of 2 squares [6, 9]. To derive the standard forms in the general case we let S be T_n or P_n, and let $f: S \to K$ be as in the proof of Theorem 5.1. We then coalesce all of the type 1 disks in S as we did in Theorem 3.1.

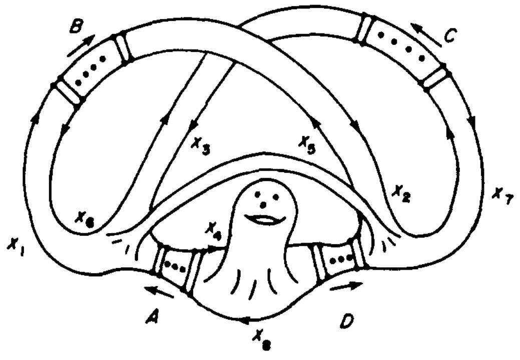

Fig. 4. A possibility for $f^{-1}(p)$, f: $T_3 \to K$.

We will leave the reader with the project of describing the standard forms for elements of a given genus in a free product. For the sake of illustration we will derive one of the many forms for elements of genus 3. If $f^{-1}(p)$ is as shown in Fig. 4, then g is conjugate to a reduced word of the form

$$Ax_1Bx_2Cx_3A^{-1}x_4Dx_5B^{-1}x_6C^{-1}x_7D^{-1}x_8$$

where x_1, x_2, x_3, x_5, x_6, and x_7 are elements of U with $x_1x_6x_3$ conjugate to $x_2^{-1}x_5^{-1}x_7^{-1}$, and x_4 and x_8 are elements of V such that x_4x_8 is a commutator in V.

Acknowledgements—Most of the results in this paper appeared in part of my doctoral dissertation. I thank my advisor, John Stallings, for providing guidance and inspiration.

REFERENCES

1. G. Baumslag, W. W. Boone and B. H. Neumann: Some unsolvable problems about elements and subgroups of groups. *Math. Scand.* **7** (1959), 191–291.
2. C. C. Edmunds: On the endomorphism problem for free groups. *Comm. in Algebra* **3** (1957), 1–20.
3. C. C. Edmunds: On the endomorphism problem for free groups II. *Proc. London Math. Soc.* **38**(3) (1979), 153–168.

4. W. S. MASSEY: *Algebraic Topology: an introduction.* Harcourt-Brace-World, New York (1967).
5. J. HEMPEL: 3-*Manifolds, Annals of Math. Study No.* 86. Princeton University Press, Princeton, New Jersey (1976).
6. M. J. WICKS: Commutators in free products, *J. London Math. Soc.* **37** (1962), 433–444.
7. R. Z. GOLDSTEIN and E. C. TURNER: Applications of topological graph theory to group theory. *Mathematics Zeitschrift* **165** (1979), 1–10.
8. M. CULLER: Genus of elements in a free group. Thesis. U.C. Berkeley (1978).
9. M. J. WICKS: The equation $X^2Y^2 = g$ over free products. *Proc. Cong. Singapore Nat. Acad. Sci.* (1971), 238–248.
10. R. C. LYNDON and P. E. SCHUPP: *Combinatorial Group Theory.* Springer-Verlag, Berlin (1976).
11. R. C. LYNDON and M. NEWMAN: Commutators as products of squares. *Proc. Am. Math. Soc.* **18** (1967), 267–272.
12. JU. I. HMELEVSKII: Systems of equations in a Free Group I. *Izv. Akad. Nauk. USSR, ser. Mat.* **35**(6), (1971) (A.M.S. Translation: *Math. USSR Izvestija* **5**(6), (1971) 267–272.
13. R. G. BURNES, C. C. EDMUNDS and I. H. FAROUQU: On Commutator equalities and stabilizers in free groups. *Canad. Math. Bull.* **19** (1976), 263–367.
14. R. C. LYNDON and M. J. WICKS: *Commutators in Free Groups.* (Preprint).
15. J. H. C. WHITEHEAD: On certain sets of elements in a free group. *Proc. London Math. Soc.* **41** (1936), 48–56.
16. L. P. COMERFORD JR. and C. C. EDMUNDS: Quadratic equations over free groups and free products. (Preprint).
17. L. P. COMERFORD JR.: Quadratic equations over small cancellation groups. (Preprint).

Rice University, Houston
Texas 77001, *U.S.A.*