

**PARADOXICAL DECOMPOSITIONS,
2-GENERATOR KLEINIAN GROUPS,
AND VOLUMES OF HYPERBOLIC 3-MANIFOLDS**

MARC CULLER AND PETER B. SHALEN

INTRODUCTION

The ε -thin part of a hyperbolic manifold, for an arbitrary positive number ε , is defined to consist of all points through which there pass homotopically nontrivial curves of length at most ε . For small enough ε , the ε -thin part is geometrically very simple: it is a disjoint union of standard neighborhoods of closed geodesics and cusps. (Explicit descriptions of these standard neighborhoods are given in §1.) If ε is small enough so that the ε -thin part of M has this structure then ε is called a *Margulis number* of M . There is a positive number, called a 3-dimensional *Margulis constant*, that serves as a Margulis number for every hyperbolic 3-manifold.

The results of this paper provide surprisingly large Margulis numbers for a wide class of hyperbolic 3-manifolds. In particular, we obtain the following result, which is stated as Theorem 10.3:

Let M be a closed orientable hyperbolic 3-manifold whose first Betti number is at least 3. Then $\log 3 = 1.09\dots$ is a Margulis number for M .

Knowing a Margulis number for a given manifold provides important geometric information. For example, one can give an estimate of the volume of a hyperbolic manifold M in terms of a Margulis number ε for M . One first observes that M must contain a hyperbolic ball of radius $\varepsilon/2$; indeed, any point in the thick part of M is the center of such a ball. The volume of this ball is a lower bound for the volume of M . This bound can be improved by using an observation of Meyerhoff's [Me]: the ratio of the volume of M to that of the ball is at least $d(\varepsilon/2)$, where $d(r)$ is an explicitly given function that, by a theorem of Böröczky, is a bound for the local density of a radius r sphere-packing in hyperbolic space. This technique gives Corollary 10.4 to Theorem 10.3:

Let M be a closed orientable hyperbolic 3-manifold whose first Betti number is at least 3. Then M contains a hyperbolic ball of radius $\frac{1}{2} \log 3 = .54\dots$ and the volume of M is greater than .92.

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The estimates given by the above results should be compared with other known estimates and with known examples. The greatest known lower bound for the volume of an arbitrary closed orientable hyperbolic 3-manifold, due to F. Gehring and G. Martin [GeM], is .00115, improving an earlier estimate of .00082 by R. Meyerhoff. The smallest known orientable hyperbolic 3-manifold in terms of volume was discovered by J. Weeks; it has Betti number 0 and has a volume of approximately .94. A result of Shalen and Wagreich [ShW] implies, for a 3-manifold M satisfying somewhat weaker hypotheses than those of Theorem 10.3, that $\frac{1}{2} \log 3$ is a Margulis number for M (so that M contains a hyperbolic ball of radius $\frac{1}{4} \log 3$).

It can be proved that $\log 3$ is *not* a Margulis number for the complement of the figure 8 knot, and hence that it is not a Margulis constant. Furthermore, there is overwhelming evidence, based on computer experiments by C. Hodgson and J. Weeks, that the largest hyperbolic ball in the Weeks manifold has radius $.51 \dots < \frac{1}{2} \log 3$. Thus Theorem 10.3, and—according to the experimental evidence—Corollary 10.4 as well, become false if one drops the topological restrictions on the manifold.

Theorem 10.3 is an application of a theorem on 2-generator Kleinian groups. We may regard the orientable hyperbolic 3-manifold M as the quotient of the hyperbolic 3-space \mathbf{H}^3 by a discrete group Γ of orientation-preserving isometries; from this point of view, ε is a Margulis number for M if and only if, for any two noncommuting elements ξ and η of Γ , every point of \mathbf{H}^3 is moved a distance at least ε by either ξ or η .

A 3-manifold is said to be *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold; a torsion free Kleinian group Γ is *topologically tame* if \mathbf{H}^3/Γ is topologically tame. We shall prove the following result, which is stated as Theorem 9.1:

Let ξ and η be noncommuting isometries of \mathbf{H}^3 . Suppose that ξ and η generate a torsion free discrete group that is topologically tame, is not cocompact, and contains no parabolics. Then every point of hyperbolic space is moved a distance at least $\log 3$ by either ξ or η ; that is, we have

$$\max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) \geq \log 3$$

for any $z \in \mathbf{H}^3$.

In §10 we show (Proposition 10.2) that if $M = \mathbf{H}^3/\Gamma$ has first Betti number at least 3 then every two-generator subgroup of Γ is noncocompact and topologically tame. Thus Theorem 9.1 on two-generator groups implies Theorem 10.3 about Margulis numbers.

It is conjectured that any hyperbolic 3-manifold with finitely generated fundamental group is topologically tame. If this were true it would allow one to generalize the above result about Margulis numbers to any hyperbolic 3-manifold whose fundamental group has no 2-generator subgroup of finite index.

We now describe the ingredients of the proof of our main theorem. The proof begins with the observation (Proposition 9.2) that if ξ and η are noncommuting isometries of \mathbf{H}^3 and if the group that they generate is torsion free,

is not cocompact, and contains no parabolics, then it is free of rank 2.

One novel feature of the proof is the use of a construction, due to Patterson and studied extensively by Sullivan, of a geometrically natural measure on the limit set of a Kleinian group. We use such a measure to relate the combinatorial structure of the free group of rank 2 to its action on its limit set. This part of the argument bears an intriguing resemblance to the construction of a decomposition of the sphere in the Hausdorff-Banach-Tarski paradox [Wag]. By viewing the free group of rank 2 as a set of reduced words in the alphabet $\{x, y, x^{-1}, y^{-1}\}$, one obtains a decomposition of the group into four infinite sets and a singleton $\{1\}$, where each of the infinite sets consists of all words beginning with a given letter. This decomposition has the peculiar property that each of the infinite sets is mapped onto the complement of another of the infinite sets under left multiplication by a certain letter. For example, left multiplication by x^{-1} sends a word that begins with x to a word that does not begin with x^{-1} .

In the Hausdorff-Banach-Tarski construction one has a rank-2 free group acting by isometries on the sphere. A choice of a point in a given free orbit determines an identification of the group with the orbit and hence induces a decomposition of the orbit. By choosing a point from each orbit one obtains a decomposition of the sphere (ignoring the countable subset of points with nontrivial stabilizer). Since in this situation the group action preserves the area measure of the sphere, one obtains a paradoxical conclusion.

In our situation we have a free group of rank 2 acting by isometries on hyperbolic space, and hence acting by Möbius transformations on the sphere at infinity. Using a generalization of Patterson's construction we are able to construct a measure on the limit set of our group that decomposes as a sum of four measures, each of which is transformed to the complement of another by a generator or the inverse of a generator. The key to this construction is the fact that the Patterson measures are obtained as limits of measures that are supported on an orbit of the group, and hence reflect the combinatorial structure of the group itself. It is necessary here to identify the group with an orbit by making a choice of a point z in hyperbolic space.

This construction is especially interesting in the case where the Patterson measure that admits the decomposition is equal to the area measure on the sphere at infinity. (Here the sphere is given the round metric obtained by identifying \mathbf{H}^3 with a ball so that the given point z is the center, and the area measure is normalized so as to have total mass 1.) In this case the "paradoxical" decomposition leads to an elementary proof of the inequality $\max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) \geq \log 3$. By symmetry considerations together with a measure-theoretic lemma, one obtains a subset of the sphere that has area at most $\frac{1}{4}$ but whose image under one of the generators has area at least $\frac{3}{4}$. A direct computation then shows that this element moves the center a distance at least $\log 3$.

Given a Patterson measure associated to a discrete group Γ , one can construct a positive, Γ -invariant (generalized) eigenfunction of the hyperbolic Laplacian by integrating the hyperbolic Poisson kernel against the measure. This eigenfunction has a positive eigenvalue and, therefore, determines a

superharmonic function on \mathbf{H}^3/Γ (which is a hyperbolic manifold if Γ is torsion free). This allows one to prove that if every superharmonic function on \mathbf{H}^3/Γ is constant then the area measure is the unique Patterson measure associated to Γ .

An argument due to Thurston shows that certain geometric conditions on the ends of a complete hyperbolic manifold $M = \mathbf{H}^3/\Gamma$ imply that all positive superharmonic functions on M are constant. In our setting the appropriate condition is that there exist a sequence of singular surfaces having metrics of intrinsic curvature ≤ -1 and bounded genera, such that every compact subset of M is “enclosed” in a homological sense by a surface in the sequence. By a theorem of Canary’s, such a sequence exists if Γ is topologically tame, contains no parabolics and has limit set equal to the entire sphere at infinity of hyperbolic space. Thus we need only to establish the inequality in the conclusion of Theorem 9.1 in the case that the limit set of Γ is a proper subset of the sphere at infinity, i.e., when Γ has nonempty set of discontinuity.

Notice, however, that if the set of discontinuity is nonempty then the area measure cannot be a Patterson measure. Our strategy is to reduce the case where Γ has nonempty set of discontinuity to the more exotic case where the area measure is a Patterson measure by considering limits of groups with nonempty set of discontinuity. There is an open subset $\mathfrak{G}\mathfrak{F}$ of the variety $\mathrm{PSL}_2(\mathbf{C}) \times \mathrm{PSL}_2(\mathbf{C})$ consisting of all pairs of elements (ξ, η) such that $\langle \xi, \eta \rangle$ is a free group of rank 2 having a nonempty set of discontinuity and containing no parabolics. If (ξ, η) is any point in the closure $\overline{\mathfrak{G}\mathfrak{F}}$ (in the complex topology) of $\mathfrak{G}\mathfrak{F}$, then $\langle \xi, \eta \rangle$ is still free and discrete. It is easy to show that for any point $z \in \mathbf{H}^3$ the function $(\xi, \eta) \mapsto \max(\mathrm{dist}(z, \xi \cdot z), \mathrm{dist}(z, \eta \cdot z))$ is a proper continuous function on $\overline{\mathfrak{G}\mathfrak{F}}$ with no local minimum on $\mathfrak{G}\mathfrak{F}$. Hence it takes a minimum value at some point (ξ, η) in the frontier \mathfrak{B} of $\mathfrak{G}\mathfrak{F}$. This reduces the proof of the inequality $\max(\mathrm{dist}(z, \xi \cdot z), \mathrm{dist}(z, \eta \cdot z)) \geq \log 3$ for the case $(\xi, \eta) \in \mathfrak{G}\mathfrak{F}$ to the proof in the case $(\xi, \eta) \in \mathfrak{B}$. We then complete the proof by showing that there is a dense G_δ in \mathfrak{B} consisting of points for which the area measure is indeed a Patterson measure. We use Thurston’s criterion here as well; we show that, for (ξ, η) in a dense G_δ subset of \mathfrak{B} , there exists a sequence of negatively curved singular surfaces of bounded genera enclosing every compact set in $\mathbf{H}^3/\langle \xi, \eta \rangle$. The proof of the existence of this dense G_δ contains much of the technical work in the paper.

Each of the singular surfaces that we construct is a map of a closed surface into $M = \mathbf{H}^3/\langle \xi, \eta \rangle$. The domain surface is divided into “pairs of pants” by a family of “waist” curves that are mapped to closed geodesics in M . Furthermore, the surfaces in the sequence are *girded* in the sense that the lengths of the waist geodesics tend to 0 through the sequence. The existence of a sequence of singular surfaces of this type is proved by means of a recent result of Curt McMullen’s asserting the density of maximal cusps in \mathfrak{B} . The singular surfaces are chosen to be invariant with respect to a canonical involution of M ; such an involution exists because $\pi_1(M)$ is a 2-generator group. By combining this invariance property with the fact that the waist lengths tend to 0, we show that any compact subset of M is enclosed by some surface in the family.

Our construction of girded surfaces works only in the 2-generator case because it uses the involution. However, we believe that a similar picture holds more generally. Let Γ be any geometrically finite Kleinian group without parabolics, and let M be the manifold defined by a generic point in the frontier of the set of quasi-conformal deformations of Γ . We conjecture that M is *girded*, meaning that there exists a sequence of girded surfaces in M such that every compact subset of M is enclosed by a surface in the sequence. For a precise definition of a girded manifold, see 7.4.

It is worth noting that the singular surfaces that appear in the proof of Canary's theorem are not required to contain short geodesics. In general a topologically tame manifold with geometrically infinite ends need not contain arbitrarily short geodesics, and thus need not be girded. Conversely, it is not clear in general whether a girded manifold with finitely generated fundamental group is topologically tame.

The paper is organized as follows. In the first section we establish notation and collect the basic facts about hyperbolic manifolds that will be needed in the paper. The next two sections discuss the correspondence, via the Poisson kernel, between measures on the sphere at infinity and eigenfunctions of the hyperbolic Laplacian in a Γ -equivariant setting. In essence this material is classical potential theory, with the Poincaré ball replacing the Euclidean ball.

In §4 we present Patterson's ideas in a generalized setting to give a construction of a Γ -invariant conformal density that respects a given decomposition of a discrete group Γ . Section 5 contains the argument that is used in proving the main estimate in the case where the area measure is the unique Patterson measure. In §6 we describe Thurston's geometric criterion that implies that every positive superharmonic function is constant. We introduce the notion of a Bonahon surface and formulate the criterion in terms of these surfaces.

In §7 we develop the notions that we use in constructing our girded surfaces. Section 8 contains the proof of the existence of a dense G_δ in \mathfrak{B} consisting of girded groups. In §9 we prove the main theorem, and in §10 we give the applications to closed manifolds.

The preprint of this paper contains a more self-contained and detailed discussion of the topics in §§4 and 6 of the current version. In addition the preprint contains a full development of the geometry of ultra-hyperbolic surfaces. The point of view taken in the preprint is somewhat different from the one taken in [Bo], being based on "packing" arguments similar to some that have been used by Thurston and Gromov.

1. HYPERBOLIC SPACE AND DISCRETE GROUPS

We establish some notation and conventions that will be used throughout the paper.

1.1. Hyperbolic space. Whenever we consider a metric space, we shall write $\text{dist}(x, y)$ to denote the distance function, provided that no confusion can result. In a metric space we also write $\text{dist}(x, S)$ to denote the distance from a point x to a closed set S . If S is a subset of a metric space and r is a positive number, we shall denote the closed r -neighborhood of S by $\text{nbhd}_r(S)$.

By a *model of hyperbolic n -space*, for $n \geq 2$, we shall mean a complete, 1-connected Riemannian n -manifold of constant sectional curvature -1 . Throughout this paper we shall fix, abstractly, a model \mathbf{H}^n of hyperbolic space. Any two models of hyperbolic n -space are isometric, and the isometry group of any model acts transitively on its orthonormal frame bundle. It will frequently be useful to identify \mathbf{H}^n , via a suitable isometry, with one of the standard concrete models. The concrete models that will be used most often are the *upper half-space model*, which has underlying space $\mathbf{R}^{n-1} \times [0, \infty)$, and the *Poincaré model*, for which the underlying space is the open unit ball $B^n \in \mathbf{R}^n$. The geodesics in these models are intersections with lines or circles in the ambient euclidean space that are orthogonal, respectively, to the plane $x_{n+1} = 0$ or the unit sphere. In studying convex sets in \mathbf{H}^n it is useful to consider the *projective model*, which also has underlying space B^n , but with a metric for which the geodesics are intersections of euclidean geodesics with B^n . For detailed descriptions of these spaces, and for general information on hyperbolic geometry, the reader is referred to [F].

The sphere at infinity of \mathbf{H}^n will be denoted by S_∞^{n-1} or simply S_∞ . The canonical compactification of \mathbf{H}^n , of which S_∞ is the boundary, will be denoted $\overline{\mathbf{H}}^n$. Any self-isometry γ of \mathbf{H}^n extends to a conformal automorphism of $\overline{\mathbf{H}}^n$, which we denote $\overline{\gamma}$. We denote by γ_∞ the conformal automorphism of S_∞ obtained by restricting $\overline{\gamma}$.

Given any point $z \in \mathbf{H}^n$, there exists an isometry h from \mathbf{H}^n onto the Poincaré model B^n that maps z to 0. Furthermore, h is unique modulo composition with orthogonal linear transformations of \mathbf{R}^n . It follows that if we pull back the metric of $S^{n-1} = \partial B^n$ via \overline{h} , we obtain a metric on S_∞ that is uniquely determined by the point z ; we call it the *round metric centered at z* .

Any isometry h of the upper half-space model onto \mathbf{H}^n extends uniquely to a conformal diffeomorphism of $\mathbf{R}^{n-1} \times [0, \infty)$ onto $\overline{\mathbf{H}}^n - \{\zeta\}$, where $\zeta \in S_\infty$ is a point determined by h . We shall say that h *maps ∞ to ζ* (or that h^{-1} maps ζ to ∞). Given any point $\zeta \in S_\infty$, there exists an isometry of $\mathbf{R}^{n-1} \times (0, \infty)$ onto \mathbf{H}^n that maps ∞ to ζ .

If A_1 and A_2 are points of $\overline{\mathbf{H}}^n$ and B is a point of \mathbf{H}^n , we denote by $\angle A_1 B A_2$ the angle between the ray from B to A_1 and the ray from B to A_2 ; here by the ray from B to A_i we mean the (hyperbolic) ray starting at B and passing through A_i (if $A_i \in \mathbf{H}^n$) or having A_i as an end point (if $A_i \in S_\infty$).

1.2. Convexity. A subset K of $\overline{\mathbf{H}}^n$ is termed *convex* if $K \neq \emptyset$ and if every line segment in \mathbf{H}^n with end points in K is itself contained in K ; here a “line segment” may be an entire line or a half-infinite ray. The *convex hull* of a nonempty subset X of $\overline{\mathbf{H}}^n$ is defined to be the intersection of all convex subsets of $\overline{\mathbf{H}}^n$ containing X and will be denoted $\overline{\text{hull}}(X)$. We set $\text{hull}(X) = \overline{\text{hull}}(X) \cap \mathbf{H}^n$. Clearly $\overline{\text{hull}}(X)$ and $\text{hull}(X)$ are convex.

Any isometry of \mathbf{H}^n onto the projective model B^n extends to a diffeomorphism of $\overline{\mathbf{H}}^n$ onto \overline{B}^n that maps convex subsets of $\overline{\mathbf{H}}^n$ onto convex subsets (in the usual Euclidean sense) of \overline{B}^n . Hence any compact convex subset of $\overline{\mathbf{H}}^n$ that

has nonempty interior is homeomorphic to a closed n -ball. It follows that any closed convex subset of \mathbf{H}^n is the intersection of \mathbf{H}^n with a topological closed n -ball in $\overline{\mathbf{H}^n}$ and, therefore, is a contractible n -manifold-with-boundary.

Proposition. *If K is a convex subset of \mathbf{H}^n , then for any positive number R the set $\text{nbhd}_R(K)$ is convex.*

Proof. It suffices to show that if $\alpha \subset \mathbf{H}^n$ is a compact line segment then $\text{nbhd}_R(\alpha)$ is convex. This is, in turn, equivalent to showing that for any line $l \subset \mathbf{H}^n$ the set $l \cap \text{nbhd}_R(\alpha)$ is connected. If $\phi: \mathbf{H}^n \rightarrow \mathbf{R}$ is the continuous function that assigns to each point of \mathbf{R}^n its minimum distance from α , we have $l \cap \text{nbhd}_R(\alpha) = (\phi|l)^{-1}([0, R])$. It therefore suffices to prove that the function $\phi|l$ has no local maximum. This is an exercise in hyperbolic geometry, which we leave to the reader. \square

1.3. Let γ be any nontrivial orientation-preserving isometry of \mathbf{H}^3 . For any $\varepsilon > 0$ let us write

$$C_\varepsilon(\gamma) = \{z \in \mathbf{H}^3 : \text{dist}(z, \gamma \cdot z) \leq \varepsilon\}.$$

If γ is parabolic, i.e., if γ_∞ has a unique fixed point $\zeta \in S_\infty$, then $C_\varepsilon(\gamma)$ is a closed horoball based at ζ .

If γ is loxodromic then γ_∞ has exactly two fixed points, say ζ_- and ζ_+ , in S_∞ . The line A joining ζ_- and ζ_+ is by definition the axis of γ , and $\gamma|A$ is a translation through some distance $\text{length}(\gamma)$. The distance from a point to its image under γ depends on the twist angle of γ as well as on $\text{length}(\gamma)$ and the distance of the point from the axis A . However, the following result is sufficient to permit us to describe $C_\varepsilon(\gamma)$ in this case.

Proposition. *Let γ be a loxodromic isometry of \mathbf{H}^3 with axis A . Then there is a monotonically increasing continuous function $f = f_\gamma: [0, \infty) \rightarrow [0, \infty)$ such that $\text{dist}(z, \gamma \cdot z) = f(\text{dist}(z, A))$ for every point $z \in \mathbf{H}^n$.*

Proof. Left to the reader. \square

It follows from the above proposition that if γ is loxodromic with axis A then we have $C_\varepsilon(\gamma) = \emptyset$ for any $\varepsilon < \text{length}(\gamma)$, and for any $\varepsilon \geq \text{length}(\gamma)$ we have $C_\varepsilon(\gamma) = \text{nbhd}_\delta(A_\gamma)$, where δ is a nonnegative number depending on γ and ε . (In the notation of the proposition we have $\delta = f_\gamma(\varepsilon)$.)

1.4. By a *Kleinian group* we shall mean a discrete group of orientation-preserving isometries of \mathbf{H}^3 that is nonelementary in the sense that it has no abelian subgroup of finite index. If Γ is a Kleinian group we shall write Λ_Γ for its limit set.

When Γ is a torsion free Kleinian group we shall write $M(\Gamma)$ for the complete hyperbolic 3-manifold \mathbf{H}^3/Γ . The set $\text{hull}(\Lambda_\Gamma)$ is closed since Λ_Γ is compact, and nonempty since Γ is nonelementary. Thus the set $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma)) \subset \mathbf{H}^3$ is closed and has nonempty interior. It is convex by Proposition 1.2 and is clearly Γ -invariant. Hence by 1.2, $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma))$ is a contractible, Γ -invariant 3-manifold-with-boundary. We shall write

$$N(\Gamma) = (\text{nbhd}_1(\text{hull}(\Lambda_\Gamma)))/\Gamma \subset M(\Gamma).$$

By the discussion above, $N(\Gamma)$ is a 3-manifold-with-boundary and a deformation retract of $M(\Gamma)$.

1.5. Margulis numbers. The group of all orientation-preserving isometries of \mathbf{H}^3 is isomorphic to $\mathrm{PSL}_2(\mathbf{C})$. In $\mathrm{PSL}_2(\mathbf{C})$, every element of order > 2 has an abelian centralizer. Hence if Γ is a torsion free Kleinian group, every nontrivial element of Γ lies in a unique maximal abelian subgroup. If H is any maximal abelian subgroup of Γ then either H is cyclic and $H - \{1\}$ consists of loxodromic elements with a common axis, or H is free abelian of rank 1 or 2 and $H - \{1\}$ consists of parabolic elements with a common fixed point in S_∞ . In the latter case H will be called a *cuspidal subgroup* of Γ .

For any point $P \in M(\Gamma)$ we shall denote by $\mathrm{short}(P)$ the infimum of the lengths of all homotopically nontrivial loops based at P . Thus for any $z \in \mathbf{H}^3$ lying in the fiber above P we have

$$\mathrm{short}(P) = \inf_{1 \neq \gamma \in \Gamma} \mathrm{dist}(z, \gamma \cdot z).$$

We have $0 < \mathrm{short}(P) \leq \infty$, and $\mathrm{short}(P) = \infty$ if and only if $\Gamma = \{1\}$. When $\Gamma \neq \{1\}$, the infimum in the definition of $\mathrm{short}(P)$ is realized. For any interval $I \subset (0, \infty)$ we denote by $M_I(\Gamma)$ the set of all points $P \in M(\Gamma)$ such that $\mathrm{short}(P) \in I$. Thus for any $\varepsilon > 0$, the set $M_{(0, \varepsilon]}$ consists of all points of M through which there pass homotopically nontrivial closed curves of length $\leq \varepsilon$. We call $M_{(0, \varepsilon]}$ the ε -thin part of M .

Let $\widetilde{M}_{(0, \varepsilon]}$ denote the preimage of $M_{(0, \varepsilon]}$ in \mathbf{H}^n . Then by definition, $\widetilde{M}_{(0, \varepsilon]}$ consists of all points $z \in \mathbf{H}^3$ such that $\mathrm{dist}(z, \gamma \cdot z) \leq \varepsilon$ for some $\gamma \in \Gamma - \{1\}$. Hence we have

$$\widetilde{M}_{(0, \varepsilon]} = \bigcup_{1 \neq \gamma \in \Gamma} C_\varepsilon(\gamma).$$

For any maximal abelian subgroup H of Γ and for any fixed $\varepsilon > 0$, the family of sets $\{C_\varepsilon(\gamma) : \gamma \in \Gamma\}$ is totally ordered by inclusion. It follows from the discreteness of Γ that there is always a maximal set $C_\varepsilon(H)$ in this family. Thus for any $\varepsilon > 0$ we may write

$$\widetilde{M}_{(0, \varepsilon]} = \bigcup_H C_\varepsilon(H),$$

where H ranges over all maximal abelian subgroups of Γ . The discreteness of Γ also implies that the sets $C_\varepsilon(H)$ form a locally finite family. A positive number ε will be called a *Margulis number* for $M(\Gamma)$ if the sets $C_\varepsilon(H)$, where ε ranges over the distinct maximal abelian subgroups of Γ , are pairwise disjoint. Equivalently, ε is a Margulis number for M if for every point $z \in \mathbf{H}^3$ and every pair of noncommuting elements ξ, η of Γ , we have $\max(\mathrm{dist}(z, \xi \cdot z), \mathrm{dist}(z, \eta \cdot z)) \geq \varepsilon$. There exists a positive number that is a 3-dimensional *Margulis constant* in the sense that it is a Margulis number for every complete hyperbolic 3-manifold. The existence of such a constant follows, for example, from [Be, Theorem 5.4.5].

If ε is a Margulis number for M then each component of $M_{(0, \varepsilon]}$ has the form $C_\varepsilon(H)/H$ for some maximal abelian subgroup H of Γ . When H is

cuspidal, so that $C_\varepsilon(H)$ is a horoball, we call $C_\varepsilon(H)/H$ a *standard cusp neighborhood*. When H consists of loxodromic elements, it is cyclic. In this case it follows from Proposition 1.3 that $C_\varepsilon(H) = \emptyset$ when ε is less than the length of a generator of H , and that otherwise $C_\varepsilon(H) = \text{nbhd}_\delta(A)$, where A is the axis of a generator of H and δ is a nonnegative number depending on ε and H . In particular, A/H is a simple closed geodesic of length $\leq \varepsilon$ and $C_\varepsilon(H)/H = \text{nbhd}_\delta(A/H)$. Topologically, $\mathcal{S} = C_\varepsilon(H)/H$ is a solid torus unless A/H has length ε , in which case $\mathcal{S} = A/H$. When H consists of loxodromic elements and $C_\varepsilon(H) \neq \emptyset$, we call $\mathcal{S} = C_\varepsilon(H)/H$ a *tube*.

1.6. If Γ is any Kleinian group and ε is any Margulis number for $M(\Gamma)$, we shall denote by $M_{(0, \varepsilon]}^c(\Gamma)$ the union of all components of $M_{(0, \varepsilon]}(\Gamma)$ that are standard cusp neighborhoods. We shall write

$$M_{\{\varepsilon\}}^c(\Gamma) = \partial M_{(0, \varepsilon]}^c(\Gamma) \subset M_{\{\varepsilon\}}(\Gamma)$$

and

$$M_{[\varepsilon, \infty)}^c(\Gamma) = M(\Gamma) - \text{int } M_{(0, \varepsilon]}^c(\Gamma) \supset M_{[\varepsilon, \infty)}(\Gamma).$$

Furthermore, if I is any of the intervals $(0, \varepsilon]$, $\{\varepsilon\}$, or $[\varepsilon, \infty)$, we set

$$N_I^c(\Gamma) = N(\Gamma) \cap M_I^c(\Gamma).$$

The preimage of $N_I^c(\Gamma)$ in \mathbf{H}^3 will be denoted $\tilde{N}_I^c(\Gamma)$.

For any Margulis number ε , the set $N_{(0, \varepsilon]}^c(\Gamma)$ is a 3-manifold-with-boundary and a deformation retract of $M_{(0, \varepsilon]}^c(\Gamma)$. Indeed, each component of $M_{(0, \varepsilon]}^c(\Gamma)$ has the form $C_\varepsilon(H)/H$ for some maximal abelian subgroup H of Γ consisting of parabolic elements. The fixed point $\zeta \in S_\infty$ belongs to Λ_Γ , and we have $\Lambda_\Gamma \neq \{\zeta\}$ since Γ is nonelementary; hence $\text{hull}(\Lambda_\Gamma) \cap C_\varepsilon(H) \neq \emptyset$. Since $\text{hull}(\Lambda_\Gamma)$ and $C_\varepsilon(H)$ are compact and convex, it follows that $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma)) \cap C_\varepsilon(H)$ is a closed convex subset of \mathbf{H}^3 with nonempty interior, and by 1.2 is, therefore, a contractible 3-manifold-with-boundary. Hence the quotient of $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma)) \cap C_\varepsilon(H)$ by the action of H is a 3-manifold-with-boundary and a deformation retract of $C_\varepsilon(H)/H$; our assertion follows.

1.7. **Geometric finiteness.** The Kleinian group Γ is said to be *geometrically finite* if $N_{[\varepsilon, \infty)}^c(\Gamma)$ is compact for some Margulis number ε .

If Γ is geometrically finite then it has only finitely many conjugacy classes of cuspidal subgroups. Indeed, if ε is a Margulis constant, the sets $C_\varepsilon(H)$, where H ranges over the cuspidal subgroups of Γ , form a locally finite family (1.5). Hence the corresponding cusp neighborhoods $C_\varepsilon(H)/H$ form a locally finite family of subsets of $M(\Gamma)$. For each H the fixed point ζ of H belongs to Λ_Γ , and $\Lambda_\Gamma \neq \{\zeta\}$ since Γ is nonelementary; hence $\text{hull}(\Lambda_\Gamma)$ contains a geodesic with one end point at ζ . It follows that $(C_\varepsilon(H)/H) \cap N_{[\varepsilon, \infty)}^c(\Gamma) \neq \emptyset$ for every H . Hence if $N_{[\varepsilon, \infty)}^c(\Gamma)$ is compact then the family of cusp neighborhoods $C_\varepsilon(H)/H$ is finite, establishing our assertion.

Lemma. *Let H be a discrete group of parabolic isometries of the upper half-space model of hyperbolic space, with common fixed point ∞ . Let Λ be a*

closed nonempty H -invariant subset of $\mathbf{R}^2 \times \{0\}$ such that Λ/H is compact.

- (i) If H is free abelian of rank 2 then $\mathbf{R}^2 \times \{h\} \subset \text{hull}(\Lambda \cup \{\infty\})$ for all sufficiently large h .
- (ii) If H is infinite cyclic then there is an H -invariant strip S bounded by two (possibly equal) parallel lines in \mathbf{R}^2 such that $\Lambda \subset S$ and $\mathbf{R}^2 \times \{h\} \cap \text{hull}(\Lambda \cup \{\infty\}) = S \times \{h\}$ for all sufficiently large h .

Proof. By convexity, if a point P of upper half-space is contained in $\text{hull}(\Lambda \cup \{\infty\})$ then so is the entire vertical ray beginning at P . Now suppose that α and β are points of Λ and that the Euclidean distance from α to β is d . Let σ be the Euclidean line segment from α to β . Since the hyperbolic geodesic from α to β is a Euclidean circle of radius $d/2$, it follows that for all $h > d/2$ we have $\sigma \times \{h\} \subset \text{hull}(\Lambda \cup \{\infty\})$.

Let $\gamma \in H$ and $\alpha \in \Lambda$ and let l be the line in \mathbf{R}^2 that contains the points $\bar{\gamma}^n(\alpha)$ for $n \in \mathbf{Z}$. Let d be the Euclidean distance from $\bar{\gamma}^n(\alpha)$ to $\bar{\gamma}^{n+1}(\alpha)$. By the observation above, applied to each of the pairs $\{\bar{\gamma}^n(\alpha), \bar{\gamma}^{n+1}(\alpha)\}$ we have $l \times \{h\} \subset \text{hull}(\Lambda \cup \{\infty\})$ for all $h > d/2$.

Suppose that H is free abelian of rank 2 with generators γ_1 and γ_2 . Let d_1 and d_2 be the respective translation lengths of $\bar{\gamma}_1$ and $\bar{\gamma}_2$ acting on $\mathbf{R}^2 \times \{0\}$. Let α be a point of Λ . Let l_n denote the $\bar{\gamma}_1$ -invariant line through $\bar{\gamma}_2^n(\alpha)$. For all $h > d_1/2$ we have that $l_n \times \{h\} \subset \mathbf{R}^2 \times \{h\} \cap \text{hull}(\Lambda \cup \{\infty\})$. Every point of \mathbf{R}^2 lies on a line segment of length d_2 with end points in l_n and l_{n+1} for some $n \in \mathbf{Z}$. It follows that for $h > d_1/2 + d_2/2$ we have $\mathbf{R}^2 \times \{h\} \subset \text{hull}(\Lambda \cup \{\infty\})$.

Suppose that H is infinite cyclic with generator γ and that $\bar{\gamma}$ acts on $\mathbf{R}^2 \times \{0\}$ as a Euclidean translation through a distance d . The assumption that Λ/H is compact implies that Λ is contained in a strip S bounded by $\bar{\gamma}$ -invariant lines l_1 and l_2 . Since Λ is closed, the lines l_1 and l_2 may be taken to contain points of Λ . If the Euclidean distance from l_1 to l_2 is w then for all $h > d/2 + w/2$ we have $S \times \{h\} \subset \text{hull}(\Lambda \cup \{\infty\})$. On the other hand, $S \times [0, \infty)$ is a convex set containing $\Lambda \cup \{\infty\}$. Therefore $\mathbf{R}^2 \times \{h\} \cap \text{hull}(\Lambda \cup \{\infty\}) = S \times \{h\}$. \square

Recall that a topological 3-manifold N is said to be *irreducible* if every locally flat 2-sphere in N is the boundary of a 3-ball in N .

Proposition. *Let Γ be a geometrically finite, torsion free Kleinian group; set $M = M(\Gamma)$ and $N = N(\Gamma)$. Then $N_{[\varepsilon, \infty)}^c$ is compact whenever ε is a Margulis number for M . Furthermore, for every sufficiently small Margulis number ε_0 the following conclusions hold:*

- (i) $N_{[\varepsilon_0, \infty)}^c$ is an orientable, irreducible 3-manifold-with-boundary and is a deformation retract of N and hence of M ; and
- (ii) $N_{\{\varepsilon_0\}}^c$ is a compact 2-manifold-with-boundary, properly embedded in N , and is a deformation retract of $N_{(0, \varepsilon_0]}$ and hence of $M_{(0, \varepsilon_0]}$.

Proof. Since Γ is geometrically finite, there is a Margulis number ε_1 for M such that $N_{[\varepsilon_1, \infty)}^c$ is compact.

Let ε be any Margulis number. In order to show that $N_{[\varepsilon, \infty)}^c$ is compact, we observe that for any sufficiently small $\varepsilon' > 0$ we have

$$\tilde{N}_{[\varepsilon, \infty)}^c \subset \text{nbhd}_1(\text{hull}(\Lambda_\Gamma) \cap \tilde{M}_{[\varepsilon', \infty)}^c).$$

Hence we need only show that $(\text{hull}(\Lambda_\Gamma) \cap \tilde{M}_{[\varepsilon', \infty)}^c)/\Gamma$ is compact for sufficiently small ε' . In particular, we may suppose that $\varepsilon' < \varepsilon_1$. Since $(\text{hull}(\Lambda_\Gamma) \cap \tilde{M}_{[\varepsilon_1, \infty)}^c)/\Gamma$ is compact, and since Γ has only finitely many conjugacy classes of cuspidal subgroups by 1.7, we need only show that for any cuspidal subgroup H of Γ the set $(\text{hull}(\Lambda_\Gamma) \cap (C_{\varepsilon_1}(H) - C_\varepsilon(H)))/H$ has compact closure in M . For this we identify \mathbf{H}^3 with the upper half-space model by an isometry mapping the fixed point of H to ∞ and set $\Lambda = \Lambda_\Gamma - \{\infty\}$. Let h and h_1 be real numbers so that the boundaries of the horoballs $C_\varepsilon(H)$ and $C_{\varepsilon_1}(H)$ are identified with the horizontal planes $\mathbf{R}^2 \times \{h\}$ and $\mathbf{R}^2 \times \{h_1\}$, respectively. If H has rank 2 then

$$(C_{\varepsilon_1}(H) - C_\varepsilon(H))/H \subset (\mathbf{R}^2 \times [h_1, h])/H,$$

the set on the right being compact. If H has rank 1 then

$$(\text{hull}(\Lambda_\Gamma) \cap (C_{\varepsilon_1}(H) - C_\varepsilon(H)))/H \subset (S \times [h_1, h])/H,$$

where S is the strip provided by the lemma. Again, the set on the right is compact.

Next we prove that for a sufficiently small Margulis number ε_0 the set $N_{\{\varepsilon_0\}}^c$ is a compact 2-manifold-with-boundary properly embedded in N (which by 1.4 is a 3-manifold-with-boundary), and that $N_{\{\varepsilon_0\}}^c$ is a deformation retract of $N_{(0, \varepsilon_0]}$. This will establish assertion (ii) of the proposition, since $N_{(0, \varepsilon_0]}$ is a deformation retract of $M_{(0, \varepsilon_0]}$ according to 1.6. Since Γ has only finitely many conjugacy classes of cuspidal subgroups, it suffices to show that for any cuspidal subgroup H of Γ there is a Margulis number ε_H such that for any $\varepsilon \leq \varepsilon_H$ the set $N \cap (\partial C_\varepsilon(H)/H)$ is a properly embedded 2-manifold-with-boundary in N and is a deformation retract of $C_\varepsilon(H)/H$. Equivalently, we must show that for small enough ε the set $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma) \cap \partial C_\varepsilon(H))$ is a contractible, properly embedded 2-manifold-with-boundary in $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma))$.

We prove this by applying the lemma as before. The statement follows immediately in the case where H is free abelian of rank 2. In the case that H is isomorphic to \mathbf{Z} let C denote the horoball $\mathbf{R}^2 \times [e^c, \infty)$, and D denote the region $S \times (0, \infty)$ where S is the strip provided by the conclusion of the lemma. We have $\text{hull}(\Lambda_\Gamma) \cap C = D \cap C$. Hence if C' denotes the horoball $\mathbf{R}^2 \times [e^{c+1}, \infty) \subset C$ so that the minimum distance between ∂C and $\partial C'$ is 1, we have $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma) \cap C') = \text{nbhd}_1(D) \cap C'$. But $\text{nbhd}_1(D)$ is bounded by two equidistant surfaces [F, p. 39] with the two boundary components of D as axial planes; these equidistant surfaces are intersections of the upper half-space with nonhorizontal Euclidean planes E_1 and E_2 in \mathbf{R}^3 .

We have $C' = C_\varepsilon(H)$ for some Margulis number ε_H ; and for any $\varepsilon \leq \varepsilon_H$, the set $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma) \cap \partial C_\varepsilon(H))$ is the closed strip in the horizontal

Euclidean plane $\partial C_\varepsilon(H)$ bounded by the lines $\partial C_\varepsilon(H) \cap E_1$ and $\partial C_\varepsilon(H) \cap E_2$. Hence $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma)) \cap \partial C_\varepsilon(H)$ is a contractible 2-manifold-with-boundary. By inspection we have that $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma)) \cap \partial C_\varepsilon(H)$ is properly embedded in $\text{nbhd}_1(\text{hull}(\Lambda_\Gamma))$. This proves (ii) for small ε_0 .

We shall complete the proof by showing that (ii) implies (i). If (ii) holds for a given ε_0 then, in particular, $N_{\{\varepsilon_0\}}^c$ is a deformation retract of $N_{(0, \varepsilon_0]}^c$; thus $N_{[\varepsilon_0, \infty)}^c$ is a deformation retract of N , and hence (by 1.4) of M . In particular $N_{[\varepsilon_0, \infty)}^c$ is aspherical. Since M is covered by \mathbf{H}^3 , every locally flat 2-sphere S in $M(\Gamma)$ bounds a unique ball $B \subset M$; if $S \subset N_{[\varepsilon_0, \infty)}^c$ then $B \subset N_{[\varepsilon_0, \infty)}^c$ by asphericity. Hence $N_{[\varepsilon_0, \infty)}^c$ is irreducible. Since M is orientable, so is $N_{[\varepsilon_0, \infty)}^c$. This establishes (i). \square

1.8. The following proposition about 2-generator groups of isometries of \mathbf{H}^3 follows easily from results that can be found in [Jør] or [F, V.1].

If l is any (geodesic) line in \mathbf{H}^3 , we denote by τ_l the 180° rotation about l . Thus τ_l is the unique element of order 2 in the group $\text{Isom}(\mathbf{H}^3)$ whose fixed point set is l .

Proposition. *Let ξ and η be two orientation-preserving isometries of \mathbf{H}^3 . Suppose that ξ_∞ and η_∞ have no common fixed point in S_∞ . Then there is a unique line $l \subset \mathbf{H}^3$ such that $\tau = \tau_l$ satisfies $\tau\xi\tau = \xi^{-1}$ and $\tau\eta\tau = \eta^{-1}$.*

If ξ and η satisfy the hypotheses of the above proposition, the line l given by the proposition will be denoted $l(\xi, \eta)$. This line is the common perpendicular to the axes of ξ and η , suitably interpreted in the degenerate cases where one or both of these isometries are parabolic [Fe, III.3 and V.1].

1.9. The line $l(\xi, \eta)$ is easily seen to depend continuously on ξ and η in the following sense. Suppose that ξ_∞ and η_∞ satisfy the hypotheses of Proposition 1.11, and let (ξ_i) and (η_i) be sequences of isometries of \mathbf{H}^3 converging to ξ_∞ and η_∞ , respectively, so that ξ_i and η_i also satisfy the hypotheses of the proposition for large i . Let ζ_∞^+ and ζ_∞^- denote the end points of $l(\xi_\infty, \eta_\infty)$. Then we may label the end points of $l(\xi_i, \eta_i)$ (for large i) as ζ_i^+ and ζ_i^- in such a way that $\zeta_i^+ \rightarrow \zeta_\infty^+$ and $\zeta_i^- \rightarrow \zeta_\infty^-$ as $i \rightarrow \infty$.

2. THE POISSON KERNEL

We shall need some elementary facts about the Poisson kernel for hyperbolic space. In this section and the next we present the relevant material from a geometric point of view that is natural for the applications in this paper.

2.1. We define a continuous function $\Pi: \mathbf{H}^n \times \mathbf{H}^n \times \mathbf{H}^n \rightarrow \mathbf{R}$ by

$$\Pi(z, z', w) = \exp(\text{dist}(z, w) - \text{dist}(z', w)).$$

From the hyperbolic law of cosines [F, p. 91] we have

$$\frac{\cosh \operatorname{dist}(z, w)}{\cosh \operatorname{dist}(z', w)} = (\cosh \operatorname{dist}(z, z') - \tanh \operatorname{dist}(z, w) \sinh \operatorname{dist}(z, z') \cos \angle z' z w)^{-1}.$$

It follows that the function Π has a unique positive continuous extension $\overline{\Pi}: \mathbf{H}^n \times \mathbf{H}^n \times \overline{\mathbf{H}^n} \rightarrow \mathbf{R}$. The restriction of $\overline{\Pi}$ to a function $\mathbf{H}^n \times \mathbf{H}^n \times S_\infty$ will be denoted \mathcal{P} . This restriction is given explicitly by

$$(2.1.1) \quad \mathcal{P}(z, z', \zeta) = (\cosh \operatorname{dist}(z, z') - \sinh \operatorname{dist}(z, z') \cos \angle z' z \zeta)^{-1}$$

for any $z \neq z' \in \mathbf{H}^n$ and any $\zeta \in S_\infty^{n-1}$; and by $\mathcal{P}(z, z, \zeta) = 1$ for any $z \in \mathbf{H}^n$ and any $\zeta \in S_\infty^{n-1}$.

2.2. It is immediate from the definition of the function Π that for any points $z, z', z'', w \in \mathbf{H}^n$ we have $\Pi(z, z'', w) = \Pi(z, z', w)\Pi(z', z'', w)$. Hence in view of the continuity of $\overline{\Pi}$ we have

$$\mathcal{P}(z, z'', \zeta) = \mathcal{P}(z, z', \zeta)\mathcal{P}(z', z'', \zeta)$$

for any $z, z', z'' \in \mathbf{H}^n$ and any $\zeta \in S_\infty$.

2.3. There is a simple formula for the function \mathcal{P} in terms of the upper half-space model. Given any point $\zeta \in S_\infty^{n-1}$, let us fix an isometry J of \mathbf{H}^n onto the upper half-space model $\mathbf{R}^{n-1} \times (0, \infty)$ that maps ζ to ∞ (see 1.1). The n th coordinate function on $\mathbf{R}^{n-1} \times (0, \infty)$ pulls back via J to a function $\operatorname{Im}: \mathbf{H}^n \rightarrow (0, \infty)$. The effect of replacing J by another isometry mapping ζ to ∞ would be to multiply the function Im by a constant; this is because a self-isometry of the upper half-space model that fixes ∞ is of the form $x \mapsto \alpha\sigma(x)$, where α is a positive constant and σ is a Euclidean isometry σ of \mathbf{R}^n that fixes the n th standard basis vector. In particular, for any two points $z, z' \in \mathbf{H}^n$, the value of the expression $\operatorname{Im} z / \operatorname{Im} z'$ depends only on the point ζ and not on the choice of J .

Proposition. *With the above conventions we have*

$$\mathcal{P}(z, z', \zeta) = \operatorname{Im} z' / \operatorname{Im} z$$

for any points $z, z' \in \mathbf{H}^n$.

Proof. We identify \mathbf{H}^n with $\mathbf{R}^{n-1} \times (0, \infty)$ via the diffeomorphism J , and write $z = (x, t)$ and $z' = (x', t')$, where $x, x' \in \mathbf{R}^{n-1}$ and $t, t' > 0$. We define two sequences $(w_j)_{j \geq 1}$ and $(w'_j)_{j \geq 1}$ of points of \mathbf{H}^n by $w_j = (x, j)$ and $w'_j = (x', j)$. For each j , the vertical line segments zw_j and $z'w'_j$ are hyperbolic geodesic arcs; hence $\operatorname{dist}(z, w_j) = \log |j - z_n|$ and $\operatorname{dist}(z', w'_j) = \log |j - z'_n|$. On the other hand, the horizontal line segment from w_j to w'_j is an arc of hyperbolic length $1/j$, and hence $\lim_{j \rightarrow \infty} \operatorname{dist}(w_j, w'_j) = 0$. From

the definition of \mathcal{P} we then have

$$\begin{aligned} \mathcal{P}(z, z', \zeta) &= \lim_{j \rightarrow \infty} \exp(\text{dist}(z, w_j) - \text{dist}(z', w_j)) \\ &= \lim_{j \rightarrow \infty} \exp(\text{dist}(z, w_j) - \text{dist}(z', w'_j)) \\ &= t'/t = \text{Im } z' / \text{Im } z. \quad \square \end{aligned}$$

The function \mathcal{P} is our version of the Poisson kernel. If \mathbf{H}^n is identified with the Poincaré model via an isometry that maps a given point z_0 to the origin, then the hyperbolic Poisson kernel $P: B^n \times S_\infty \rightarrow \mathbf{R}$, discussed, for example, in [N] or [P], is given by $P(z, \zeta) = \mathcal{P}(z_0, z, \zeta)$. This follows from the above proposition together with the calculation [N, Theorem 5.1.3] that if V is the isometry from the Poincaré model to the upper half-space model that sends the origin to the point $(0, 1) \in \mathbf{R}^n \times (0, \infty)$ and maps ζ to ∞ then $P(V^{-1}(z), \zeta) = \text{Im } z$.

2.4. Conformal expansion factors. Consider a point $z \in \mathbf{H}^n$ and an isometry $\gamma: \mathbf{H}^n \rightarrow \mathbf{H}^n$. By 1.1, γ induces a conformal diffeomorphism γ_∞ of S_∞ . If we equip S_∞ with the round metric centered at z , then at each point $\zeta \in S_\infty$, the map γ_∞ has a well-defined conformal expansion factor $\lambda > 0$. (This means that the tangent map $d\gamma_\infty: T_\zeta(S_\infty) \rightarrow T_{\gamma_\infty(\zeta)}(S_\infty)$ satisfies $|d\gamma_\infty(v)| = \lambda|v|$ for every tangent vector v at ζ , where $|\cdot|$ denotes length in the round metric.) We shall denote the conformal expansion factor of γ_∞ at ζ by $\lambda_{\gamma, z}(\zeta)$. Thus $\lambda_{\gamma, z}$ is a smooth positive-valued function on S_∞ .

Note that if ζ is a fixed point of γ then $d\gamma_\infty(\zeta)$ is a linear automorphism of $T_\zeta(S_\infty)$ and we have $\lambda_{\gamma, z}(\zeta) = |\det d\gamma_\infty(\zeta)|^{1/(n-1)}$. Thus in this case $\lambda_{\gamma, z}(\zeta)$ may be calculated without reference to the round metric or the point z .

Proposition. *Let z be any point of \mathbf{H}^n and let γ be any isometry of \mathbf{H}^n . Then we have $\lambda_{\gamma, z}(\zeta) = \mathcal{P}(z, \gamma^{-1}z, \zeta)$ for every point $\zeta \in S_\infty$.*

Proof. The corresponding statement for the Poincaré model is given in [N, Lemma 3.4.2]. This together with 2.2 gives the proposition. \square

2.5. The Laplacian. For any Riemannian n -manifold M we have a Laplacian operator

$$\Delta = *d * d: C^\infty(M) \rightarrow C^\infty(M),$$

where $*$ is the Hodge star and d denotes exterior differentiation.

Using Proposition 2.3, one shows the following by a straightforward calculation in the upper half-space model. (See [N, Theorem 5.1.3].)

Proposition. *For any $z \in \mathbf{H}^n$, $\zeta \in S_\infty$, and any real number r , the function $\phi_\zeta: \mathbf{H}^n \rightarrow \mathbf{R}$ defined by $\phi_\zeta(z) = \mathcal{P}(z_0, z, \zeta)^r$ satisfies the equation $\Delta\phi_\zeta = -r(n - r - 1)\phi_\zeta$.*

Proposition 2.5 is the starting point for the use of the Poisson kernel in constructing solutions to the equation $\Delta\phi_\zeta = -r(n - r - 1)\phi_\zeta$ (“generalized eigenfunctions” for the Laplacian) with prescribed “boundary values.” From the point of view of the present paper this construction is most naturally described

in terms of the notion, first formalized by Sullivan, of a “conformal density.” We shall discuss this in the next section.

3. CONFORMAL DENSITIES

In this section we discuss a Γ -equivariant potential theory in hyperbolic space, where Γ is a Kleinian group. Our treatment, which roughly parallels the one given in Nicholls’s recent book [N], is formulated in terms of Sullivan’s notion of a conformal density. This makes the equivariant theory work out quite neatly. Our approach to the uniqueness of a Γ -invariant conformal density when all Γ -invariant superharmonic functions are constant is somewhat different from the approach used in [N, Su1]. We show (Proposition 3.7) that an arbitrary $(n - 1)$ -conformal density on \mathbf{H}^n , not necessarily Γ -invariant, is determined by the associated harmonic function. This works only when the degree of the conformal density is $n - 1$, but is simpler than the arguments in [N, Su1] that depend on the ergodicity of the action of Γ on its limit set with respect to the measure class determined by the conformal density.

3.1. By a *Borel measure* on a locally compact metrizable space X we shall mean a measure on the Borel subsets of X that is finite on the compact subsets of X . Measures will always be understood to be nonnegative: we shall have no occasion to consider signed measures. If μ is a finite measure on a space X , the number $\mu(X)$ will be called the *total mass* of μ . If μ is a measure and f is a measurable function on a measurable space (X, \mathcal{B}) , and if ν is the measure defined by $\nu(E) = \int_E f d\mu$ for every $E \in \mathcal{B}$, then we write $d\nu = f d\mu$.

If x is a point of locally compact metrizable space X , we denote by δ_x the Dirac measure supported at x : by definition, for any Borel set $E \subset X$ we have $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ otherwise.

If X and Y are locally compact metrizable spaces and $f: X \rightarrow Y$ is a homeomorphism, then any Borel measure μ on Y gives rise to a measure on X , called the *pull-back* of μ and denoted $f^*\mu$. It is defined by $(f^*\mu)(E) = \mu(f(E))$ for any Borel set $E \subset X$.

3.2. **Conformal densities.** Let n be an integer ≥ 2 , and let D be a number in $[0, n - 1]$. By a *D-conformal density* for the sphere at infinity $S_\infty^{n-1} \subset \overline{\mathbf{H}^n}$ we mean a family $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^n}$ of finite Borel measures on $S_\infty = S_\infty^{n-1}$, indexed by the points of \mathbf{H}^n , such that for any two points $z, z' \in \mathbf{H}^n$ we have $d\mu_{z'} = \mathcal{P}(z, z', \cdot)^D d\mu_z$.

A family $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^n}$ of finite Borel measures on S_∞ will be called a *conformal density* if it is a *D-conformal density* for some $D \in [0, n - 1]$.

The *trivial* conformal density is $(0)_{z \in \mathbf{H}^n}$; it will be denoted by 0. Note that 0 is a *D-conformal density* for every $D \in [0, n - 1]$. On the other hand, it is clear that a nontrivial conformal density \mathcal{M} can be a *D-conformal density* for only one value of D ; we call D the *degree* of \mathcal{M} .

If z_0 is a point of \mathbf{H}^n and μ is a finite Borel measure on S_∞ , then for any $D \geq 0$ there is a unique *D-conformal density* $\mathcal{M} = (\mu_z)$ such that $\mu_{z_0} = \mu$. Indeed, uniqueness is clear since for any $z \in \mathbf{H}^n$ we must have $d\mu_z = \mathcal{P}(z_0, z, \cdot)^D d\mu$. To prove existence, we define μ_z , for each $z \in \mathbf{H}^n$, to be

the unique Borel measure such that $d\mu_z = \mathcal{P}(z_0, z, \cdot)^D d\mu$; using 2.3 we find that for any points $z, z' \in \mathbf{H}^n$ we have

$$\mathcal{P}(z_0, z', \cdot)^D = \mathcal{P}(z_0, z, \cdot)^D \mathcal{P}(z, z', \cdot)^D,$$

and hence

$$d\mu_{z'} = \mathcal{P}(z_0, z', \cdot)^D d\mu = \mathcal{P}(z_0, z, \cdot)^D \mathcal{P}(z, z', \cdot)^D d\mu = \mathcal{P}(z, z', \cdot)^D d\mu_z,$$

so that (μ_z) is the required conformal density.

If $\mathcal{M} = (\mu_z)$ is a conformal density then all the measures $\mu_z, z \in \mathbf{H}^n$, have the same support; this follows from the definition, since the function $\mathcal{P}(z, z', \cdot)$ is strictly positive on S_∞ for all $z, z' \in \mathbf{H}^n$. The common support of the μ_z will be called the *support* of \mathcal{M} and will be denoted $\text{supp } \mathcal{M}$.

If $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^n}$ and $\mathcal{M}' = (\mu'_z)_{z \in \mathbf{H}^n}$ are two D -conformal densities for a given D , it is clear that $(\mu_z + \mu'_z)_{z \in \mathbf{H}^n}$ is also a conformal density; it will be denoted $\mathcal{M} + \mathcal{M}'$. More generally we can define the sum $\sum_{i=1}^n \mathcal{M}_i$ of any finite family $(\mathcal{M}_i)_{1 \leq i \leq n}$ of D -conformal densities.

Now let $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^n}$ be a D -conformal density and f a continuous positive-valued function on S_∞ . For each $z \in \mathbf{H}^n$ let μ'_z be the Borel measure defined by $d\mu'_z = f d\mu_z$. It is clear that $\mathcal{N} = (\mu'_z)_{z \in \mathbf{H}^n}$ is a D -conformal density; we shall indicate that \mathcal{N} is defined in this way from f and \mathcal{M} by writing $d\mathcal{N} = f d\mathcal{M}$.

3.3. The area density. For any point $z \in \mathbf{H}^n$, the round metric centered at z determines an area measure on S_∞ . We let A_z denote such an area measure normalized so as to have total mass 1. It follows from Proposition 2.4 and the change-of-variable formula for the Lebesgue integral that $(A_z)_{z \in \mathbf{H}^n}$ is an $(n - 1)$ -conformal density. This will be called the *area density* and will be denoted \mathcal{A} .

3.4. Invariant conformal densities. Suppose that $\mathcal{M} = (\mu_z)$ is a D -conformal density and that $\gamma: \mathbf{H}^n \rightarrow \mathbf{H}^n$ is an isometry. Then the family of Borel measures $(\gamma_\infty^* \mu_{\gamma z})_{z \in \mathbf{H}^n}$ is also a D -conformal density. Indeed, for any points $z, z' \in \mathbf{H}^n$ and any continuous function f on S_∞ , we have

$$\begin{aligned} \int f d(\gamma_\infty^* \mu_{\gamma z'}) &= \int (f \circ \gamma_\infty^{-1}) d\mu_{\gamma z'} \\ &= \int f \circ \gamma_\infty^{-1}(w) \mathcal{P}(\gamma z, \gamma z', w)^D d\mu_{\gamma z}(w) \\ &= \int f \circ \gamma_\infty^{-1}(w) \mathcal{P}(z, z', \gamma^{-1} w)^D d\mu_{\gamma z}(w) \\ &= \int f \mathcal{P}(z, z', \cdot)^D d(\gamma_\infty^* \mu_{\gamma z}). \end{aligned}$$

We call $(\gamma_\infty^* \mu_{\gamma z})$ the *pull-back* of the conformal density \mathcal{M} , and we shall denote it by $\gamma^* \mathcal{M}$.

Now let Γ be any group of isometries of \mathbf{H}^n . A conformal density \mathcal{M} will be termed Γ -invariant if we have $\gamma^* \mathcal{M} = \mathcal{M}$ for every $\gamma \in \Gamma$. The area density is clearly invariant under the full group of isometries of \mathbf{H}^n .

If a D -conformal density $\mathcal{M} = (\mu_z)$ is Γ -invariant then for any $z \in \mathbf{H}^n$ and any $\gamma \in \Gamma$ we have $d(\gamma^* \mu_z) = d\mu_{\gamma^{-1}z} = \mathcal{P}(z, \gamma^{-1}z, \cdot)^D d\mu_z$; hence in view of 2.4 we have

$$(3.4.1) \quad d(\gamma^* \mu_z) = \lambda_{\gamma, z}^D d\mu_z.$$

3.5. Eigenfunctions of the Laplacian. Let $n \geq 2$ be an integer, and let D be a number in the interval $[0, n-1]$. Let $\mathcal{M} = (\mu_z)_{z \in \mathbf{H}^n}$ be a D -conformal density. Let us consider the nonnegative-valued function $u = u_{\mathcal{M}}$ on \mathbf{H}^n defined by $u(z) = \mu_z(S_\infty)$. If we fix any point $z_0 \in \mathbf{H}^n$ and set $\mu = \mu_{z_0}$, then according to the definition of a conformal density, u may be written as

$$(3.5.1) \quad u(z) = \int_{S_\infty} \mathcal{P}(z_0, z, \zeta)^D d\mu(\zeta).$$

For the area density \mathcal{A} we clearly have $u_{\mathcal{A}} \equiv 1$.

According to Proposition 2.5, for any $\zeta \in S_\infty$ the function $\phi_\zeta = \mathcal{P}(z_0, \cdot, \zeta)^D$ is C^∞ and satisfies $\Delta\phi_\zeta = -D(n-D-1)\phi_\zeta$. By differentiating under the integral sign in (3.5.1) it follows that for any D -conformal density \mathcal{M} the function $u = u_{\mathcal{M}}$ is C^∞ and satisfies the equation

$$(3.5.2) \quad \Delta u = -D(n-D-1)u.$$

In particular, if $D = n - 1$ then (3.5.2) says that u is harmonic.

If \mathcal{M} is invariant (3.4) under a group Γ of isometries of \mathbf{H}^n , then for any $\gamma \in \Gamma$ and any $z \in \mathbf{H}^n$ we have

$$u(\gamma z) = \mu_{\gamma z}(S_\infty) = (\gamma^* \mu_z)(S_\infty) = \mu_z(S_\infty) = u(z).$$

Thus u is Γ -invariant (i.e., constant on the orbits of Γ).

3.6. Proposition. *Let f be a continuous function on S_∞ , and let \mathcal{N} be the conformal density defined by $d\mathcal{N} = f d\mathcal{A}$ where \mathcal{A} is area density. Then the function $\bar{u}: \mathbf{H}^n \rightarrow \mathbf{R}$, defined by $\bar{u}|_{\mathbf{H}^n} = u_{\mathcal{N}}$ and $\bar{u}|_{S_\infty} = f$, is continuous.*

Proof. We identify \mathbf{H}^n via a conformal diffeomorphism with the Poincaré model. Thus S_∞ is identified with the unit sphere S^{n-1} . We write $A = A_0$. Since f is continuous and $u = u_{\mathcal{N}}$ is C^∞ , we need only show that if (z_i) is a sequence of points \mathbf{H}^n converging to a point $\zeta_0 \in S_\infty$, then $u(z_i)$ converges to $f(\zeta_0)$. We set $C = f(\zeta_0)$. For each i we have

$$u(z_i) = \int_{S_\infty} \mathcal{P}(0, z_i, \zeta)^{n-1} f(\zeta) dA(\zeta).$$

On the other hand, the integral $\int_{S_\infty} \mathcal{P}(0, z_i, \zeta)^{n-1} dA(\zeta)$ represents the total mass of A_{z_i} , which is 1. Hence

$$u(z_i) - C = \int_{S_\infty} \mathcal{P}(0, z_i, \zeta)^{n-1} (f(\zeta) - C) dA(\zeta).$$

To show that $u(z_i) \rightarrow C$, it suffices to show that $\limsup |u(z_i) - C| < \varepsilon$ for any prescribed $\varepsilon > 0$. Given ε , let U be a neighborhood of ζ_0 in S_∞ such that

$|f(\zeta) - C| < \varepsilon$ for every $\zeta \in U$. We may take U to be the δ -neighborhood for some $\delta > 0$ in the standard metric on S^{n-1} ; thus a point ζ lies in U if and only if $\angle(\zeta_0, 0, \zeta) < \delta$. We may write $u(z_i) - C$ as

$$\int_U \mathcal{P}(0, z_i, \zeta)^{n-1} (f(\zeta) - C) dA(\zeta) + \int_{S_\infty - U} \mathcal{P}(0, z_i, \zeta)^{n-1} (f(\zeta) - C) dA(\zeta).$$

The first term in the above expression is bounded above in absolute value by ε . We shall complete the proof by showing that the second term tends to 0 as $i \rightarrow \infty$.

Since f is continuous on S_∞ , it is enough to show that $\sup_{\zeta \in S_\infty - U} \mathcal{P}(0, z_i, \zeta)$ tends to 0 as $i \rightarrow \infty$. This follows easily from the formula (2.1.1) for \mathcal{P} . \square

3.7. Proposition. *If \mathcal{M} and \mathcal{M}' are $(n - 1)$ -conformal densities for \mathbf{H}^n such that $u_{\mathcal{M}} = u_{\mathcal{M}'}$, then $\mathcal{M} = \mathcal{M}'$.*

Proof. For the purpose of this argument we identify \mathbf{H}^n with the Poincaré model B^n so that $\bar{\mathbf{H}}^n$ is identified with \bar{B}^n . Then any point in \mathbf{H}^n may be written in the form $t\theta$, where $\theta \in S^{n-1}$ and $0 \leq t < 1$.

We will use the following fundamental symmetry of the function \mathcal{P} that follows directly from the formula (2.1.1):

$$(3.7.1) \quad \mathcal{P}(0, t\zeta, 0) = \mathcal{P}(0, t\theta, \zeta)$$

for any $t \in [0, 1)$ and any $\zeta, \theta \in S^{n-1}$.

Now let $\mathcal{M} = (\mu_z)$ be any $(n - 1)$ -conformal density and let f be any continuous function on S_∞ . Let \mathcal{N} be the conformal density defined by $d\mathcal{N} = f d\mathcal{A}$, where \mathcal{A} is the area density. By Proposition 3.6 the function $\bar{u}: \bar{\mathbf{H}}^n \rightarrow \mathbf{R}$, defined by $\bar{u}|_{\mathbf{H}^n} = u_{\mathcal{N}}$ and $\bar{u}|_{S_\infty} = f$, is continuous. Hence

$$(3.7.2) \quad \int f d\mu_0 = \lim_{t \rightarrow 1^-} \int u_{\mathcal{N}}(t\zeta) d\mu_0(\zeta).$$

For any $t < 1$ we have, by (3.7.1) and the Fubini Theorem,

$$\begin{aligned} \int u_{\mathcal{N}}(t\zeta) d\mu_0(\zeta) &= \iint f(\theta) \mathcal{P}(0, t\zeta, \theta)^{n-1} dA(\theta) d\mu_0(\zeta) \\ &= \iint \mathcal{P}(0, t\theta, \zeta)^{n-1} d\mu_0(\zeta) f(\theta) dA(\theta) \\ &= \int u_{\mathcal{M}}(t\theta) f(\theta) dA(\theta). \end{aligned}$$

If $\mathcal{M}' = (\mu'_z)$ is a second conformal density such that $u_{\mathcal{M}} = u_{\mathcal{M}'}$, then for any $t < 1$ we have

$$\begin{aligned} \int u_{\mathcal{N}}(t\zeta) d\mu'_0(\zeta) &= \int u_{\mathcal{M}'}(t\theta) f(\theta) dA(\theta) \\ &= \int u_{\mathcal{M}}(t\theta) f(\theta) dA(\theta) = \int u_{\mathcal{N}}(t\zeta) d\mu_0(\zeta). \end{aligned}$$

Taking limits as $t \rightarrow 1$, it follows by (3.7.2) that $\int f d\mu_0 = \int f d\mu'_0$ for every continuous function f on S_∞ . Hence $\mu_0 = \mu'_0$. By 3.2 it follows that $\mathcal{M} = \mathcal{M}'$. \square

3.8. We shall need the following elementary result concerning the degree of a Γ -invariant conformal density, when Γ is a discrete group (cf. [N, Corollary 3.4.5]).

Proposition. *Let Γ be a nonelementary discrete group of isometries of \mathbf{H}^n . Then every nontrivial Γ -invariant conformal density for \mathbf{H}^n has strictly positive degree.*

Proof. Suppose that there exists a nontrivial Γ -invariant conformal density of degree 0. By definition, this means that there is a nonzero Γ -invariant finite Borel measure μ on S_∞ .

Since Γ is nonelementary, some element γ of Γ is loxodromic; let P and Q be the fixed points of γ_∞ . Then any point $\zeta \in S_\infty - \{P, Q\}$ has a neighborhood U in S_∞ such that $\gamma^k(U) \cap U = \emptyset$ for all $k > 0$. Since μ is finite and countably additive, it follows that $\mu(U) = 0$. This shows that $\text{supp } \mu \subset \{P, Q\}$. But $\text{supp } \mu$ is nonempty and invariant under Γ ; thus S_∞ contains a Γ -invariant subset of cardinality 1 or 2. Since Γ is discrete, it follows that Γ has an abelian subgroup of finite index. This contradicts the hypothesis that Γ is nonelementary, and the proposition is proved. \square

3.9. A C^∞ function f on a Riemannian manifold is termed *superharmonic* if it satisfies the inequality $\Delta f \leq 0$.

Proposition. *Let Γ be a nonelementary discrete group of isometries of \mathbf{H}^n . Suppose that every Γ -invariant positive-valued superharmonic function on \mathbf{H}^n is constant. Then any Γ -invariant conformal density for \mathbf{H}^n is a constant multiple of the area density.*

Proof. Suppose that $\mathcal{M} = (\mu_z)$ is a Γ -invariant D -conformal density for some $D \in [0, n - 1]$. Since Γ is nonelementary, it follows from Proposition 3.9 that $D \neq 0$. We may assume that $\mathcal{M} \neq 0$, so that the function $u = u_{\mathcal{M}}$ is positive-valued. By 3.5 u is Γ -invariant and satisfies the equation $\Delta u = -D(n - D - 1)u$; in particular, u is superharmonic. Hence by the hypothesis u is a constant C .

Thus $0 = \Delta u \equiv D(n - D - 1)u$. Since $D \neq 0$ it follows that $D = n - 1$. Since $u_{\mathcal{M}} = C = u_{C\mathcal{A}}$, Proposition 3.7 then guarantees that $\mathcal{M} = C\mathcal{A}$. \square

Note that our approach to this uniqueness question is based on Proposition 3.7, which applies to an arbitrary $(n - 1)$ -conformal density. The support of the conformal density \mathcal{M} plays no role in the argument.

Another, somewhat less elementary, approach is found in the literature. Suppose that $\mathcal{M} = (\mu_z)$ is a Γ -invariant D -conformal density with $\text{supp } \mathcal{M} = \Lambda_\Gamma$. (A construction, due to Patterson, of conformal densities with these properties will be discussed in the next section.) Then any Γ -invariant D -conformal density supported on Λ_Γ is a constant multiple of \mathcal{M} provided that Γ acts ergodically on S_∞ with respect to the measure class determined by \mathcal{M} [N, Theorem 4.2.1]. Suppose that A is a Γ -invariant subset of S_∞ such that A and $S_\infty - A$ are sets of positive measure with respect to this measure class. Let \mathcal{M}' be given by $d\mathcal{M}' = \chi_A d\mathcal{M}$. Using points of density one can show that the ratio $u_{\mathcal{M}'} / u_{\mathcal{M}}$ tends to χ_A along almost all (μ_0) rays from the origin in the Poincaré model. Given this, the ergodicity follows if one knows that the ratio of any two positive Γ -invariant eigenfunctions of Δ , with eigenvalue $-D(n - D - 1)$, is bounded above and below by positive constants. In [Su3]

Sullivan uses this method to give examples of Kleinian groups Γ , for which there exist unique Γ -invariant conformal densities of degree 2 supported on Λ_Γ , but where these conformal densities are not constant multiples of the area density. (In fact, for these groups Λ_Γ has area 0, although it follows from the existence of such conformal densities that Λ_Γ has Hausdorff dimension 2.)

4. THE PATTERSON CONSTRUCTION

In this section we describe a construction first introduced by Patterson [P], and extensively studied by Sullivan [Su1–Su3], for associating to a Kleinian group Γ a Γ -invariant conformal density supported on the limit set of Γ . By taking a slightly more general point of view than Patterson’s, we are able to construct decompositions of a Patterson density corresponding to decompositions of the group. In §5, by specializing to the case of a free group, we shall obtain the paradoxical decomposition described in the introduction.

4.1. Uniformly discrete sets. A subset W of \mathbf{H}^n will be called *uniformly discrete* if there is a number $\varepsilon > 0$ such that $\text{dist}(z, w) > \varepsilon$ for any two distinct points $z, w \in W$. Such a number ε will be called a *modulus of discreteness* for W .

If Γ is any discrete group of isometries of \mathbf{H}^n , then for any $w \in \mathbf{H}^n$, the orbit $W = \Gamma w$ is a uniformly discrete set. (Indeed, since Γ is discrete and w has compact stabilizer in the isometry group of \mathbf{H}^n , the orbit Γw is discrete. Thus there is a number $\varepsilon > 0$ such that $\text{nbhd}_\varepsilon(w) \cap \Gamma w = \{w\}$. Since Γ consists of isometries it follows that ε is a modulus of discreteness.)

If $W \subset \mathbf{H}^n$ is any uniformly discrete set, the closure of W in $\overline{\mathbf{H}^n}$ has the form $W \cup \Lambda$, where Λ is a closed subset of S_∞ . We call Λ the *limit set* of W . Note that Λ is empty if and only if W is finite.

If Γ is a discrete group of isometries, the limit set of any orbit of Γ is the limit set of Γ .

4.2. The main result of this section is:

Proposition. *Let W be an infinite, uniformly discrete subset of \mathbf{H}^n , and let \mathfrak{W} be a countable collection of subsets of W . Suppose that $W \in \mathfrak{W}$. Then there exist a number $D \in [0, n - 1]$ and a family $(\mathcal{M}_V)_{V \in \mathfrak{W}}$ of D -conformal densities for \mathbf{H}^n , indexed by the collection \mathfrak{W} , satisfying the following conditions.*

- (i) $\mathcal{M}_W \neq 0$.
- (ii) For any finite family $(V_i)_{1 \leq i \leq m}$ of disjoint sets in \mathfrak{W} such that $V = \coprod_{i=1}^m V_i \in \mathfrak{W}$, we have $\mathcal{M}_V = \sum_{i=1}^m \mathcal{M}_{V_i}$.
- (iii) For any $V \in \mathfrak{W}$ and any isometry $\gamma: \mathbf{H}^n \rightarrow \mathbf{H}^n$ such that $\gamma V \in \mathfrak{W}$, we have $\gamma_\infty^*(\mathcal{M}_{\gamma V}) = \mathcal{M}_V$.
- (iv) For any $V \in \mathfrak{W}$, the support of \mathcal{M}_V is contained in the limit set of \mathcal{M}_V . In particular, for any finite set $V \in \mathfrak{W}$ we have $\mathcal{M}_V = 0$.

Before turning to the proof of the proposition, let us point out one of its consequences. If Γ is any nonelementary discrete group of isometries of \mathbf{H}^n , then any orbit W of Γ is uniformly discrete by 4.1 and infinite because Γ is nonelementary. Thus we may apply Proposition 4.2 to W , taking \mathfrak{W} to be any countable collection of subsets of W with $W \in \mathfrak{W}$ (e.g., $\mathfrak{W} = \{W\}$). This gives

a conformal density $\mathcal{M} = \mathcal{M}_W$. It follows from 4.2(iii) that \mathcal{M} is Γ -invariant. In particular, the support of \mathcal{M} is Γ -invariant. On the other hand, we have $\text{supp } \mathcal{M} \neq \emptyset$ by 4.2(i); and by 4.2(iv) we have $\text{supp } \mathcal{M} \subset \Lambda_\Gamma$. Since the action of Γ on Λ_Γ is topologically minimal, it follows that $\text{supp } \mathcal{M} = \Lambda_\Gamma$.

In particular, taking $\mathfrak{W} = \{W\}$, we obtain the following

Corollary (Patterson [P]). *If Γ is any nonelementary discrete subgroup of isometries of \mathbf{H}^n , there is a Γ -invariant conformal density for \mathbf{H}^n whose support is Λ_Γ . \square*

The rest of this section is devoted to the proof of the proposition. We are given an infinite, uniformly discrete set $W \subset \mathbf{H}^n$ and a countable collection \mathfrak{W} of subsets of W , with $W \in \mathfrak{W}$.

4.3. The Poincaré series. For any point $z \in \mathbf{H}^n$ and any $s \geq 0$ we denote by $\Sigma(z, s)$ the sum of the “Poincaré series” for the uniformly discrete set W . That is,

$$\Sigma(z, s) = \sum_{w \in W} e^{-s \text{dist}(w, z)}$$

where we interpret the sum as a nonnegative number or $+\infty$.

The following properties of $\Sigma(z, s)$ are immediate from the definition.

4.3.1. For any $z \in \mathbf{H}^n$ and any $s \geq s' \geq 0$ we have $\Sigma(z, s') \geq \Sigma(z, s)$.

4.3.2. For any $z, z' \in \mathbf{H}^n$ and any $s \geq 0$ we have

$$\Sigma(z', s) \leq e^{s \text{dist}(z, z')} \Sigma(z, s).$$

Lemma. *There is a unique nonnegative number $D \leq n - 1$ such that, for every point $z \in \mathbf{H}^n$, the sum $\Sigma(z, s)$ is finite for all $s > D$ and infinite whenever $0 < s < D$.*

Proof. First we note that $\Sigma(z, s)$ is finite for $s > n - 1$. This is proved, in the case that W is an orbit of a discrete group, in [N, Theorem 1.6.1]. The proof applies as well to an arbitrary uniformly discrete set. Briefly, the idea is that the Poincaré series converges as long as s is larger than the exponential growth rate of the number N_R of points of W in the ball of radius R about z . The uniform discreteness of W implies that N_R cannot grow faster than the volume $V(R)$ of a ball of radius R . However, we have $V(R) = A_n \int_0^R \sinh^{n-1} x dx$, where A_n is a number depending only on the dimension n . Hence for all R we have $V(R) \leq B_n e^{(n-1)R}$, where B_n is again a number depending only on n .

It now follows from (4.3.1) and the discussion in 4.3 that for each $z \in \mathbf{H}^n$ there is a unique number $D_z \leq n - 1$ such that $\Sigma(z, s) < \infty$ for all $s > D_z$ and $\Sigma(z, s) = \infty$ whenever $0 < s < D_z$. It follows from 4.3.2 that D_z is independent of z . \square

The number D given by the lemma is called the *critical exponent* of W .

4.4. The critically divergent case. The construction of the conformal densities in the statement of Proposition 4.2 is simplest in the case where $\Sigma(z, D)$ is infinite for every $z \in \mathbf{H}^n$. We sketch the construction in this case before discussing the general case.

First we define a family of measures $\mu_{V, z, s}$ on $\overline{\mathbf{H}^n}$ by

$$\mu_{V, z, s} = \frac{1}{\Sigma(z, s)} \sum_{w \in V} e^{-s \operatorname{dist}(w, z)} \delta_w,$$

for $z \in \mathbf{H}^n$ and $s > D$.

Now fix an arbitrary point $z_0 \in \mathbf{H}^n$ and let $(s_j)_{j \geq 0}$ be a decreasing sequence of real numbers converging to D such that the sequence of measures (μ_{V, s_j, z_0}) is a weakly convergent sequence. Let μ_{V, z_0} denote the limit. The condition $\Sigma(z, D) = \infty$ implies that the support of μ_{V, z_0} is contained in S_∞ .

It is immediate from the definition that for all $s > D$

$$d\mu_{V, z', s} = \Pi(z, z', \cdot)^s d\mu_{V, z, s}.$$

This implies first of all that the sequence (μ_{V, z, s_j}) is weakly convergent for each $z \in \mathbf{H}^n$. Furthermore, letting $\mu_{V, z}$ denote the limit of the sequence (μ_{V, z, s_j}) , and viewing these measures as measures on S_∞ , we obtain the formula

$$d\mu_{V, z'} = \mathcal{P}(z, z', \cdot)^D d\mu_{V, z}.$$

In particular, $\mathcal{M}_V = \mu_{V, z}$ is a D -conformal density.

4.5. Adjustment functions. The above construction must be modified in the case where $\Sigma(z, D) < \infty$; the idea, invented by Patterson, is to perturb the terms in the sum $\Sigma(z, s)$ to obtain an expression $\Sigma'(z, s)$ that is always finite when $s > D$ and infinite when $0 \leq s \leq D$. In our formulation, we show that for a suitably chosen real-valued function α the construction sketched above can be carried out with $\operatorname{dist}(z, w)$ replaced by $\alpha(\operatorname{dist}(z, w))$ throughout.

If α is any real-valued function on $[0, \infty)$, then for any point $z \in \mathbf{H}^n$ and any $s > 0$ we define

$$\Sigma(\alpha, z, s) = \sum_{w \in W} e^{-s\alpha(\operatorname{dist}(w, z))}$$

where we again interpret the sum as a nonnegative number or $+\infty$. Note that if id denotes the identity function on $[0, \infty)$ then $\Sigma(\operatorname{id}, z, s) = \Sigma(z, s)$ for any z and s .

We define an *adjustment function* to be a C^1 real-valued function α on $[0, \infty)$ such that (i) $\alpha(0) = 0$, (ii) $0 \leq \alpha'(t) \leq 1$ for all $t > 0$, and (iii) $\lim_{t \rightarrow \infty} \alpha'(t) = 1$. In particular, an adjustment function α must satisfy $0 \leq \alpha(t) \leq t$ for all $t \geq 0$. The identity function on $[0, \infty)$ is obviously an adjustment function.

The following generalizations of 4.3.1 and 4.3.2 hold for any adjustment function α .

4.5.1. For any $z \in \mathbf{H}^n$ and any $s \geq s' \geq 0$ we have $\Sigma(\alpha, z, s') \geq \Sigma(\alpha, z, s)$.

4.5.2. For any $z, z' \in \mathbf{H}^n$ and any $s \geq 0$ we have

$$\Sigma(\alpha, z', s) \leq e^{s \operatorname{dist}(z, z')} \Sigma(\alpha, z, s).$$

Indeed, 4.5.1 is immediate; and 4.5.2 follows from the observation that, by virtue of condition (ii) in the definition of an adjustment function, we have

$$|\alpha(\text{dist}(w, z)) - \alpha(\text{dist}(w, z'))| \leq |\text{dist}(w, z) - \text{dist}(w, z')| \leq \text{dist}(z, z')$$

for any $w \in W$.

We also have the following extension of 2.1.

Lemma. *For any adjustment function α , the function $\bar{\Pi}_\alpha : \mathbf{H}^n \times \mathbf{H}^n \times \bar{\mathbf{H}}^n \rightarrow \mathbf{R}$, defined by*

$$\bar{\Pi}_\alpha(z, z', w) = \exp(\alpha(\text{dist}(z, w)) - \alpha(\text{dist}(z', w)))$$

for $(z, z', w) \in \mathbf{H}^n \times \mathbf{H}^n \times \mathbf{H}^n$ and $\bar{\Pi}_\alpha(z, z', \zeta) = \mathcal{P}(z, z', \zeta)$ for $(z, z', \zeta) \in \mathbf{H}^n \times \mathbf{H}^n \times S_\infty$, is continuous.

Proof. It is enough to prove that if (z_i) , (z'_i) , and (w_i) are sequences of points in \mathbf{H}^n that converge, respectively, to points $z \in \mathbf{H}^n$, $z' \in \mathbf{H}^n$, and $\zeta \in S_\infty$, then

$$\lim_{i \rightarrow \infty} \exp(\alpha(\text{dist}(z_i, w_i)) - \alpha(\text{dist}(z'_i, w_i))) = \mathcal{P}(z, z', \zeta).$$

By definition, we have

$$\lim_{i \rightarrow \infty} \exp(\text{dist}(z_i, w_i) - \text{dist}(z'_i, w_i)) = \mathcal{P}(z, z', \zeta).$$

In particular, $\text{dist}(z_i, w_i) - \text{dist}(z'_i, w_i)$ converges. Since $\alpha'(t) \rightarrow 1$ as $t \rightarrow \infty$, we have

$$\lim_{i \rightarrow \infty} \frac{\alpha(\text{dist}(z_i, w_i)) - \alpha(\text{dist}(z'_i, w_i))}{\text{dist}(z_i, w_i) - \text{dist}(z'_i, w_i)} = 1.$$

The result follows. \square

4.6. Lemma. *There exists an adjustment function α such that, for every $z \in \mathbf{H}^n$, we have $\Sigma(\alpha, z, s) < \infty$ for all $s > D$ and $\Sigma(\alpha, z, s) = \infty$ whenever $0 \leq s \leq D$.*

Proof. First consider an arbitrary adjustment function α . Since $\alpha(t) \leq t$ for all $t \geq 0$, we have $\Sigma(\alpha, z, s) \geq \Sigma(z, s)$ for all s ; hence $\Sigma(\alpha, z, s) = \infty$ when $0 \leq s < D$. Now suppose that $s > D$ and choose a positive number $k < 1$ such that $sk < D$. Since $\lim_{t \rightarrow \infty} \alpha'(t) = 1$, there is a constant C such that $\alpha(t) \geq kt - C$ for all $t \geq 0$. Hence, for $s > D$, we have

$$\Sigma(\alpha, z, s) \leq e^{Cs} \Sigma(z, sk) < \infty.$$

Thus we need only to construct α so that $\Sigma(\alpha, z, D) = \infty$. We first dispose of the special case $D = 0$. Since W is infinite, it is clear that $\Sigma(z, 0) = +\infty$ for every z . Hence in the case $D = 0$ we may take α to be the identity function.

Now suppose that $D > 0$. If α is any adjustment function, it follows from 4.5.2 that $\Sigma(\alpha, z, D)$ is either finite for all $z \in \mathbf{H}^n$ or infinite for all $z \in \mathbf{H}^n$. Hence we need only show that for a given point $z \in \mathbf{H}^n$ there is an adjustment function α such that $\Sigma(\alpha, z, D) = +\infty$.

Let $(\theta_m)_{m \geq 1}$ be a strictly increasing sequence of positive numbers that converge to 1. For each $m \geq 1$ we set $s_m = \theta_m D$. Since $0 < s_m < D$ the sum

$$\Sigma(z, s_m) = \sum_{w \in W} e^{-s_m \operatorname{dist}(w, z)}$$

is infinite. Hence there is a finite subset X_m of W such that

$$\sum_{w \in X_m} e^{-s_m \operatorname{dist}(w, z)} \geq m.$$

Let $(R_m)_{m \geq 1}$ be a monotone increasing sequence of positive numbers tending to $+\infty$, such that for each $m \geq 1$ and each $w \in X_m$ we have $\operatorname{dist}(z, w) \leq R_m$. Let β be any continuous, monotone increasing real-valued function on $[0, \infty)$ such that $\beta(R_m) = \theta_m$ for all $m > 0$ and $\beta(t) < 1$ for all $t \geq 0$. Then $\alpha(t) = \int_0^t \beta(u) du$ is an adjustment function and $\alpha(t) \leq \theta_m t$ whenever $0 \leq t \leq R_m$. For each $m > 0$ and each $w \in X_m$ we have $\operatorname{dist}(w, z) < R_m$, and hence

$$\alpha(\operatorname{dist}(w, z))D \leq \theta_m D \operatorname{dist}(w, z) = s_m \operatorname{dist}(w, z).$$

Thus

$$\Sigma(\alpha, z, D) \geq \sum_{w \in X_m} e^{-D\alpha(\operatorname{dist}(w, z))} \geq \sum_{w \in X_m} e^{-s_m \operatorname{dist}(w, z)} \geq m.$$

Since this holds for all $m \geq 1$, we must have $\Sigma(\alpha, z, D) = +\infty$. \square

For the rest of this section, we fix an adjustment function α constructed as in the lemma. We shall write $\Sigma'(z, s) = \Sigma(\alpha, z, s)$ for all z and s .

4.7. For any $s > D$ and any $V \in \mathfrak{V}$ we have

$$\sum_{w \in V} e^{-s\alpha(\operatorname{dist}(w, z))} \leq \Sigma'(z, s) < \infty$$

by Lemma 4.6. Hence we may define a Borel measure

$$\sum_{w \in V} e^{-s\alpha(\operatorname{dist}(w, z))} \delta_w$$

on \overline{H}^n . This measure will be denoted $\mu_{V, z, s}$. Its total mass is $\leq \Sigma'(z, s)$, with equality when $V = W$.

4.8. For the rest of this section we fix, arbitrarily, a point $z_0 \in H^n$.

Lemma. *There exists a sequence $(s_j)_{j \geq 0}$ of real numbers with the following properties:*

- (i) $s_j > D$ for all j , and $\lim_{j \rightarrow \infty} s_j = D$;
- (ii) for every $V \in \mathfrak{V}$, the sequence $(\Sigma'(z_0, s_j)^{-1} \mu_{V, z_0, s_j})_{j \geq 0}$ is a weakly convergent sequence of Borel measures on \overline{H}^n .

Proof. Let (t_i) be any sequence of numbers $> D$ such that $\lim_{i \rightarrow \infty} t_i = D$. For any $V \in \mathfrak{V}$, we have a sequence of measures $(\Sigma'(z_0, t_i)^{-1} \mu_{V, z_0, t_i})$ that by

4.7 all have total mass ≤ 1 . Given any $V \in \mathfrak{V}$, there is a subsequence (t_{i_j}) such that the sequence $(\Sigma'(z_0, t_{i_j})^{-1} \mu_{V, z_0, t_{i_j}})$ is weakly convergent. Since \mathfrak{V} is countable, the conclusion of the lemma follows via diagonalization. \square

For the rest of the section we fix a sequence (s_j) having the above properties (i) and (ii). For each $j \geq 0$ we set $C_j = \Sigma'(z_0, s_j)$. For each $V \in \mathfrak{V}$ we let μ_V^0 denote the weak limit of the sequence $(\frac{1}{C_j} \mu_{V, z_0, s_j})$; by definition, μ_V^0 is a Borel measure on $\overline{\mathbf{H}}^n$.

4.9. **Lemma.** *We have $C_j \rightarrow +\infty$ as $j \rightarrow \infty$.*

Proof. By 4.6 we have $\Sigma'(z_0, D) = +\infty$. This implies the lemma since the terms of the sum $\Sigma'(z, s) = \sum_{i \in I} e^{-s\alpha(\text{dist}(w_i, z))}$ are positive-valued continuous functions of s . \square

4.10. **Lemma.** *For each $V \in \mathfrak{V}$, the measure μ_V^0 has support contained in S_∞^{n-1} .*

Proof. It is enough to show that $\mu_V^0(\mathcal{E}) = 0$ for any compact set $\mathcal{E} \subset \mathbf{H}^n$. The set $V \cap \mathcal{E}$ is finite since V is a subset of the uniformly discrete set W . We have

$$\mu_{V, z_0, s_j}(\mathcal{E}) = \sum_{w \in V} \exp(-s_j \alpha(\text{dist}(w, z))) \leq \text{card}(V \cap \mathcal{E})$$

for each $j \geq 0$. Thus $\frac{1}{C_j} \mu_V^0(\mathcal{E})$ is at most $\frac{1}{C_j} \text{card}(V \cap \mathcal{E})$, which tends to 0 as $j \rightarrow \infty$ by virtue of Lemma 4.9. Hence $\mu_V^0(\mathcal{E}) = 0$. \square

4.11. **Lemma.** *For any $V \in \mathfrak{V}$ and any point $z \in \mathbf{H}^n$, the sequence $\frac{1}{C_j} \mu_{V, z, s_j}$ converges weakly to $\mu_{V, z}$.*

Proof. It is immediate from the definition that

$$d\mu_{V, z, s_j} = \overline{\Pi}_\alpha(z_0, z, \cdot)^{s_j} d\mu_{V, z_0, s_j},$$

where $\overline{\Pi}_\alpha$ is the function given by Lemma 4.5. Setting $\mu_j^0 = \frac{1}{C_j} \mu_{V, z_0, s_j}$ and $\mu_j = \frac{1}{C_j} \mu_{V, z, s_j}$, we therefore obtain, for any continuous function f on $\overline{\mathbf{H}}^n$ and any $j \geq 0$,

$$\int f d\mu_j = \int \overline{\Pi}_\alpha(z_0, z, \cdot)^{s_j} f d\mu_j^0.$$

Now since $\lim_{j \rightarrow \infty} s_j = D$, and since f and $\overline{\Pi}$ are continuous on the compact space $\overline{\mathbf{H}}^n$ we have $\overline{\Pi}_\alpha(z_0, z, \cdot)^{s_j} f \rightarrow \overline{\Pi}_\alpha(z_0, z, \cdot)^D f$ uniformly. Since the μ_j^0 converge weakly to μ_V^0 , it follows that

$$\lim_{j \rightarrow \infty} \int \overline{\Pi}_\alpha(z_0, z, \cdot)^{s_j} f d\mu_j^0 = \int \overline{\Pi}_\alpha(z_0, z, \cdot)^D f d\mu_V^0 = \int \mathcal{P}^D f d\mu_V^0.$$

The right-hand side of the last equation is equal to $\int f d\mu_{V,z}$ since $\mathcal{M}_V = (\mu_{V,z})$ is a D -conformal density. So we have

$$\lim_{j \rightarrow \infty} \int f d\mu_j = \int f d\mu_{V,z}.$$

This proves the lemma. \square

Proof of Proposition 4.2. In light of Lemma 4.10, for each $V \in \mathfrak{V}$, the measure μ_V^0 may be regarded as a measure on S_∞ . According to 3.3, there exists for each $V \in \mathfrak{V}$ a unique D -conformal density $\mathcal{M}_V = (\mu_{V,z})$ such that $\mu_{V,z_0} = \mu_V^0$. We shall prove Proposition 4.2 by showing that the family (\mathcal{M}_V) of D -conformal densities satisfies conditions (i)–(iv) of the proposition.

To prove (i) it is enough to show that $\mu_{W,z_0} \neq 0$. But as we observed in 4.7, for each $s > D$ and each $z \in \mathbf{H}^n$, the measure $\mu_{W,z,s}$ has total mass $\Sigma'(z, s)$; in particular, for each $j \geq 0$ the total mass of μ_{W,z_0,s_j} is C_j . Thus μ_{W,z_0} is the weak limit of a sequence of measures of total mass 1, and therefore itself has total mass 1.

To prove (ii) we observe that if $V = \coprod_{i=1}^m V_i$, where V and the V_i belong to \mathfrak{V} , then for each $s > D$ and each $z \in \mathbf{H}^n$ we have $\mu_{V,z,s} = \sum_{i=1}^m \mu_{V_i,z,s}$ by the definitions of $\mu_{V,z,s}$ and $\mu_{V_i,z,s}$. Hence for each $j \geq 0$ we have

$$\frac{1}{C_j} \mu_{V,z,s} = \frac{1}{C_j} \sum_{i=1}^m \mu_{V_i,z,s}.$$

Using Lemma 4.11 we can take weak limits of both sides to obtain $\mu_{V,z} = \sum_{i=1}^m \mu_{V_i,z}$. This proves (ii).

To prove (iii) we must show that if V is a set in \mathfrak{V} and γ is a hyperbolic isometry such that $\gamma V \in \mathfrak{V}$, then for each $z \in \mathbf{H}^n$ we have $\gamma_\infty^* \mu_{\gamma V, \gamma z} = \mu_{V,z}$. If we think of $\mu_{\gamma V, \gamma z}$ and $\mu_{V,z}$ as measures on $\overline{\mathbf{H}}^n$, then $\gamma_\infty^* \mu_{\gamma V, \gamma z}$ may be rewritten as $\bar{\gamma}^* \mu_{\gamma V, \gamma z}$. For each $s > D$ we have

$$\bar{\gamma}^* \mu_{\gamma V, \gamma z, s} = \sum_{w \in V} e^{-s\alpha(\text{dist}(\gamma w, \gamma z))} \bar{\gamma}^*(\delta_{\gamma w}) = \sum_{w \in V} e^{-s\alpha(\text{dist}(w, z))} \delta_w = \mu_{V,z,s}.$$

Hence for each $j \geq 0$ we have

$$\bar{\gamma}^* \left(\frac{1}{C_j} \mu_{\gamma V, \gamma z, s_j} \right) = \frac{1}{C_j} \mu_{V,z,s_j}.$$

Using Lemma 4.11 we can take weak limits of both sides to obtain $\bar{\gamma}^* \mu_{\gamma V, \gamma z} = \mu_{V,z}$, as required.

To prove (iv) we consider an arbitrary set $V \in \mathfrak{V}$. For any $s > D$, it follows from the definition of the measure $\mu_{V,z_0,s}$ that the support of $\mu_{V,z_0,s}$ is exactly $V \cup \Lambda_V$, the closure of V in $\overline{\mathbf{H}}^n$. Hence $\text{supp}(\frac{1}{C_j} \mu_{V,z_0,s_j}) = V \cup \Lambda_V$ for each $j \geq 0$. By Lemma 4.11 it follows that $\text{supp} \mu_V^0 \subset V \cup \Lambda_V$. But $\text{supp} \mu_V^0 \subset S_\infty$ by 4.10, and hence $\text{supp} \mu_V^0 \subset \Lambda_V$. This implies (iv). \square

5. THE PARADOXICAL DECOMPOSITION

This section is devoted to the proof of the following two results.

5.1. **Proposition.** *Let Γ be a Kleinian group that is free on a generating set Ξ with k elements, where $2 \leq k < \infty$. Suppose that every Γ -invariant conformal density for \mathbf{H}^3 is a constant multiple of the area density. Then for any $z \in \mathbf{H}^3$ we have*

$$\max_{\xi \in \Xi} \text{dist}(z, \xi \cdot z) \geq \frac{1}{2} \log((k - 1)(2k - 1)).$$

5.2. **Proposition.** *Let ξ and η be two orientation-preserving isometries of \mathbf{H}^3 . Suppose that the group Γ generated by ξ and η is discrete and is free on the generators ξ and η . Suppose in addition that every Γ -invariant conformal density for \mathbf{H}^3 is a constant multiple of the area density. Then for any $z \in \mathbf{H}^3$ we have*

$$\max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) \geq \log 3.$$

5.3. The key step in the proofs of Propositions 5.1 and 5.2 is the following lemma. When the hypotheses of 5.1 or 5.2 are satisfied, the lemma will give the “paradoxical” decomposition of the normalized area measure on S_∞ , which was described in the introduction.

Lemma. *Let Γ be a Kleinian group that is free on a generating set Ξ . Set $\Psi = \Xi \amalg \Xi^{-1} \subset \Gamma$. Let z_0 be any point of \mathbf{H}^3 . Then there exist a number $D \in [0, 2]$, a Γ -invariant D -conformal density $\mathcal{M}(\mu_z)$ for \mathbf{H}^3 , and a family $(\nu_\psi)_{\psi \in \Psi}$ of Borel measures on S_∞ such that*

- (i) $\mu_{z_0}(S_\infty) = 1$;
- (ii) $\mu_{z_0} = \sum_{\psi \in \Psi} \nu_\psi$; and
- (iii) for each $\psi \in \Psi$ we have

$$\int (\lambda_{\psi, z_0})^D d\nu_{\psi^{-1}} = 1 - \int d\nu_\psi.$$

If Ξ is a two-element set $\{\xi, \eta\}$ and if $z_0 \in l(\xi, \eta)$ (see 1.8) then we also have

$$(iv) \int d\nu_{\xi^{-1}} = \int d\nu_\xi \text{ and } \int d\nu_{\eta^{-1}} = \int d\nu_\eta.$$

Proof. Every element γ of Γ can be written uniquely as a reduced word $\psi_1 \cdots \psi_m$, where $m \geq 0$, each ψ_i is an element of Ψ , and $\psi_{i+1} \neq \psi_i^{-1}$ for $i = 1, \dots, m - 1$. If $\gamma \neq 1$, i.e., if $m > 0$, we shall call ψ_1 the initial letter of γ . For each $\psi \in \Psi$, let J_ψ denote the set of nontrivial elements of Γ that have initial letter ψ . Then Γ is a finite disjoint union $\{1\} \amalg \coprod_{\psi \in \Psi} J_\psi$. Since Γ is discrete and torsion free, it acts freely on \mathbf{H}^3 . Hence the orbit $W = \Gamma z_0$ is a disjoint union

$$(5.3.1) \quad W = \{z_0\} \amalg \coprod_{\psi \in \Psi} V_\psi$$

where $V_\psi = \{\gamma z : \gamma \in J_\psi\}$. Let \mathfrak{A} denote the finite collection of all subsets of W that are subunions of the above disjoint union; that is, W consists of all sets

of the form $\coprod_{\psi \in \Psi'} V_\psi$ or $\{z_0\} \amalg \coprod_{\psi \in \Psi'} V_\psi$ for $\Psi' \subset \Psi$. We apply Proposition 4.2 with these choices of W and \mathfrak{A} . We take D to be a number, and $(\mathcal{M}_V)_{V \in \mathfrak{A}}$ a family of conformal densities, for which conditions (i)–(iv) of 4.2 are satisfied. We write $\mathcal{M}_V = (\mu_{V,z})_{z \in \mathbb{H}^n}$. We set $\mathcal{M} = \mathcal{M}_W$ and $\nu_\psi = \mu_{V_\psi, z_0}$ for each $\psi \in \Psi$. According to 4.2, \mathcal{M} is Γ -invariant. It follows from condition (i) of 4.2 and the definition of a conformal density that $\mu_{z_0} = \mu_{W, z_0} \neq 0$; hence after multiplying by a constant we may assume that μ_{z_0} has total mass 1. This is conclusion (i) of the present lemma. We shall show that the other conclusions of the lemma also hold with these definitions of \mathcal{M} and of the ν_ψ .

By (5.3.1) above and condition (ii) of 4.2, we have

$$\mu_{z_0} = \mu_{W, z_0} = \mu_{\{z_0\}, z_0} + \sum_{\psi \in \Psi} \mu_{V_\psi, z_0}.$$

But $\mu_{\{z_0\}, z_0} = 0$ by condition (iv) of 4.2. Hence

$$\mu_{z_0} = \sum_{\psi \in \Psi} \mu_{V_\psi, z_0} = \sum_{\psi \in \Psi} \nu_\psi,$$

which is conclusion (ii) of the lemma.

It follows immediately from the definition of the sets J_ψ that for each $\psi \in \Psi$ we have $\psi J_{\psi^{-1}} = \Gamma - J_\psi$ so that $\psi V_{\psi^{-1}} = W - V_\psi$. Since $W - V_\psi = \{z_0\} \amalg \coprod \psi' \neq \psi V_{\psi'}$, $\in \mathfrak{A}$, condition (iii) of 4.2 gives

$$(5.3.2) \quad \mathcal{M}_{V_{\psi^{-1}}} = \psi_\infty^*(\mathcal{M}_{W - V_\psi}).$$

On the other hand, by 4.2(ii) we have $\mathcal{M} = \mathcal{M}_{W - V_\psi} + \mathcal{M}_{V_\psi}$. Thus (5.3.2) can be rewritten in the form $\mathcal{M}_{V_{\psi^{-1}}} = \psi_\infty^*(\mathcal{M} - \mathcal{M}_{V_\psi})$; in particular, we have

$$(5.3.3) \quad \mu_{V_{\psi^{-1}}, \psi(z_0)} = \psi_\infty^*(\mu_{z_0} - \mu_{V_\psi, z_0}) = \psi_\infty^*(\mu_{z_0} - \nu_\psi).$$

Since $\mathcal{M}_{V_{\psi^{-1}}}$ is a D -conformal density, we have

$$d\mu_{V_{\psi^{-1}}, \psi(z_0)} = \mathcal{P}(z_0, \psi^{-1}z_0, \cdot)^D d\mu_{V_{\psi^{-1}}} = \lambda_{\psi, z_0}^D d\mu_{V_{\psi^{-1}}},$$

where the last equality follows from Proposition 2.4. Hence, equating the total masses of the two sides of (5.3.3), we obtain

$$\int \lambda_{\psi, z_0}^D d\mu_{V_{\psi^{-1}}} = \int d(\psi_\infty^*(\mu_{z_0} - \nu_\psi)) = \int d(\mu_{z_0} - \nu_\psi) = 1 - \int d\nu_\psi,$$

and conclusion (iii) of the lemma is established.

It remains to show that conclusion (iv) holds if $\Xi = \{\xi, \eta\}$ and $z_0 \in l(\xi, \eta)$. Since the involution $\tau = \tau_{\xi, \eta}$ satisfies $\tau\xi\tau = \xi^{-1}$ and $\tau\eta\tau = \eta^{-1}$, we have $\tau\Gamma\tau = \Gamma$; furthermore, $\tau J_\xi\tau = J_{\xi^{-1}}$ and $\tau J_\eta\tau = J_{\eta^{-1}}$. On the other hand, since $z_0 \in l(\xi, \eta)$ we have $\tau z_0 = z_0$. Hence $V_{\xi^{-1}} = \{\gamma z_0 : \gamma \in J_{\xi^{-1}}\} = \{\tau\gamma\tau z_0 : \gamma \in J_\xi\} = \tau V_\xi$. By condition (iii) of 4.2 it follows that $\tau_\infty^* \mathcal{M}_{V_\xi} = \mathcal{M}_{V_{\xi^{-1}}}$. In particular, $\tau_\infty^* \nu_\xi = \nu_{\xi^{-1}}$. Taking total masses of both sides gives

$\int d\nu_{\xi^{-1}} = \int d\nu_{\xi}$. The same argument shows that $\int d\nu_{\eta^{-1}} = \int d\nu_{\eta}$. This establishes (iv) and completes the proof of the lemma. \square

The proofs of Propositions 5.1 and 5.2 will require combining Lemma 5.3.3 with the elementary considerations covered by the following two lemmas.

5.4. Lemma. *Let μ and μ_0 be measures of finite total mass on a measurable space (X, \mathcal{B}) . Suppose that $0 \leq \mu_0 \leq \mu$. Let C be a Borel set such that $\mu(C) \geq \mu_0(X)$. Let f be a measurable, nonnegative real-valued function on X such that $\inf f(C) \geq \sup f(X - C)$. Then $\int_X f d\mu_0 \leq \int_C f d\mu$.*

Proof. We write $\mu = \mu_0 + \mu_1$, where μ_1 is again a measure on (X, \mathcal{B}) . We have

$$\mu_0(X - C) = \mu_0(X) - \mu_0(C) \leq \mu(C) - \mu_0(C) = \mu_1(C),$$

so that

$$\int_{X-C} f d\mu_0 \leq (\sup f(X - C))\mu_0(X - C) \leq (\inf f(C))\mu_1(C) \leq \int_C f d\mu_1.$$

Hence

$$\int_X f d\mu_0 = \int_C f d\mu_0 + \int_{X-C} f d\mu_0 \leq \int_C f d\mu_0 + \int_C f d\mu_1 = \int_C f d\mu.$$

\square

5.5. Lemma. *Let a and b be numbers such that $0 \leq a \leq \frac{1}{2}$ and $0 \leq b \leq 1$, let γ be an isometry of \mathbf{H}^3 and let z be a point in \mathbf{H}^3 . Suppose that ν is a measure on S_∞ such that*

- (i) $\nu \leq A_z$,
- (ii) $\nu(S_\infty) \leq a$, and
- (iii) $\int_{S_\infty} \lambda_{\gamma, z}^2 d\nu \geq b$.

Then

$$\text{dist}(z, \gamma \cdot z) \geq \frac{1}{2} \log \frac{a(1-b)}{b(1-a)}.$$

Proof. We let h denote the constant $\text{dist}(z, \gamma \cdot z)$, and set $c = \cosh h$ and $s = \sinh h$. We let λ denote the function $\lambda_{\gamma, z}$. We identify $\overline{\mathbf{H}}^3$ conformally with the unit ball in \mathbf{R}^3 in such a way that z is the origin (so that S_∞ has the round metric centered at z) and $\gamma^{-1} \cdot z$ is on the positive vertical axis.

According to 2.4, we have $\lambda(\zeta) = \mathcal{P}(z, \gamma^{-1} \cdot z, \zeta)$ for all $\zeta \in S_\infty$. Hence by (2.1.1), λ is given by the formula

$$\lambda(\zeta) = (c - s \cos \phi)^{-1}$$

where $\phi = \phi(\zeta)$ is the angle between the positive vertical axis and the ray from the origin through ζ ; thus ϕ is the polar angle of ζ in spherical coordinates.

Set $A = A_z$. Since S_∞ has the round metric centered at z , the measure A is obtained by dividing the area measure on the unit sphere by the area 4π of the sphere. In spherical coordinates θ and ϕ on the unit sphere we have $dA = (1/4\pi) \sin \phi d\phi d\theta$.

Set $\phi_0 = \arccos(1 - 2a)$, and let $C \subset S_\infty$ denote the spherical cap defined by the inequality $\phi \geq \phi_0$. Then we have

$$A(C) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\phi_0} \sin \phi \, d\phi \, d\theta = \frac{1}{2}(1 - \cos \phi_0) = a.$$

Thus by hypothesis (ii) we have $A(C) \geq \nu(S_\infty)$. Observe also that since λ is given by the function $(c - s \cos \phi)^{-1}$, which is positive and monotone decreasing for $0 \leq \phi \leq \pi$, we have $\inf \lambda(C) \geq \sup \lambda(\Xi - C)$. Since we also have $\nu \leq A$ by hypothesis (i), we may apply Lemma 5.4 with $f = \lambda^2$ to obtain

$$\begin{aligned} \int_{S_\infty} \lambda^2 \, d\nu &\leq \int_C \lambda^2 \, dA = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\phi_0} \frac{\sin \phi}{(c - s \cos \phi)^2} \, d\phi \, d\theta \\ &= \frac{1}{2} \int_0^{\phi_0} \frac{\sin \phi}{(c - s \cos \phi)^2} \, d\phi = \frac{1}{2s} \left(\frac{1}{c - s} - \frac{1}{c - \cos \phi_0} \right) \end{aligned}$$

where the last step follows from the substitution $u = c - s \cos \phi$. Recalling that $\cos \phi_0 = 1 - 2a$ and using hypothesis (iii), we find that

$$b \leq \int_{S_\infty} \lambda^2 \, d\nu \leq \frac{a}{(c - s)(c - s + 2as)}.$$

After rewriting c and s respectively as $\frac{1}{2}(e^h + e^{-h})$ and $\frac{1}{2}(e^h - e^{-h})$, and simplifying, we conclude that $e^{-2h} \leq a(1 - b)/b(1 - a)$. \square

Proof of Proposition 5.1. Suppose that Γ and Ξ satisfy the hypotheses of the proposition, and let $z_0 \in \mathbf{H}^n$ be an arbitrary point.

We wish to show that

$$\max_{\xi \in \Xi} \text{dist}(z_0, \xi \cdot z_0) \geq \frac{1}{2} \log((k - 1)(2k - 1)).$$

In particular, Γ and Ξ satisfy the hypotheses of Lemma 5.3, and so there exist a number $D \in [0, 2]$, a conformal density $\mathcal{M} = (\mu_z)$, and a family of Borel measures $(\nu_\psi)_{\psi \in \Psi}$ (where $\Psi = \Xi \Pi \Xi^{-1}$), for which conclusions (i)–(iii) of 5.3 hold. But the hypotheses of 5.1 imply that \mathcal{M} is a constant multiple of the area density \mathcal{A} ; hence $D = 2$. Since μ_{z_0} has total mass 1 by 5.3(i), we must in fact have $\mathcal{M} = \mathcal{A}$.

By 5.3(ii) we have

$$1 = \sum_{\psi \in \Psi} \nu_\psi(S_\infty) = \sum_{\xi \in \Xi} (\nu_\xi(S_\infty) + \nu_{\xi^{-1}}(S_\infty)).$$

Hence for some $\xi_0 \in \Xi$ we have $\nu_{\xi_0}(S_\infty) + \nu_{\xi_0^{-1}}(S_\infty) \leq 1/k$. It follows that for some $\psi_0 \in \{\xi_0, \xi_0^{-1}\}$ we have $\nu_{\psi_0}(S_\infty) \leq 1/2k$.

We set $\lambda = \lambda_{\psi_0, z}$ and $\nu = \nu_{\psi_0^{-1}}$. By 5.3(iii) we have

$$\int_S \lambda^2 \, d\nu = 1 - \nu_{\psi_0}(S_\infty) \geq 1 - (\nu_{\xi_0}(S_\infty) + \nu_{\xi_0^{-1}}(S_\infty)) \geq 1 - \frac{1}{k}.$$

Furthermore, it follows from 5.3(ii) that $\nu \leq \mu_{z_0} = A_{z_0}$. Hence we may apply Lemma 5.5 with $z = z_0$, $\gamma = \psi_0$, $a = 1/2k$, and $b = 1 - 1/k$. This gives

$$\text{dist}(z, \psi_0 \cdot z) \geq \frac{1}{2} \log((k - 1)(2k - 1)).$$

But since $\psi_0 = \xi_0^{\pm 1}$, we have $\text{dist}(z, \psi_0 \cdot z) = \text{dist}(z, \xi_0 \cdot z)$, and the conclusion follows. \square

Proof of Proposition 5.2. Suppose that Γ , ξ , and η satisfy the hypotheses of the proposition. Let $z_0 \in \mathbf{H}^n$ be any point; we wish to show that

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \log 3.$$

Set $l = l(\xi, \eta)$. It follows from Proposition 1.15 that

$$\max(\text{dist}(z_0, \xi \cdot z_0), \text{dist}(z_0, \eta \cdot z_0)) \geq \max(\text{dist}(z_1, \xi \cdot z_1), \text{dist}(z_1, \eta \cdot z_1)),$$

where z_1 denotes the orthogonal projection of z_0 onto l . Hence we may assume without loss of generality that $z_0 \in l$.

In particular Γ and $\Xi = \{\xi, \eta\}$ satisfy the hypotheses of Lemma 5.3, and so there exist a number $D \in [0, 2]$, a conformal density $\mathcal{M} = (\mu_z)$ and a family of Borel measures $(\nu_\psi)_{\psi \in \Psi}$ (where $\Psi = \{\xi, \xi^{-1}, \eta, \eta^{-1}\}$), for which conclusions (i)–(iv) of 5.3 hold. But the hypotheses of 5.2 imply that \mathcal{M} is a constant multiple of the area density \mathcal{A} ; hence $D = 2$. Since μ_{z_0} has total mass 1 by 5.3(i), we must, in fact, have $\mathcal{M} = \mathcal{A}$.

By conclusions (ii) and (iv) of 5.3 we have $1 = 2\nu_\xi(S_\infty) + 2\nu_\eta(S_\infty)$ so that either $\nu_\xi(S_\infty)$ or $\nu_\eta(S_\infty)$ is $\leq 1/4$. By symmetry we may assume that $\nu_\xi(S_\infty) \leq 1/4$. We set $\lambda = \lambda_{\xi^{-1}, z_0}$. By 5.3(iii) and (iv) we have

$$\int_S \lambda^2 d\nu_\xi = 1 - \nu_{\xi^{-1}}(S_\infty) = 1 - \nu_\xi(S_\infty) \geq \frac{3}{4}.$$

Furthermore, it follows from 5.3(i) that $\nu_\xi \leq A$. Hence we may apply Lemma 5.5 with $\nu = \nu_\xi$, $\gamma = \psi_0$, $a = \frac{1}{4}$, and $b = \frac{3}{4}$. This gives $\text{dist}(z_0, \xi^{-1} \cdot z_0) \geq \log 3$. Since $\text{dist}(z_0, \xi \cdot z_0) = \text{dist}(z_0, \xi^{-1} \cdot z_0)$, the conclusion follows. \square

6. BONAHOON SURFACES AND SUPERHARMONIC FUNCTIONS

The results of §5 depend on the hypothesis that every Γ -invariant conformal density is a constant multiple of the area density, where Γ is the given free Kleinian group. According to Proposition 3.9, this is always true if every positive, Γ -invariant superharmonic function on \mathbf{H}^3 is constant. Thus in order to apply the results of §5 one needs a geometric condition that guarantees that the hyperbolic manifold $M(\Gamma)$ admits no nonconstant positive superharmonic functions. In this section we describe such a condition. Very similar conditions have been used in Thurston [Th, 8.12] and, later, in Sullivan [Su3], Bohanon [Bo], and Canary [Ca1].

For our applications we need only consider Kleinian groups Γ that contain no parabolics and satisfy $M(\Gamma) = N(\Gamma)$. We will restrict attention to these groups

whenever convenient. While the complications that arise from the existence of parabolics can be dealt with, it would unnecessarily complicate the exposition to do so here.

6.1. Let Σ be a closed 2-manifold, not necessarily connected. Let $u \in H_2(\Sigma; \mathbf{Z}/2)$ denote the fundamental class of Σ , i.e., the sum of the images under inclusion of the fundamental classes of the components of Σ . A map $f: \Sigma \rightarrow M$ will be said to be *null-homologous* if the class $f_*(u) \in H_2(M; \mathbf{Z}/2)$ is trivial. Suppose that $f: \Sigma \rightarrow M$ is a null-homologous map and that P is a point of $M - f(P)$. We will say that P is *strictly enclosed* by f if the class $f_*(u) \in H_2(M - \{P\}; \mathbf{Z}/2)$ is nontrivial, where we are regarding f as a map from Σ to $M - \{P\}$. We will say that a point $P \in M$ is *enclosed* by the null-homologous map f if either $P \in f(\Sigma)$ or P is strictly enclosed by f . A compact set K will be said to be enclosed by f if every point of K is enclosed by f .

Proposition. *Let Γ be a Kleinian group without parabolics such that $M(\Gamma) = N(\Gamma)$. Suppose that $f_j: \Sigma_j \rightarrow M$ is a sequence of maps from closed 2-manifolds into M such that every compact subset of $M(\Gamma)$ is enclosed by some f_j . If there exists a constant V such that the volume of $\text{nbhd}_1 f_j(\Sigma_j)$ is less than V for all j then every positive superharmonic function on M is constant.*

Proof. This statement is proved by an argument given in [Th, 8.12] and [Ca1, Theorem 7.1]. In that argument it is assumed that M has k simply degenerate ends with neighborhoods homeomorphic to $S_1 \times [0, \infty), \dots, S_k \times [0, \infty)$. The argument is applied to a sequence of maps $f_j: \Sigma \rightarrow M$ where $\Sigma = \Sigma_j$ is the disjoint union of S_1, \dots, S_k and where the restriction of f_j to S_i is homotopic to the inclusion of $S_j \times \{0\}$ into M . The argument uses two properties of these maps. First that there is a bound on the volume of the sets $\text{nbhd}_1(f_j(\Sigma_j))$ and second that a flow line that exits an end of M must pass through infinitely many of these sets. The second property follows immediately from our hypothesis that every compact set in M is enclosed by one of the maps f_j . \square

The rest of this section is devoted to a description of a geometric construction, due to Bonahon, of maps that satisfy the hypotheses of Proposition 6.1. The construction is a simplification of Thurston's construction of "pleated surfaces."

6.2. **Virtual triangulations.** A mapping ϕ of an affine k -simplex Δ^k into a topological space X will be said to define a *virtual k -simplex* in X if ϕ is one-to-one on the interior of each face of Δ^k . Two maps $\phi: \Delta \rightarrow X$ and $\phi': \Delta' \rightarrow X$ will be said to define the same virtual simplex if there is an affine isomorphism $I: \Delta \rightarrow \Delta'$ such that $\phi \circ I = \phi'$. By the *interior* of the virtual simplex (defined by) $\phi: \Delta \rightarrow X$ we mean the set $\phi(\text{int} \Delta)$. The *dimension* of ϕ is the dimension of Δ . A virtual simplex $\phi_1: \Delta_1 \rightarrow X$ is called a *face* of a virtual simplex ϕ if there is an affine isomorphism J of Δ_1 onto a face of Δ such that $\phi \circ J = \phi_1$.

Let Σ be a topological 2-manifold-with-boundary. (We may have $\partial\Sigma = \emptyset$, and Σ may be disconnected.) A *virtual triangulation* of Σ is a collection Φ of

virtual simplices in Σ such that

- (i) Φ contains the faces of each simplex in Φ , and
- (ii) as a set, Σ is the disjoint union of the interiors of the virtual simplices in Φ .

(Note that two faces of a simplex may coincide.)

For any integer i , the i -skeleton of Σ in a given virtual triangulation is defined to be the union of the interiors of all the virtual simplices of dimension $\leq i$; we denote it by $\text{skel}_i(K)$. The virtual 0-simplices of Σ will be called *vertices*.

A virtual 1-simplex ϕ in the 2-manifold-with-boundary Σ may be regarded as a (nonoriented) path in Σ . If ϕ is a homotopically trivial loop we shall say that it is *degenerate*. A virtual triangulation will be termed *nondegenerate* if it has no degenerate 1-simplices.

6.3. Piecewise hyperbolic structures. Let $\phi: \Delta \rightarrow \Sigma$ be a virtual 2-simplex in a topological surface Σ . A homeomorphism h of $\Delta = \text{domain } \phi$ onto a triangle in \mathbb{H}^2 that maps vertices onto vertices will be said to define a *hyperbolic structure* on ϕ . Two such homeomorphisms h and h' will be said to define the same hyperbolic structure on ϕ if $h'h^{-1}$ is an isometry of hyperbolic triangles. If $\psi: \Delta_1 \rightarrow \Sigma$ is a 1-dimensional face of ϕ , any hyperbolic structure on ϕ induces a linear metric on Δ_1 .

A virtual 1-simplex in the interior of a virtually triangulated surface Σ either occurs as a face of two distinct 2-simplices or as two coincident faces of a single 2-simplex. Thus if each 2-simplex in Σ is given a hyperbolic structure then for each interior 1-simplex there are two linear metrics induced on Δ_1 . A *piecewise-hyperbolic (PH) structure* on Σ is a family of hyperbolic structures on the virtual 2-simplices of Σ such that, for every interior 1-simplex of Σ , the two induced linear metrics on Δ_1 agree. A PH-structure on Σ determines, in a natural way, the structure of a (necessarily incomplete) hyperbolic 2-manifold with geodesic boundary on $\Sigma - \text{skel}_0(\Sigma)$.

If $\phi: \Delta \rightarrow \Sigma$ is a virtual 2-simplex and w is a vertex of Δ , we shall denote by $\theta_w(\phi)$ the angle in the hyperbolic triangle Δ at the vertex w . For any vertex x of Σ we define the *angle sum at x* in a given triangulation to be $\sum_{\phi, w} \theta_w(\phi)$, where ϕ ranges over all virtual 2-simplices of the given virtual triangulation and w ranges over all the vertices of domain ϕ that are mapped by ϕ to x . (Note that a given virtual 2-simplex ϕ can contribute as many as three terms to this sum.) If x is a point of Σ that is not a vertex, we define the angle sum at x to be 2π if $x \in \text{int } \Sigma$ and π if $x \in \partial \Sigma$.

Let x be any point of Σ and let a denote the angle sum at x . If $x \in \text{int } \Sigma$ we set $\text{excess}(x) = a - 2\pi$. If $x \in \partial \Sigma$ we set $\text{excess}(x) = a - \pi$. We call x a *singular point* if $\text{excess}(x) \neq 0$. If $x \in \partial \Sigma$ and $\text{excess}(x) < 0$, we call x a *corner*. We shall denote by $\text{sing } \Sigma$ the set of all singular points of Σ . It is clear that the natural hyperbolic structure on $\Sigma - \text{skel}_0(\Sigma)$ admits a unique extension to a hyperbolic structure on $\Sigma - \text{sing } \Sigma$.

Given a virtual 2-simplex $\phi: \Delta \rightarrow \Sigma$ in the PH surface Σ , a path in $\phi(\Delta)$ will be called *geodesic* if it has the form $\phi \circ \alpha$ for some geodesic path α (parametrized proportionally to arc length) in Δ . A path in Σ will be termed

piecewise geodesic (PG) if it is a composition $\gamma = (\phi_1 \circ \alpha_1) * \cdots * (\phi_n \circ \alpha_n)$ of finitely many geodesic paths supported in 2-simplices of Σ . The *length* of γ is the sum of the lengths of the α_i . Any two points in the same component of Σ can be joined by a PG path, and it is not hard to show that a metric on each component of Σ can be constructed by defining $\text{dist}(x, y)$ to be the infimum of the lengths of PG paths joining x and y . It is also not difficult to verify that this metric agrees with the hyperbolic metric on the components of $\Sigma - \text{sing } \Sigma$.

6.4. Ultrahyperbolic surfaces. We define an *ultrahyperbolic structure* on a virtually triangulated surface Σ to be a PH structure such that $\text{excess}(x) \geq 0$ for every $x \in \text{int } \Sigma$. The ultrahyperbolic surfaces form a rich class of geometric objects that are natural generalizations of hyperbolic surfaces. For example, by a generalization of the Gauss-Bonnet theorem that applies to PH surfaces (c.f. [Bo, Lemme 1.9]), each component of a surface with an ultrahyperbolic structure must have negative Euler characteristic. The fundamental geometric property of these surfaces that we will need here is the version of Bonahon's Bounded Diameter Lemma stated below.

Given an ultrahyperbolic surface Σ and a point $P \in \Sigma$ we define $\text{short}(P)$ to be the infimum of the lengths of homotopically nontrivial PG loops based at P . If I is an interval contained in $(0, \infty)$ we define Σ_I as in 1.5. In particular, for $\varepsilon > 0$, the ε -thin part of Σ is the subset $\Sigma_{(0, \varepsilon]}$. Given a PG path γ in Σ we define the *length of γ modulo $\Sigma_{(0, \varepsilon]}$* to be the length, i.e., the Lebesgue measure, of $\gamma^{-1}(\Sigma_{(\varepsilon, \infty)})$. If P and Q are points of Σ then the *distance from P to Q modulo $\Sigma_{(0, \varepsilon]}$* is the infimum of the set of lengths modulo $\Sigma_{(0, \varepsilon]}$ of PG paths from P to Q . The *diameter of Σ modulo $\Sigma_{(0, \varepsilon]}$* is the supremum over all $P, Q \in \Sigma$ of the distance modulo $\Sigma_{(0, \varepsilon]}$ from P to Q .

The following result is due to Bonahon [Bo, Lemme 1.10].

Proposition. *For each positive integer g and each positive number ε there exists a constant $D(g, \varepsilon)$ such that if Σ is a closed ultrahyperbolic surface of total genus g then the diameter of Σ modulo $\Sigma_{(0, \varepsilon]}$ is less than $D(g, \varepsilon)$.*

6.5. Hyperbolically simplicial surfaces. Let us consider a complete hyperbolic manifold $M = \mathbf{H}^n / \Gamma$. A map f of a virtually triangulated (not necessarily connected) surface Σ into M will be called *hyperbolically simplicial* if for every virtual simplex $\phi: \Delta \rightarrow \Sigma$, the map $f_\phi = f \circ \phi$ admits a continuous lift $\tilde{f}_\phi: \Delta \rightarrow \mathbf{H}^n$ such that $\tilde{f}_\phi(\Delta)$ is a hyperbolic simplex and either (i) $\dim \tilde{f}_\phi(\Delta) < \dim \Delta$ or (ii) \tilde{f}_ϕ maps Δ homeomorphically onto $\tilde{f}_\phi(\Delta)$. If (ii) holds for every simplex of Σ we shall say that f is *nondegenerate*. If f is nondegenerate then for each virtual 2-simplex ϕ the map f_ϕ induces a hyperbolic structure on the 2-simplex Δ that is obviously independent of the choice of lift. The linear metric induced on the domain of a 1-simplex ψ by an adjacent 2-simplex agrees with the linear metric determined by \tilde{f}_ψ . Thus a nondegenerate hyperbolically simplicial map from Σ to M induces a PH structure on Σ . If we equip Σ with the PH structure induced by the nondegenerate hyperbolically simplicial map f , then for every PG path γ in Σ , the path $f \circ \gamma$ has the same length as γ . Hence if we give each component of Σ the metric defined by its PH

structure, the restriction of f to each component of Σ is distance-decreasing.

By a *hyperbolically simplicial surface* in M we mean a pair (Σ, f) , where Σ is a closed, orientable surface equipped with a virtual triangulation, and $f: \Sigma \rightarrow M$ is a hyperbolically simplicial map. We shall say that (Σ, f) is *nondegenerate* if the virtual triangulation of Σ and the map f are nondegenerate. The Euler characteristic and the total genus of a hyperbolically simplicial surface (Σ, f) are defined, respectively, to be the Euler characteristic and the total genus of the surface Σ .

Suppose we are given a hyperbolically simplicial surface (Σ, f) . Then any small perturbation of $f|_{\text{skel}_0 \Sigma}$ can be extended over Σ to give a hyperbolically simplicial surface (Σ, f') such that f' is homotopic to f . One first modifies f in a neighborhood of the 0-skeleton to make it agree with f' on the vertices. The restrictions to $\text{skel}_1 \Sigma$ and $\text{skel}_2 \Sigma$ can then be successively modified to make them geodesic. The map f' can be made arbitrarily close to f in the uniform topology by choosing the initial perturbation to be sufficiently small.

We will say that a hyperbolically simplicial surface (Σ, f) encloses a compact set $K \subset M$ provided that the map $f: \Sigma \rightarrow M$ is null-homologous and encloses K .

6.6. Bonahon surfaces. A continuous map f of a topological 1-manifold S into a hyperbolic 3-manifold M will be called *geodesic* if for every component C of S there is a covering map $p: \mathbf{R} \rightarrow C$ such that $f \circ p$ is a geodesic. In the case where S is a simple closed curve, i.e., is homeomorphic to S^1 , a geodesic map $f: S \rightarrow M$ is a reparametrization of a closed geodesic in M , and thus has a well-defined length.

A nondegenerate hyperbolically simplicial surface (Σ, f) in M will be called a *Bonahon surface* if there is a closed 1-manifold $S \subset \Sigma$ such that $\text{skel}_0(\Sigma) \subset S \subset \text{skel}_1(\Sigma)$ and $f|_S$ is geodesic. The following observation is contained in [Bo, Lemme 1.8].

Proposition. *If (Σ, f) is a Bonahon surface in a complete hyperbolic manifold M , then the PH structure induced on Σ by f is ultrahyperbolic.*

Proof. We must show that the angle sum at any vertex x of Σ is at least 2π . Let e_1, \dots, e_n be the distinct oriented virtual 1-simplices of Σ having x as initial point. Then f maps each e_i to a geodesic path $\gamma_i: [0, 1] \rightarrow \Sigma$ with initial point x . Let U denote the unit sphere in the tangent space to the 3-manifold M at $f(x)$. Let $u_i \in U$ be the normalized tangent vector to γ_i at 0. Then the angle sum at x is equal to the length of a piecewise-geodesic curve γ in U whose vertices are u_1, \dots, u_n . Now according to the definition of a Bonahon surface, the vertex x lies on a simple closed curve $C \subset \text{skel}_1(\Sigma)$ such that $f|_C$ is geodesic. This implies that the curve γ passes through two antipodal points of U , and therefore has length $\geq 2\pi$. \square

6.7. Neighborhoods of Bonahon surfaces. Let (Σ, f) be a Bonahon surface in M , and let Σ be given the PH structure induced by f . Let ε be a positive number. We shall say that (Σ, f) is ε -*incompressible* if for every homotopically nontrivial PG closed curve $\gamma: S^1 \rightarrow \Sigma$ of length $< \varepsilon$, the closed curve $f \circ \gamma$ is homotopically nontrivial in M .

The following proposition is proved in [Ca, Lemma 7.2]. (The pleated surface version is found in [Th, 8.12.1].)

Proposition. *For each positive integer g and each positive number ε there exists a constant $V(g, \varepsilon)$ such that if (Σ, f) is any ε -incompressible Bonahon surface of genus g in a complete hyperbolic 3-manifold M then the volume of $\text{nbhd}_1(f(\Sigma))$ is at most $V(g, \varepsilon)$.*

6.8. The following proposition, which is an immediate consequence of Propositions 6.1 and 6.7, gives a basic criterion for every positive superharmonic function on $M(\Gamma)$ to be constant.

Proposition. *Let Γ be a Kleinian group without parabolics such that $M(\Gamma) = N(\Gamma)$. Suppose that there exist positive integers g_1, \dots, g_k and a real number ε such that every compact subset of $M(\Gamma)$ is enclosed by a system $(\Sigma_1, f_1), \dots, (\Sigma_k, f_k)$ of ε -incompressible Bonahon surfaces such that the genus of Σ_i is less than g_i . Then every positive superharmonic function on $M(\Gamma)$ is constant.*

6.9. **Topological tameness.** A 3-manifold is said to be *topologically tame* if it is homeomorphic to the interior of a compact 3-manifold. A Kleinian group Γ is topologically tame if $M(\Gamma)$ is topologically tame.

A theorem due to R. Canary [Ca1] implies that if Γ is a Kleinian group without parabolics such that $M(\Gamma) = N(\Gamma)$ and if Γ is topologically tame, then the hypotheses of Proposition 6.8 hold. In particular, we have the following special case of [Ca1, Theorem 7.2].

Proposition. *Let Γ be a Kleinian group without parabolics such that $M(\Gamma) = N(\Gamma)$. If Γ is topologically tame then every positive superharmonic function on $M(\Gamma)$ is constant.*

In §8 we show, by quite different methods, that the hypotheses of Proposition 6.8 hold for most rank-2 free Kleinian groups that have no parabolics and lie on the boundary of the space of rank-2 free geometrically finite groups. The proof of our main theorem in §9 breaks into two cases that are handled, respectively, by the results of §8 and the above proposition.

6.10. We conclude this section with a result that will be needed in §8.

Proposition. *Let M be a 3-manifold with an involution T such that every connected component of $\text{Fix } T$ is homeomorphic to \mathbf{R} . Let (Σ, f) be a null-homologous hyperbolically simplicial surface in M , and let T_Σ be an orientation-preserving involution of Σ such that $f \circ T_\Sigma = T \circ f$. Suppose that for some component L of $\text{Fix } T$ there are exactly two fixed points v_+ and v_- of T_Σ that are mapped into L by f ; let s denote the compact arc in L bounded by $f(v_+)$ and $f(v_-)$. Then (Σ, f) encloses s .*

Proof. It is enough to prove that every point of $s \cap (M - f(\Sigma))$ is strictly enclosed by (Σ, f) . Let P be any point of $s \cap (M - f(\Sigma))$. By a small perturbation, as described in 6.5, one can approximate the map f arbitrarily well, in the uniform topology on the space of maps of Σ into M , by a hyperbolically simplicial map f_1 that is topologically transverse to L and satisfies $f_1 \circ T_\Sigma = T \circ f_1$. In particular, f_1 maps the finite set $\text{Fix } T_\Sigma$ into the 1-manifold $\text{Fix } T$. Hence if

f_1 is close enough to f then f_1 maps v_+ and v_- into distinct components of $L - \{P\}$ and maps the remaining fixed points of T_Σ into $(\text{Fix } T) - L$. Thus the hypotheses of the theorem hold when f is replaced by f_1 . Furthermore, if f_1 is close enough to f then f_1 maps Σ into $M - \{P\}$ and is homotopic in $M - \{P\}$ to f . Hence f_1 strictly encloses P if and only if f does. Thus we may assume without loss of generality that f is topologically transverse to L .

In this case the set $f^{-1}(L)$ is finite and has even cardinality since (Σ, f) is null-homologous. Let L_+ and L_- denote the components of $L - \{P\}$ containing $f(v_+)$ and $f(v_-)$, respectively. To say that (Σ, f) encloses P is equivalent to saying that $f^{-1}(L_+)$ (and hence also $f^{-1}(L_-)$) has odd cardinality. But $f^{-1}(L_+)$ is invariant under the involution T_Σ (since $f \circ T_\Sigma = T \circ f$) and contains only one fixed point of T_Σ , namely, a . Hence $\text{card } f^{-1}(L_+)$ is indeed odd. \square

7. GIRDED SURFACES AND GIRDED 3-MANIFOLDS

In this section we introduce a special class of Bonahon surfaces that will be used in the proof of our main theorem.

7.1. Let ε be a positive number. By an ε -girded surface in M we mean a triple (Σ, f, S) , where (Σ, f) is a nondegenerate hyperbolically simplicial surface in M and $S \subset \Sigma$ is a closed 1-manifold, such that

- (i) $\text{skel}_0(\Sigma) \subset S \subset \text{skel}_1(\Sigma)$;
- (ii) each component of $\Sigma - S$ is homeomorphic to a sphere with three punctures;
- (iii) $f|_S$ is geodesic, and $f|_C$ has length $< \varepsilon/3$ for each component C of S ; and
- (iv) for each component R of $\Sigma - S$, the map $f|_R$ induces an injective homomorphism from $\pi_1(R)$ to $\pi_1(M)$.

The maps $f|_C$, where C ranges over the components of S , are reparametrizations of closed geodesics. These closed geodesics will be said to be *carried* by (Σ, f, S) . It follows from condition (ii) above that S has $3g - 3$ components, where g is the genus of Σ . Hence if k denotes the number of distinct closed geodesics carried by Σ , we have $1 \leq k \leq 3g - 3$. It follows from the definitions that if (Σ, f, S) is an ε -girded surface then (Σ, f) is a Bonahon surface. Note also that since no component of Σ is a sphere, condition (iv) of the above definition implies that the closed geodesics carried by (Σ, f, S) are nontrivial; in particular, the components of S are homotopically nontrivial in Σ .

7.2. **Proposition.** *Let ε be a Margulis number for M , and let (Σ, f, S) be an $\varepsilon/3$ -girded surface in M . Then (Σ, f, S) is $\varepsilon/3$ -incompressible.*

Proof. Let $\gamma: S^1 \rightarrow \Sigma$ be any homotopically nontrivial PG closed curve of length $< \varepsilon/3$. If $\gamma(S^1)$ is disjoint from S , then by 7.1(iv), $f \circ \gamma$ is homotopically nontrivial in M .

Now suppose that $\gamma(S^1)$ meets a component C of S . Then $f(\gamma(S^1))$ meets the set $f(C)$, which by 7.1(iii) is the support of a closed geodesic of length $\varepsilon/3$

in M . In particular, $f(\gamma(S^1))$ meets $M_{(0, \varepsilon/3]}$. But since γ has length $< \varepsilon/3$ and f is distance-decreasing, $f(\gamma(S^1))$ has diameter $< \varepsilon/3$ and hence $f(\gamma(S^1)) \subset \text{nbhd}_{\varepsilon/3}(M_{(0, \varepsilon/3]}) \subset M_{(0, \varepsilon)}$. Since ε is a Margulis number for M , the component \mathcal{T} of $M_{(0, \varepsilon)}$ containing $f(\gamma(S^1))$ is a Margulis tube, and the core of \mathcal{T} is the only closed geodesic meeting \mathcal{T} . Hence if $S' \supset C$ denotes the union of all components of S that meet $\gamma(S^1)$, then $f(S' \cup \gamma(S^1)) \subset \mathcal{T}$.

Let Z denote the connected set $S' \cup \gamma(S^1)$. Since Z is a finite union of geodesic arcs, it is a subcomplex of Σ in some triangulation. Hence there is a connected neighborhood U of Z in Σ such that Z carries $\pi_1(U)$. We may take U to be a compact 2-manifold-with-boundary contained in $f^{-1}(\mathcal{T})$. Now we let U' denote the 2-manifold-with-boundary obtained by adjoining to U all disk components of $\Sigma - U$, and we let U'' denote the complement in U' of a regular neighborhood of S' in $\text{int } U'$. Then $\pi_1(U')$ is carried by $Z \subset f^{-1}(\mathcal{T})$, and hence $(f|_{U'})_{\#}(\pi_1(U')) \subset \pi_1(M)$ is cyclic. On the other hand, it follows from 7.1(iv) and the construction of U'' that f induces an injection from the fundamental group of each component of U'' into $\pi_1(M)$. Hence each component of U'' has cyclic fundamental group and, therefore, must be an annulus.

This implies that U' has Euler characteristic zero. Since Σ has genus > 1 , U' must be an annulus. Since C is a homotopically nontrivial simple closed curve in U' , it is a topological core of U' . But γ is a homotopically nontrivial closed curve in U' and, therefore, is homotopic in U' to a power of C . As $f|_C$ is a nontrivial closed geodesic, and as the complete hyperbolic manifold M necessarily has torsion free fundamental group, we conclude that $f \circ \gamma$ is homotopically nontrivial as required. \square

7.3. Proposition. *Let M be a hyperbolic 3-manifold and let ε be a Margulis number for M . Let (Σ, f, S) be an $(\varepsilon/3)$ -girded surface of total genus g in M . Let $\alpha_1, \dots, \alpha_k$ be the distinct geodesics carried by (Σ, f, S) , and let \mathcal{T}_i denote the component of $M_{(0, \varepsilon]}$ containing α_i . Then*

$$f(\Sigma) \subset \text{nbhd}_D(\mathcal{T}_1 \cup \dots \cup \mathcal{T}_k),$$

where $D = D(g, \varepsilon/3)$ in the notation of Proposition 6.4.

Proof. In the metric defined by the PH structure of Σ induced by f , we have by Proposition 6.4 that the diameter of Σ modulo $\Sigma_{(0, \varepsilon/3]}$ is less than $D = D(g, \varepsilon/3)$.

Thus $\Sigma = \text{nbhd}_D(\Sigma_{(0, \varepsilon/3]})$. Since f is distance-decreasing, it follows that $f(\Sigma) \subset \text{nbhd}_D(f(\Sigma_{(0, \varepsilon/3]}))$. In order to prove the proposition, therefore, it suffices to show that $f(\Sigma_{(0, \varepsilon/3]}) \subset \mathcal{T}_1 \cup \dots \cup \mathcal{T}_k$.

Let x be any point of $\Sigma_{(0, \varepsilon/3]}$. Then there is a closed curve $\gamma: S^1 \rightarrow \Sigma$ that passes through x , is homotopically nontrivial in Σ and has length $< \varepsilon/3$. If $\gamma(S^1)$ intersects S then x is in the $\varepsilon/3$ -neighborhood of S ; since f is distance-decreasing, it follows that $f(x) \in \text{nbhd}_{\varepsilon/3}(\alpha_i)$ for some i . Since the

closed geodesic α_i has length $< \varepsilon/3$, it is clear that $\text{nbhd}_{\varepsilon/3}(\alpha_i)$ is contained in $M_{(0,\varepsilon]}$ and hence in \mathcal{T}_i ; thus $f(x) \in \mathcal{T}_i$ in this case.

Finally, suppose that $\gamma(S^1)$ is disjoint from S . Then γ is contained in some component R of $\Sigma - S$. Since γ is homotopically nontrivial in Σ , the graph $\mathcal{G} = \gamma(S^1)$ is noncontractible in R (i.e., the inclusion $\mathcal{G} \rightarrow R$ is not homotopic to a constant). Hence \mathcal{G} contains a simple closed curve C' that is noncontractible in R . By condition (ii) in the definition of an ε -girded surface, R is homeomorphic to a sphere with three punctures, and hence C' is parallel in Σ to a component C of S . Since γ has length $< \varepsilon/3$, the curve C' has length $< \varepsilon/3$ and x lies in the $\varepsilon/3$ -neighborhood of C' . Hence there is a loop γ' based at x that is freely homotopic to the simple closed curve C and has length $< \varepsilon$. Now $f \circ \gamma'$ has length $< \varepsilon$ and is freely homotopic to α_i . Hence $f(\gamma'(S^1)) \subset \mathcal{T}_i$. In particular, $f(x) \in \mathcal{T}_i$. \square

7.4. Girded manifolds. An ε -girded surface is said to enclose a set if its underlying hyperbolic simplicial surface is null-homologous and encloses the set. Let M be a complete hyperbolic 3-manifold. We shall say that M is *girded* if there exist an integer g and a point $P_0 \in M$ such that for every $\varepsilon > 0$ there is an ε -girded surface of total genus at most g that encloses P_0 . A Kleinian group Γ will be termed *girded* if the hyperbolic 3-manifold $M(\Gamma)$ is girded.

Proposition. *If a complete hyperbolic 3-manifold M is girded then every positive superharmonic function on M is constant.*

Proof. Let M be a complete hyperbolic 3-manifold and let ε be a Margulis constant for M . We will show that every compact set in M is enclosed by an $\varepsilon/3$ -girded surface of total genus at most g . Any $\varepsilon/3$ -girded surface is $\varepsilon/3$ -incompressible by Proposition 7.2 and is by definition a Bonahon surface. The proposition then follows immediately from Proposition 6.8.

Let K be a compact set in M . By enlarging K if necessary we may assume that K is connected and that $P_0 \in K$. Set $D = D(g, \varepsilon/3)$ in the notation of Proposition 6.4. Since $\text{nbhd}_D(K)$ is compact, there are only finitely many components of $M_{(0,\varepsilon]}$ that meet $\text{nbhd}_D(K)$. Each of these is a tube around a geodesic of length less than ε . Let $\varepsilon' < \varepsilon/3$ be a positive number that is less than the length of the core geodesic of any of the components of $M_{(0,\varepsilon]}$ that meets $\text{nbhd}_D(K)$. Thus if α is a geodesic of length less than ε' then the component \mathcal{T} of $M_{(0,\varepsilon]}$ that contains α is disjoint from $\text{nbhd}_D(K)$; equivalently, $\text{nbhd}_D(\mathcal{T})$ is disjoint from K .

Now it follows from Proposition 7.3 that any ε' -girded surface is disjoint from K . Let (Σ, f, S) be an ε' -girded surface of total genus at most g that encloses the point P_0 . Since the set of points of M that are enclosed by (Σ, f, S) is open and closed in $M - f(\Sigma)$ and since K is connected and contains P_0 , it follows that (Σ, f, S) encloses K . An ε' -girded surface is also $\varepsilon/3$ -girded. Thus we have constructed an $\varepsilon/3$ -girded surface of total genus at most g that encloses K , as desired. \square

8. GIRDED GROUPS ARE DENSE

8.1. We shall let F denote the abstract free group on two generators x and y . We fix an identification of the group of orientation-preserving isometries of \mathbf{H}^3 with $\mathrm{PSL}_2(\mathbf{C})$. The representations of F in $\mathrm{PSL}_2(\mathbf{C})$ are in 1-1 correspondence with points of the complex affine variety $V = \mathrm{PSL}_2(\mathbf{C}) \times \mathrm{PSL}_2(\mathbf{C})$; the representation $\rho = \rho_{\xi, \eta}$ corresponding to a point (ξ, η) of V is defined by $\rho(x) = \xi$, $\rho(y) = \eta$. We shall always understand V to be endowed with the complex topology. We shall denote by \mathfrak{D} the set of all points $(\xi, \eta) \in V$ such that $\rho_{\xi, \eta}$ is a faithful representation with discrete image, i.e., such that the group $\langle \xi, \eta \rangle$ generated by ξ and η is a Kleinian group that is free of rank 2. According to [Ch], \mathfrak{D} is a closed subset of V . We shall denote by \mathfrak{GF} the set of all points $(\xi, \eta) \in \mathfrak{D}$ such that $\langle \xi, \eta \rangle$ is a geometrically finite group (1.7) without parabolic elements. It follows from [Mar, Theorem 8.1] that \mathfrak{GF} is an open subset of V . We shall denote by \mathfrak{B} the frontier of \mathfrak{GF} in V ; thus $\mathfrak{B} = \overline{\mathfrak{GF}} - \mathfrak{GF}$.

8.2. The purpose of this section is to prove the following result.

Theorem. *There is a dense G_δ -set $\mathfrak{C} \subset \mathfrak{B}$ such that for every $(\xi, \eta) \in \mathfrak{C}$, the Kleinian group $\langle \xi, \eta \rangle$ is girded.*

8.3. Note that for any $(\xi, \eta) \in \mathfrak{B}$, the hyperbolic 3-manifold $M(\langle \xi, \eta \rangle) = \mathbf{H}^3 / \langle \xi, \eta \rangle$ satisfies $H_2(M(\langle \xi, \eta \rangle); \mathbf{Z}/2) \cong H_2(F; \mathbf{Z}/2) = 0$ since F is free. Thus every hyperbolically simplicial surface in $M(\langle \xi, \eta \rangle)$ is null-homologous. In order to prove that $\langle \xi, \eta \rangle$ is girded for a given $(\xi, \eta) \in \mathfrak{B}$, it suffices to show that for every $\varepsilon > 0$, every compact subset of $M(\langle \xi, \eta \rangle)$ is enclosed by some connected ε -girded surface of genus 2.

8.4. Since the details of the proof of Theorem 8.2 are rather involved, we begin with a rough sketch of the argument. There are two main issues. First one must produce, for a generic point $(\xi, \eta) \in \mathfrak{B}$ and an arbitrary $\varepsilon > 0$, a connected ε -girded surface of genus 2 in the hyperbolic 3-manifold $M = M(\langle \xi, \eta \rangle)$. Second one must choose this girded surface to enclose a prescribed point in M .

To deal with the first issue we observe that an ε -girded surface of genus 2 determines three closed geodesics of length $< \varepsilon$ in M ; since these geodesics are defined by simple closed curves in the domain surface, their homotopy classes are of a restricted type. Conversely, if one can realize a suitable triple of homotopy classes by a triple of geodesics of length $< \varepsilon$, one can construct an ε -girded genus 2 surface in M . That such a triple of geodesics exists for a generic (ξ, η) will be deduced from a recent result due to Curt McMullen, asserting the density of “maximal cusps” in \mathfrak{B} (see Theorem 8.9).

To deal with the second issue, we choose our ε -girded surfaces to be equivariant with respect to suitable involutions of M and of the domain surface. We then apply Proposition 6.10 to conclude that our girded surfaces enclose appropriately defined “reference points” in M . The existence of the appropriate involution of M follows from the results presented in 1.8–1.15. This stage of the argument depends strongly on the hypothesis that F is a 2-generator group.

8.5. For the rest of the section we fix, arbitrarily, a base point $w_0 \in \mathbf{H}^3$. We shall let $p_{\xi, \eta}$ denote the covering projection from \mathbf{H}^3 to $M(\langle \xi, \eta \rangle)$.

For any $(\xi, \eta) \in \mathcal{D}$ the group $\langle \xi, \eta \rangle$ is free of rank 2. In particular, $\langle \xi, \eta \rangle$ is not solvable, and hence ξ and η have no common fixed point. It follows by 1.8 that each $(\xi, \eta) \in \mathcal{D}$ determines a line $l(\xi, \eta)$ in \mathbf{H}^3 . We shall write $\tau_{\xi, \eta} = \tau_{l(\xi, \eta)}$; by 1.8, the involution $\tau = \tau_{\xi, \eta}$ satisfies $\tau \xi \tau = \xi^{-1}$ and $\tau \eta \tau = \eta^{-1}$. Since $\tau_{\xi, \eta}$ normalizes $\langle \xi, \eta \rangle$, it induces an involution $T_{\xi, \eta}$ of $M(\langle \xi, \eta \rangle)$.

We shall denote by \widehat{F} the group defined by the presentation

$$\langle x, y, t \mid t^2 = 1, txt = x^{-1}, tyt = y^{-1} \rangle.$$

We may regard \widehat{F} as a semidirect product of the free group $F = \langle x, y \rangle$ by the cyclic group $\langle t \mid t^2 = 1 \rangle$. Thus F may be identified with an index-2 subgroup of \widehat{F} . On the other hand, if we rewrite the presentation for \widehat{F} in terms of the generators $t_1 = t$, $t_2 = tx$, and $t_3 = ty$, we obtain $\langle t_1, t_2, t_3 \mid t_1^2 = t_2^2 = t_3^2 = 1 \rangle$ so that \widehat{F} is a free product of three groups of order 2. It follows (e.g., by the Kurosh subgroup theorem [K, p. 34]) that every element of finite order in \widehat{F} is conjugate to one of the t_j .

For every point $(\xi, \eta) \in \mathcal{D}$ the representation $\rho_{\xi, \eta}: F \rightarrow \text{PSL}_2(\mathbf{C})$ extends to a representation $\hat{\rho} = \hat{\rho}_{\xi, \eta}: \widehat{F} \rightarrow \text{PSL}_2(\mathbf{C})$ defined by $\hat{\rho}(x) = \xi$, $\hat{\rho}(y) = \eta$, $\hat{\rho}(t) = \tau_{\xi, \eta}$. The index of $\rho_{\xi, \eta}(F) = \langle \xi, \eta \rangle$ in $\widehat{F}_{\xi, \eta} = \hat{\rho}_{\xi, \eta}(\widehat{F}) = \langle \xi, \eta, \tau_{\xi, \eta} \rangle$ is exactly 2; indeed, the index is at most 2 since \widehat{F} has index 2 in F , and we cannot have $F_{\xi, \eta} = \widehat{F}_{\xi, \eta}$ since $F_{\xi, \eta}$ is free and $\widehat{F}_{\xi, \eta}$ contains an element of order 2. It follows that $\hat{\rho}$ is a faithful representation with discrete image.

Thus $\widehat{F}_{\xi, \eta}$ is a Kleinian group having exactly three conjugacy classes of elements of finite order. Hence the set $\mathcal{L}(\xi, \eta)$ of fixed points of elements of $\widehat{F}_{\xi, \eta}$ is the union of a locally finite disjoint family of lines in \mathbf{H}^3 , each of which is the fixed point set of a unique element of $\widehat{F}_{\xi, \eta}$; this family consists of exactly three $\widehat{F}_{\xi, \eta}$ -orbits of lines. Hence $\text{Fix } T_{\xi, \eta} = p_{\xi, \eta}(\mathcal{L}(\xi, \eta))$ is a disjoint union of three geodesics in $M(\langle \xi, \eta \rangle)$.

Explicitly, the three conjugacy classes of elements of finite order in $\widehat{F}_{\xi, \eta}$ are represented by the elements $\tau_{\xi, \eta, j} = \hat{\rho}_{\xi, \eta}(t_j)$ for $j = 1, 2, 3$. We have $\tau_{\xi, \eta, 1} = \tau_{\xi, \eta}$, $\tau_{\xi, \eta, 2} = \tau_{\xi, \eta} \xi$, and $\tau_{\xi, \eta, 3} = \tau_{\xi, \eta} \eta$. We set $l_j(\xi, \eta) = \text{Fix } \tau_{\xi, \eta, j}$. We have $l_1(\xi, \eta) = l(\xi, \eta)$. The components of $\text{Fix } T_{\xi, \eta}$ are $p_{\xi, \eta}(l_j(\xi, \eta))$ for $j = 1, 2, 3$. We shall write $L_j(\xi, \eta) = p_{\xi, \eta}(l_j(\xi, \eta))$.

8.6. Proposition. *For any $(\xi, \eta) \in \mathcal{D}$ and for $j = 1, 2, 3$, the line $l_j(\xi, \eta)$ is mapped homeomorphically onto $L_j(\xi, \eta)$ by $p_{\xi, \eta}$.*

Proof. Since $L_j(\xi, \eta)$ is a component of $\text{Fix } T_{\xi, \eta}$, it is a closed subset of $M(\langle \xi, \eta \rangle)$. Hence if the lemma were false, $L_j(\xi, \eta)$ would be a closed geodesic and $l_j(\xi, \eta)$ would be the axis of a loxodromic element γ of F . Let us set $l = l_j(\xi, \eta)$ and $\tau = \tau_{\xi, \eta, j}$. Then τ fixes l (pointwise) and hence so does the commutator $[\tau, \gamma] \in \widehat{F}$. Since \widehat{F} is discrete, it follows that $[\tau, \gamma]$ has

finite order. But the commutator subgroup of $\widehat{F} \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ is torsion free, and hence γ must commute with τ . But the centralizer of τ in \widehat{F} has order 2 (e.g., by the Kurosh subgroup theorem) and γ has infinite order; this is a contradiction. \square

8.7. We let $\tilde{P}_j(\xi, \eta)$ denote the point of $l_j(\xi, \eta)$ that is closest to the base point w_0 , and we set $P_j(\xi, \eta) = p_{\xi, \eta}(\tilde{P}_j(\xi, \eta)) \in L_j(\xi, \eta)$. The points $P_j(\xi, \eta)$, $j = 1, 2, 3$, will be called the *reference points* of $M(\langle \xi, \eta \rangle)$. Thus $M(\langle \xi, \eta \rangle)$ contains exactly three reference points, and there is one reference point in each component of $\text{Fix } T_{\xi, \eta}$.

8.8. We shall let \mathfrak{A} denote the set of all points $(\xi, \eta) \in \mathfrak{D}$ such that the group $\langle \xi, \eta \rangle$ contains no parabolics. The following result, which will be needed for the proof of Theorem 8.2, was announced in [Berl]. A version of Bers's λ -lemma can be used to complete the proof: see [McM, Corollary 1.5].

Proposition. *The set $\mathfrak{A} \cap \mathfrak{B}$ is a dense G_δ in \mathfrak{B} .*

8.9. The following result, due to Curt McMullen, is the deepest ingredient in the proof of Theorem 8.2. McMullen has informed us that the theorem below is proved by a generalization of the techniques of [McM].

Theorem. *There is a dense subset \mathfrak{E} of \mathfrak{B} such that for every $(\xi, \eta) \in \mathfrak{E}$, the free Kleinian group $\langle \xi, \eta \rangle$ is geometrically finite and has three distinct conjugacy classes of cuspidal subgroups (1.5).*

8.10. The proof of Theorem 8.2 will be proved by combining the results stated above with the following result.

Proposition. *Let (α, β) be a point of \mathfrak{D} such that $\langle \alpha, \beta \rangle$ is geometrically finite and contains three distinct conjugacy classes of cuspidal subgroups. Let ε be any positive number. Then (α, β) has a neighborhood $U = U_\varepsilon(\alpha, \beta)$ in \mathfrak{B} such that for every $(\xi, \eta) \in \mathfrak{A} \cap U$, there is a connected ε -girded surface of genus 2 in $M(\langle \xi, \eta \rangle)$ that encloses at least one of the reference points of $M(\langle \xi, \eta \rangle)$.*

8.11. Before proving Proposition 8.10, let us give the

Proof that Proposition 8.10 implies Theorem 8.2. According to Theorem 8.9 there is a dense set $\mathfrak{E} \subset \mathfrak{B}$ such that for every $(\alpha, \beta) \in \mathfrak{E}$, the group $\langle \alpha, \beta \rangle$ is geometrically finite and has three nonconjugate maximal cuspidal subgroups. For each $\varepsilon > 0$ and each $(\alpha, \beta) \in \mathfrak{E}$, Proposition 8.10 gives a neighborhood $U = U_\varepsilon(\alpha, \beta)$ of (α, β) in \mathfrak{B} . For every $\varepsilon > 0$ the set $W_\varepsilon = \bigcup_{(\alpha, \beta) \in \mathfrak{E}} U_\varepsilon(\alpha, \beta)$ is open and dense in \mathfrak{B} . On the other hand, by Proposition 8.8, $\mathfrak{A} \cap \mathfrak{B}$ is a dense G_δ in \mathfrak{B} . Hence the set $\mathfrak{C} = \mathfrak{A} \cap \bigcap_{0 < n \in \mathbb{Z}} W_{1/n}$ is a dense G_δ in \mathfrak{B} .

To complete the proof it suffices to show that for every $(\xi, \eta) \in \mathfrak{C}$ the group $\langle \xi, \eta \rangle$ is girded. If $(\xi, \eta) \in \mathfrak{C}$ then for every positive integer n there is a point $(\alpha_n, \beta_n) \in \mathfrak{E}$ such that $(\xi, \eta) \in \mathfrak{A} \cap U_{1/n}(\alpha_n, \beta_n)$. According to the defining property of $U_\varepsilon(\alpha_n, \beta_n)$ (see Proposition 8.10) this means that for every $n > 0$ there is a reference point z_{j_n} of $M(\langle \xi, \eta \rangle)$ that is enclosed by a connected $(1/n)$ -girded surface of genus 2 in $M(\langle \xi, \eta \rangle)$.

Since $M(\langle \xi, \eta \rangle)$ has only three reference points, it follows that there is a single reference point z of M that is enclosed by a connected $(1/n)$ -girded surface of genus 2 for arbitrarily large values of n ; it follows at once that $M(\langle \xi, \eta \rangle)$ is girded. \square

8.12. The rest of this section is devoted to the proof of Proposition 8.10. We suppose that we are given a point (α, β) of \mathcal{D} such that the free Kleinian group $\langle \alpha, \beta \rangle$ is geometrically finite and contains three distinct conjugacy classes of cuspidal subgroups. We set $\hat{\rho}_0 = \hat{\rho}_{\alpha, \beta}$, $p_0 = p_{\alpha, \beta}: \mathbf{H}^3 \rightarrow M(\langle \alpha, \beta \rangle)$, and $T_0 = T_{\alpha, \beta}$.

8.13. **Lemma.** *For each component L of $\text{Fix } T_0$ we have $N(\langle \alpha, \beta \rangle) \cap L \neq \emptyset$. Furthermore, there is a constant $\eta > 0$ such that for every $\varepsilon < \eta$ and every component L of $\text{Fix } T_0$ we have $N_{[\varepsilon, \infty)}(\langle \alpha, \beta \rangle) \cap L \neq \emptyset$.*

Proof. To prove the first assertion we choose a component l of $p_0^{-1}(L)$. Then l is a component of $\mathcal{L}(\alpha, \beta)$, and hence by 8.5 we have $l = \text{Fix } \tau$ for some order-2 element τ of $\hat{\rho}_0(\hat{F})$. Recall from 1.4 that the set $\tilde{N}(\langle \alpha, \beta \rangle) = p_0^{-1}(N(\langle \alpha, \beta \rangle)) = \text{nbhd}_1(\text{hull}(\Lambda_{\langle \alpha, \beta \rangle}))$ is $\langle \alpha, \beta \rangle$ -invariant and convex. Since τ normalizes $\langle \alpha, \beta \rangle$ (see 8.5), $\tilde{N}(\langle \alpha, \beta \rangle)$ is τ -invariant.

It follows that for any point $z \in \tilde{N}(\langle \alpha, \beta \rangle)$, the midpoint of the line segment joining z to τz lies in $N(\langle \alpha, \beta \rangle) \cap L$, and the first assertion is proved.

It follows from the first assertion that for any component L of $\text{Fix } T_0$ there is a number $\eta_L > 0$ such that $N_{[\eta_L, \infty)}(\langle \alpha, \beta \rangle) \cap L \neq \emptyset$. The second assertion now follows if we set $\eta = \min_L \eta_L$, where L ranges over the three components of $\text{Fix } T_0$. \square

8.14. Let ε_0 be a 3-dimensional Margulis constant such that conclusions (i) and (ii) of Proposition 1.7 hold with $\Gamma = \langle \alpha, \beta \rangle$. According to 1.7 we may take ε_0 to be arbitrarily small. In particular, we may suppose ε_0 to be chosen so that $N_{[\varepsilon_0, \infty)}(\langle \alpha, \beta \rangle) \cap L \neq \emptyset$ for every component L of $\text{Fix } T_0$.

By conclusion (i) of 1.7, $J = N_{[\varepsilon_0, \infty)}(\langle \alpha, \beta \rangle)$ is a compact, orientable, irreducible 3-manifold-with-boundary. Furthermore, J is a deformation retract of $M(\langle \alpha, \beta \rangle)$; in particular, J is connected and $\pi_1(J)$ is a free group of rank 2. According to [He, Theorem 5.2], any compact, connected, orientable, irreducible 3-manifold-with-boundary whose fundamental group is free of rank 2 is a genus-2 handlebody.

We set $\Sigma = \partial J$. Then Σ is a closed orientable surface of genus 2. Since the fundamental group of a handlebody is carried by its boundary and since J is a deformation retract of $M(\langle \alpha, \beta \rangle)$, the inclusion homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(M(\langle \alpha, \beta \rangle))$ is surjective.

Hence $\tilde{\Sigma} = p_0^{-1}(\Sigma)$ is a connected covering surface of Σ . Clearly $\tilde{\Sigma}$ is invariant under the action of \hat{F} on \mathbf{H}^3 defined by the representation $\hat{\rho}_0$. Restricting this action, we obtain a natural action of \hat{F} on $\tilde{\Sigma}$.

Since $\langle \alpha, \beta \rangle$ contains three conjugacy classes of cuspidal subgroups, the set $M_{(0, \varepsilon_0]}^c(\langle \alpha, \beta \rangle)$ has three components. Each component of $M_{(0, \varepsilon_0]}^c(\langle \alpha, \beta \rangle)$ has

the form C/Π , where Π is a cuspidal subgroup of Γ and C is a horoball stabilized by Π . Since $\langle\alpha, \beta\rangle$ is free, each of these groups Π is cyclic. Thus $M_{(0, \varepsilon_0]}(\langle\alpha, \beta\rangle)$ has the homotopy type of a disjoint union of three circles. Hence if we set $\mathcal{A} = N_{\{\varepsilon_0\}}(\langle\alpha, \beta\rangle)$, it follows from conclusion (ii) of 1.7 that \mathcal{A} is a disjoint union of three annuli in Σ , all homotopically nontrivial in $M(\langle\alpha, \beta\rangle)$.

We set $\tilde{\mathcal{A}} = p_0^{-1}(\mathcal{A}) \subset \tilde{\Sigma} \subset \mathbf{H}^3$. Each component \tilde{A} of $\tilde{\mathcal{A}}$ is contained in a unique horoball in \mathbf{H}^3 , which we denote by $C(\tilde{A})$. We denote the base point of $C(\tilde{A})$ in S_∞ by $\zeta(\tilde{A})$. The stabilizer of \tilde{A} in $\langle\xi, \eta\rangle$ is an infinite cyclic group consisting of parabolic elements, and this group leaves $C(\tilde{A})$ invariant and fixes $\zeta(\tilde{A})$.

Since the three annuli comprising \mathcal{A} correspond to different conjugacy classes of (cuspidal) maximal infinite cyclic subgroups of Γ , their cores are noncontractible and (freely) nonhomotopic simple closed curves in $M(\langle\alpha, \beta\rangle)$. In particular, they are noncontractible and nonhomotopic in Σ . Hence each component of $\overline{\Sigma - \mathcal{A}}$ is a planar surface with three boundary curves, and at least two of the components of \mathcal{A} are nonseparating annuli in Σ .

It is apparent from the definitions of J and \mathcal{A} that they are invariant under any self-isometry of $M(\langle\alpha, \beta\rangle)$. Thus, in particular, J is invariant under T_0 so that T_0 induces an involution T_Σ of Σ ; \mathcal{A} is invariant under T_Σ . (We shall sharpen this assertion in Lemma 8.16.)

Observe also that since T_0 is induced by an orientation-preserving isometry of \mathbf{H}^3 , it is itself orientation-preserving. It follows that $T_0|J$ is orientation-preserving and, therefore, so is T_Σ .

8.15. Lemma. *For every component L of $\text{Fix } T_0$ we have $\text{card}(L \cap \Sigma) = 2$.*

Proof. Recall from 8.14 that we chose ε_0 in such a way that $J \cap L = N_{[\varepsilon_0, \infty)}(\langle\alpha, \beta\rangle) \cap L \neq \emptyset$ for every component L of $\text{Fix } T_0$. Since $J \cap L$ is a union of components of the fixed set of the involution $T_0|J$ of the compact 3-manifold-with-boundary J , it is a properly embedded 1-manifold-with-boundary in J ; it has no closed components since L is homeomorphic to \mathbf{R} by 8.6. Thus for each component L of $\text{Fix } T_0$, the set $\Sigma \cap L = \partial(J \cap L)$ has nonzero, finite, even cardinality. On the other hand, the sum of the cardinalities of $\Sigma \cap L$ as L ranges over the three components of $\text{Fix } T_0$ is equal to $\text{card } \text{Fix } T_\Sigma$, the number of fixed points of an orientation-preserving involution of a closed connected surface which by the Hurwitz branching formula is at most 6. Hence we must have $\text{card}(\Sigma \cap L) = 2$ for each component L of $\text{Fix } T_0$. \square

8.16. Lemma. *Each component of \mathcal{A} is invariant under T_Σ .*

Proof. If the assertion is false then two components A_1 and A_2 are interchanged by T_Σ . Since \mathcal{A} has at least two nonseparating components, A_1 and A_2 cannot both separate Σ ; since they are interchanged by the homeomorphism T_Σ , they are both nonseparating. Hence $\Sigma' = \overline{\Sigma - (A_1 \cup A_2)}$ is connected. Since neither A_1 nor A_2 can contain a fixed point of T_Σ and since T_Σ has six fixed points by Lemma 8.15, $T_{\Sigma'} = T_\Sigma|_{\Sigma'}$ is an orientation-preserving involution,

with six fixed points, of a connected surface having nonempty boundary and Euler characteristic -2 . It then follows from the Hurwitz branching formula that $\Sigma'/T_{\Sigma'}$ is a connected surface having nonempty boundary and Euler characteristic 2 . But no such surface exists. \square

8.17. It follows from Lemma 8.16 that for each component A of \mathcal{A} we may choose a T_{Σ} -invariant core curve S_A . Having made such a choice, we set $S = \bigcup_A S_A$, where A ranges over the components of \mathcal{A} . It follows from 8.14 that the S_A are homotopically nontrivial curves in $M(\langle\alpha, \beta\rangle)$, that at least two of the S_A are nonseparating curves in Σ , and that each component of $\Sigma - S$ is a three-punctured sphere. We set $S = p_0^{-1}(S) \subset \tilde{\Sigma}$.

Lemma. *For each component R of $\Sigma - S$, the inclusion homomorphism from $\pi_1(R)$ to $\pi_1(M(\langle\alpha, \beta\rangle))$ is injective.*

Proof. Since J is a deformation retract of $M(\langle\alpha, \beta\rangle)$ by 8.14, it suffices to show that the inclusion homomorphism $\pi_1(R) \rightarrow \pi_1(J)$ is injective. If it is not, it follows from the loop theorem [He, Theorem 4.2] that some nontrivial element of $\ker(\pi_1(R) \rightarrow \pi_1(J))$ is represented by a simple closed curve in R . Since R is a three-punctured sphere, every simple closed curve in R is parallel to a component of $\partial R \subset S$. But every component of S is homotopically nontrivial in $M(\langle\alpha, \beta\rangle)$ and hence in J . \square

8.18. **Lemma.** *Every nonseparating component of S meets $\text{Fix } T_{\Sigma}$ in two points. If C is a separating component of S then C is disjoint from $\text{Fix } T_{\Sigma}$ and each component of $\Sigma - C$ is T_{Σ} -invariant.*

Proof. It follows from Lemmas 8.15 and 8.16 that T_{Σ} is an orientation-preserving involution of Σ with six fixed points and that each component of S is invariant under T_{Σ} . The Hurwitz branching formula implies that the orbit surface $\Psi = \Sigma/T_{\Sigma}$ is a sphere. Let $q: \Sigma \rightarrow \Psi$ denote the quotient map.

If C is any component of S , either C contains two fixed points of T_{Σ} and T_{Σ} reverses the orientation of C , or C contains no fixed point of T_{Σ} and T_{Σ} preserves the orientation of C . If C separates Σ then since the other components of S are T_{Σ} -invariant, each component of $\Sigma - C$ must be T_{Σ} -invariant; hence T_{Σ} preserves the orientation of C . Conversely, if T_{Σ} preserves the orientation of C then $q(C)$ is a simple closed curve and therefore separates Ψ . Hence $C = q^{-1}(q(C))$ separates Σ . \square

8.19. **Lemma.** *There is a T_{Σ} -invariant virtual triangulation of Σ such that*

- (i) $\text{skel}_0(\Sigma) \subset S$,
- (ii) $S \cup \text{Fix } T_{\Sigma} \subset \text{skel}_1(\Sigma)$, and
- (iii) no virtual 2-simplex has more than one edge supported in S .

Proof. Let us label the components of S as C_1, C_2 , and C_3 , where C_1 and C_2 are nonseparating curves in Σ , and C_3 may or may not separate Σ .

Consider the case in which C_3 is nonseparating. Then $\Sigma - S$ has two components Y_1 and Y_2 , each of which is the interior of a compact planar subsurface of Σ with three boundary curves. Since the C_i are nonseparating, T_{Σ} reverses the orientation of C_i ; hence T_{Σ} interchanges Y_1 and Y_2 . By Lemma 8.18,

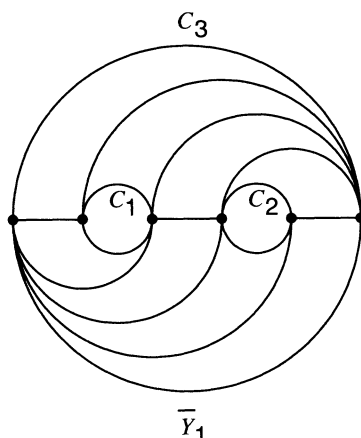


FIGURE 1

each C_i contains two fixed points of T_Σ . Thus $\text{Fix } T_\Sigma$ is a subset of $\partial \bar{Y}_1$ containing exactly two points of each component of $\partial \bar{Y}_1$. It follows (see Figure 1) that \bar{Y}_1 has a virtual triangulation with $\text{Fix } T_\Sigma$ as its 0-skeleton, and in which no virtual 2-simplex has more than one edge supported in $\partial \bar{Y}_1$.

The induced triangulation of $\partial \bar{Y}_1$ is automatically T_Σ -invariant, and hence the virtual triangulation of \bar{Y}_1 extends to a unique T_Σ -invariant virtual triangulation of Σ ; this extended triangulation clearly has the required properties.

Now consider the case in which C_3 separates Σ . For $i = 1, 2$, let J_i denote the component of $\Sigma - C_3$ containing C_i . As in the proof of Lemma 8.18 we observe that $\Psi = \Sigma/T_\Sigma$ is a sphere and we let $q: \Sigma \rightarrow \Psi$ denote the quotient map. Since J_i is T_Σ -invariant by Lemma 8.16, J_i is a 2-fold branched cover of the disk $q(J_i) \subset \Psi$. But J_i is a genus-1 surface with connected boundary and, therefore, it follows from the Hurwitz branching formula that J_i contains exactly three fixed points of T_Σ . Hence there is a unique fixed point u_i of T_Σ lying in $J_i - C_i$. Now let v be any point of C_3 . For $i = 1, 2$, there is a T_Σ -invariant, properly embedded arc α_i in J_i that has v and $T_\Sigma(v)$ as end points and u_i as an interior point: we may in fact define $\alpha_i = q^{-1}(a_i)$, where a_i is an arc in $q(J_i)$ joining $q(v)$ to $q(u_i)$. If α_i separates J_i , then since T_Σ clearly reverses the orientation of α_i , it must interchange the two components of $J_i - \alpha_i$; this implies that J_i has even genus, a contradiction. Hence α_i is a nonseparating arc in J_i .

It follows that $C'_3 = \alpha_1 \cup \alpha_2$ is a nonseparating simple closed curve in Σ . Furthermore, since C'_3 intersects each J_i in an arc, it cannot be parallel to C_1 or C_2 . Hence if we set $S' = C_1 \cup C_2 \cup C'_3$, then $\Sigma - S'$ has two components Y_1 and Y_2 , each of which is the interior of a compact planar subsurface of Σ with three boundary curves. Since the components of S' are nonseparating, T_Σ reverses the orientation of each component of S' ; hence T_Σ interchanges Y_1 and Y_2 . The set

$$K = ((\text{Fix } T_\Sigma) \cap (C_1 \cup C_2)) \cup \{v, T_\Sigma(v)\} \subset \partial \bar{Y}_1$$

contains exactly two points of each component of $\partial\overline{Y_1}$. It follows that $\overline{Y_1}$ has a virtual triangulation with K as its 0-skeleton, and in which no virtual 2-simplex has more than one edge supported in $\partial\overline{Y_1}$. Note also that the arc $a = C_3 \cap \overline{Y_1}$ is not boundary parallel in $\overline{Y_1}$; for otherwise a would be parallel to one of the α_i , and this α_i would separate J_i , a contradiction. It follows that we may choose the virtual triangulation of $\overline{Y_1}$ in such a way that a is contained in the 1-skeleton. As in the previous case, the induced triangulation of $\partial\overline{Y_1}$ is T_Σ -invariant so that the virtual triangulation of $\overline{Y_1}$ extends to a unique T_Σ -invariant virtual triangulation of Σ ; and again the extended triangulation clearly has the required properties. \square

8.20. For the rest of this section we fix a virtual triangulation of Σ having the properties stated in Lemma 8.19. Since the virtual triangulation of Σ is T_Σ -invariant, it induces an \widehat{F} -invariant virtual triangulation of $\widetilde{\Sigma}$. We shall always understand $\widetilde{\Sigma}$ to be equipped with this virtual triangulation.

Let (ξ, η) be any point in \mathcal{D} , and let \tilde{f} be a map of $\widetilde{\Sigma}$ into \mathbf{H}^3 . We shall say that \tilde{f} is (ξ, η) -good if (i) \tilde{f} is hyperbolically simplicial, (ii) $\tilde{f}|_{\widetilde{S}}$ is geodesic, and (iii) \tilde{f} is \widehat{F} -equivariant with respect to the standard action of \widehat{F} on $\widetilde{\Sigma}$ and the action on \mathbf{H}^3 associated to (ξ, η) . Any (ξ, η) -good map $\tilde{f}: \widetilde{\Sigma} \rightarrow \mathbf{H}^3$ induces a hyperbolically simplicial map $f: \Sigma \rightarrow M((\xi, \eta))$, and $f|_S$ is geodesic. The \widehat{F} -equivariance of \tilde{f} implies that f satisfies $T_{\xi, \eta} \circ f = f \circ T_\Sigma$. If \tilde{f} is nondegenerate (as a hyperbolically simplicial map) then so is f .

8.21. **Lemma.** *For every $(\xi, \eta) \in \mathcal{A}$ there exists a (ξ, η) -good map $\tilde{f} = \tilde{f}_{\xi, \eta}: \widetilde{\Sigma} \rightarrow \mathbf{H}^3$.*

Proof. It follows from Lemma 8.19 that $\text{skel}_0(\widetilde{\Sigma}) \subset \widetilde{S} \subset \text{skel}_1(\widetilde{\Sigma})$. The first step in the construction of \tilde{f} is to construct a geodesic, \widehat{F} -equivariant map $\tilde{f}_S: \widetilde{S} \rightarrow \mathbf{H}^3$.

Let us denote the components of S by C_1, C_2 , and C_3 . For $i = 1, 2, 3$, we choose a component \widetilde{S}_i of $p_0^{-1}(C_i)$. We let G_i and \widehat{G}_i denote the stabilizers of \widetilde{S}_i in F and \widehat{F} , respectively. Then G_i is infinite cyclic, and it has index 2 in \widehat{G}_i since C_i is T_Σ -invariant. Hence \widehat{G}_i is either infinite cyclic or infinite dihedral. If γ_i denotes a generator of G_i then γ_i has infinite order in F , and γ_i is not parabolic since $(\xi, \eta) \in \mathcal{A}$. Hence γ_i has a unique axis $Z_i \subset \mathbf{H}^3$. Since \widehat{G}_i normalizes G_i , the line Z_i is \widehat{G}_i -invariant.

Up to equivariant homeomorphism, each of the groups \mathbf{Z} and D_∞ has a unique properly discontinuous action on \mathbf{R} . Hence there is a \widehat{G}_i -equivariant homeomorphism $\tilde{f}_i: \widetilde{S}_i \rightarrow Z_i$. Using the maps \tilde{f}_i we define \tilde{f}_S as follows. Given any point $u \in \widetilde{S}$ there exist an index $i \in \{1, 2, 3\}$ and an element γ of \widehat{F} such that $\gamma \cdot u \in \widetilde{S}_i$; the index i is uniquely determined by u , but the element γ is not. We wish to define $\tilde{f}_S(u) = \gamma^{-1} \cdot \tilde{f}_i(\gamma \cdot u)$. If γ_1 is another element of \widehat{F} such that $\gamma_1 \cdot u \in \widetilde{S}_i$, then $\delta = \gamma_1 \gamma^{-1}$ belongs to \widehat{G}_i , and by the

\tilde{G}_i -equivariance of \tilde{f}_i we have

$$\gamma_1^{-1} \cdot \tilde{f}_i(\gamma_1 \cdot u) = \gamma^{-1} \delta^{-1} \cdot \tilde{f}_i(\delta \gamma \cdot u) = \gamma^{-1} \cdot \tilde{f}_i(\gamma \cdot u).$$

Thus f_S is well defined. It follows readily from the definition that it is \hat{F} -equivariant. The next step is to extend \tilde{f}_S to an \hat{F} -equivariant map of $\text{skel}_1(\tilde{\Sigma})$ into \mathbf{H}^3 . Let $\phi: \Delta \rightarrow \tilde{\Sigma}$ be a virtual 1-simplex in the virtual triangulation of $\tilde{\Sigma}$ that is not carried by \tilde{S} . Let $f_\phi: \Delta \rightarrow \mathbf{H}^3$ denote the map that agrees with $\tilde{f}_S \circ (\phi|_{\partial\Delta})$ on the boundary of Δ and is geodesic in the sense that if we identify Δ with an interval in \mathbf{R} by any affine homeomorphism then f_ϕ becomes a geodesic path (parametrized proportionally to arc length). (In particular, if $\tilde{f}_S \circ (\phi|_{\partial\Delta})$ maps the two boundary points of Δ to the same point, then f_ϕ is constant.) There is a unique map $\tilde{f}^{(1)}: \text{skel}_1(\tilde{\Sigma}) \rightarrow \mathbf{H}^3$ such that $\tilde{f}^{(1)}|_{\tilde{S}} = \tilde{f}_S$ and such that for every virtual 1-simplex ϕ in the virtual triangulation of $\tilde{\Sigma}$ that is not carried by \tilde{S} we have $\tilde{f}^{(1)} \circ \phi = f_\phi$. Since \tilde{f}_S is \hat{F} -equivariant, so is $\tilde{f}^{(1)}$.

To extend $\tilde{f}^{(1)}$ to $\hat{\Sigma}$ we may use the following construction. Let $\phi: \Delta \rightarrow \tilde{\Sigma}$ be any 2-simplex. Let m denote the barycenter of Δ . Let δ denote the convex hull of $\tilde{f}^{(1)}(\phi(\Delta^{(0)}))$, where $\Delta^{(0)}$ denotes the set of vertices of Δ . Then δ is either a hyperbolic triangle, a hyperbolic line segment, or a point. If δ is a hyperbolic triangle we let μ denote its centroid [F, p. 125]. If δ is a line segment we define μ to be its midpoint, and if δ consists of a single point we define μ to be this point. Now we define a map $f_\phi: \Delta \rightarrow \delta$ by stipulating that $f_\phi|_{\partial\Delta} = \tilde{f}^{(1)} \circ (\phi|_{\partial\Delta})$, that $f_\phi(m) = \mu$, and that for each affine segment $\sigma \subset \Delta$ joining m to a point in $\partial\Delta$, the map $f_\phi|_\sigma$ be geodesic (in the above sense). Now there is a unique map $\tilde{f}: \hat{\Sigma} \rightarrow \mathbf{H}^3$ such that $\tilde{f}|_{\text{skel}_1(\tilde{\Sigma})} = \tilde{f}^{(1)}$ and such that for every virtual 2-simplex ϕ in the virtual triangulation of $\tilde{\Sigma}$ we have $\tilde{f} \circ \phi = f_\phi$. Since $\tilde{f}^{(1)}$ is \hat{F} -equivariant, so is \tilde{f} . It is clear from the construction that \tilde{f} is hyperbolically simplicial and that $\tilde{f}|_{\tilde{S}}$ is geodesic. \square

8.22. Lemma. *Let $(\xi^i, \eta^i)_{i \geq 1}$ be a sequence of points in \mathfrak{A} that converges in \mathfrak{D} to (α, β) . For each $i \geq 1$ let $\tilde{f}^i: \tilde{\Sigma} \rightarrow \mathbf{H}^3$ be a (ξ^i, η^i) -good map. Let \tilde{C} be any component of \tilde{S} and let \tilde{A} denote the unique component of $\tilde{\mathcal{A}}$ containing \tilde{C} . Then for any point $z \in \tilde{S}$ we have $\lim_{i \rightarrow \infty} \tilde{f}^i(z) = \zeta(\tilde{A})$ in $\overline{\mathbf{H}^3}$. Furthermore, if z and z' are distinct points of \tilde{C} and U is any neighborhood of $\zeta(\tilde{A})$ in $\overline{\mathbf{H}}$, then we have $\tilde{f}^i(z) \neq \tilde{f}^i(z')$ for every i , and for sufficiently large i the line joining $\tilde{f}^i(z)$ to $\tilde{f}^i(z')$ is contained in U .*

(The definition of $\zeta(\tilde{A})$ was given in 8.14.)

Proof. Let $F_{\tilde{C}}$ denote the stabilizer of \tilde{C} under the action of F on \mathbf{H}^3 defined by ρ_0 . By 8.14 and 8.17, $F_{\tilde{C}}$ is cyclic and is generated by a parabolic element h , and $\rho_0(h)$ fixes $\zeta = \zeta(\tilde{A})$. For each $i \geq 1$, since \tilde{f}^i is a (ξ, η) -good map,

$\tilde{f}^i(\tilde{C})$ is a geodesic that is invariant under $\rho_{\xi^i, \eta^i}(h)$. Thus $\tilde{f}^i(\tilde{C})$ is the axis of the isometry $\rho_{\xi^i, \eta^i}(h)$ (which is loxodromic since $(\xi^i, \eta^i) \in \mathfrak{D}$). As $i \rightarrow \infty$ we have $\rho_{\xi^i, \eta^i}(h) \rightarrow \rho_0(h)$. Hence if U is any neighborhood of the fixed point ζ of $\rho_0(h)$ in \bar{H} , the end points of the line $\tilde{f}^i(\tilde{C})$ lie in U for all sufficiently large i . Hence for large i we have $\tilde{f}^i(\tilde{C}) \subset U$. This implies the first assertion of the lemma.

To prove the second assertion we observe that since the geodesic map $\tilde{f}^i|_{\tilde{C}}$ is $F_{\tilde{C}}$ -equivariant, it is nonconstant and, therefore, is a homeomorphism. This shows that for every i we have $\tilde{f}^i(z) \neq \tilde{f}^i(z')$ and that $\tilde{f}^i(\tilde{C})$ is the unique line joining $\tilde{f}^i(z)$ to $\tilde{f}^i(z')$. Since we have shown that $\tilde{f}^i(\tilde{C}) \subset U$ for large i , the second assertion follows. \square

8.23. Lemma. *Let $(\xi^i, \eta^i)_{i \geq 1}$ be a sequence of points in \mathfrak{A} that converges in \mathfrak{D} to (α, β) . For each $i \geq 1$ let $\tilde{f}^i: \Sigma \rightarrow \mathbf{H}^3$ be a (ξ^i, η^i) -good map, and let $f^i: \Sigma \rightarrow M(\langle \xi^i, \eta^i \rangle)$ be the map induced by \tilde{f}^i . Let ε be any positive number. Then for all sufficiently large i the triple (Σ, f^i, S) is an ε -girded surface.*

Proof. We must show that for sufficiently large i the hyperbolically simplicial surface (Σ, f^i) is nondegenerate and conditions (i)–(iv) of the definition 7.1 of an ε -girded surface hold. Conditions (i) and (ii) are properties of the virtually triangulated surface Σ and the 1-manifold $S \subset \Sigma$ that have been established in 8.17 and 8.19. We claim that condition (iv) holds for every i . Indeed, the isomorphisms $\rho_0: F \rightarrow \langle \alpha, \beta \rangle$ and $\rho_{(\xi^i, \eta^i)}: F \rightarrow \langle \xi^i, \eta^i \rangle$ give rise to an isomorphism (well-defined modulo inner automorphisms) between $\langle \alpha, \beta \rangle = \pi_1(M(\langle \alpha, \beta \rangle))$ and $\langle \xi^i, \eta^i \rangle = \pi_1(M(\langle \xi^i, \eta^i \rangle))$. If we identify $\pi_1(M(\langle \alpha, \beta \rangle))$ and $\pi_1(M(\langle \xi^i, \eta^i \rangle))$ via this isomorphism, then the equivariance condition in the definition of a good map implies that $f_{\#}: \pi_1(\Sigma) \rightarrow \pi_1(M(\langle \xi^i, \eta^i \rangle))$ is identified (modulo inner automorphisms) with the inclusion homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(M(\langle \alpha, \beta \rangle))$. Our claim therefore follows from Lemma 8.17.

Next we show that condition (iii) in the definition of an ε -girded surface holds for large i . It follows from the definition of a (ξ, η) -good map that for every i the map $f^i|_S$ is geodesic. Now let S be any component of S . Then S is a core curve of some component A of \mathcal{A} . Choose any component \tilde{A} of $p_0^{-1}(A)$, and let $F_{\tilde{A}}$ denote the stabilizer of \tilde{A} in F . Then $F_{\tilde{A}}$ is also the stabilizer of the component \tilde{C} of \tilde{S} contained in \tilde{A} . By 8.14, $F_{\tilde{A}}$ is cyclic and is generated by an element h such that $\rho_0(h)$ is parabolic. Since $(\xi^i, \eta^i) \rightarrow (\alpha, \beta)$ as $i \rightarrow \infty$, it follows that $\text{trace } \rho_{\xi^i, \eta^i}(h) \rightarrow \text{trace } \rho_0(h) = \pm 2$. Hence the translation length of $\rho_{\xi^i, \eta^i}(h)$ approaches 0 as $i \rightarrow \infty$. But the closed geodesic defined by the map $f^i|_S$ is $\tilde{C}/F_{\tilde{A}}$. Hence the length of $f^i|_S$ approaches 0 as $i \rightarrow \infty$ and, in particular, is $< \varepsilon$ for large i .

It remains to show that (Σ, f^i) is nondegenerate for large i . Recall from the definition (6.5) that for every virtual simplex $\phi: \Delta \rightarrow \Sigma$ and every i , the map $f_{\phi}^i = f^i \circ \phi$ admits a continuous lift $\tilde{f}_{\phi}^i: \Delta \rightarrow \mathbf{H}^n$ such that $\tilde{f}_{\phi}^i(\Delta)$ is

a hyperbolic simplex and either (i) $\dim \tilde{f}_\phi^i(\Delta) < \dim \Delta$ or (ii) \tilde{f}_ϕ^i maps Δ homeomorphically onto $\tilde{f}_\phi^i(\Delta)$. To say that (Σ, f^i) is nondegenerate for a given i means that (ii) holds for every virtual simplex Δ of Σ . It is clearly enough to show this for the case of a virtual 2-simplex. Since Δ has only finitely many simplices, it suffices to show that for each virtual 2-simplex ϕ there is an integer i_ϕ such that (ii) holds for every $i \geq i_\phi$.

Let $\tilde{\phi}$ be a virtual 2-simplex in $\tilde{\Sigma}$ that covers ϕ . Let v_j ($j = 1, 2, 3$) denote the vertices of $\tilde{\phi}$. If (i) holds for a given i then the points $\tilde{f}^i(v_1)$, $\tilde{f}^i(v_2)$, and $\tilde{f}^i(v_3)$ are collinear in \mathbf{H}^3 . Thus we need only show that the $\tilde{f}^i(v_j)$ are noncollinear for large i . According to conclusion (i) of Lemma 8.19, each v_j lies in some component of \tilde{S} . It follows from conclusion (iii) of Lemma 8.19 that at most two of the v_j can lie in any given component of \tilde{S} .

Consider first the case in which all the v_j lie in three distinct components of \tilde{S} . Let \tilde{C}_j denote the component of \tilde{S} containing v_j , and let \tilde{A}_j denote the component of $\tilde{\mathcal{A}}$ containing \tilde{C}_j . Then the horospheres $C(\tilde{A}_1)$, $C(\tilde{A}_2)$, and $C(\tilde{A}_3)$ are distinct components of $p_0^{-1}(M_{\{\epsilon_0\}}((\alpha, \beta)))$. This implies that the base points $\zeta(\tilde{A}_j)$ of the $C(\tilde{A}_j)$ are three mutually distinct points in S_∞ . But by Lemma 8.22 we have for $j = 1, 2, 3$ that $\tilde{f}^i(v_j) \rightarrow \zeta(\tilde{A}_j)$ as $i \rightarrow \infty$. It follows that the three points $\tilde{f}^i(v_j)$, $j = 1, 2, 3$, are noncollinear when i is large.

There remains the case in which two of the v_j , say v_1 and v_2 , lie in the same component \tilde{C} of \tilde{S} , and the third, say v_3 , lies in a component $\tilde{C}' \neq \tilde{C}$ of \tilde{S} . Let \tilde{A} and \tilde{A}' denote the components of $\tilde{\mathcal{A}}$ containing \tilde{C} and \tilde{C}' , respectively. Then $\zeta(\tilde{A})$ and $\zeta(\tilde{A}')$ are distinct points of S_∞ and hence have disjoint neighborhoods U and U' in $\bar{\mathbf{H}}$. By the first assertion of Lemma 8.22 we have $\tilde{f}^i(v_3) \in U'$ for large i . On the other hand, the second assertion of Lemma 8.22 implies that $\tilde{f}^i(v_1) \neq \tilde{f}^i(v_2)$ for all i and that for large i the line joining $\tilde{f}^i(v_1)$ to $\tilde{f}^i(v_2)$ is contained in U . Since $U \cap U' = \emptyset$, it follows that the $\tilde{f}^i(v_j)$ are noncollinear for large i . \square

8.24. Lemma. *Let (ξ, η) be any point of \mathfrak{A} , let \tilde{f} be any (ξ, η) -good map, and let $f: \Sigma \rightarrow M((\xi, \eta))$ be the map induced by \tilde{f} . Let L be any component of $\text{Fix } T_{\xi, \eta}$. Then $\text{Fix } T_\Sigma \subset \Sigma$ contains exactly two points, say P_+ and P_- , which are mapped into L by f . Furthermore, the arc in L with end points $f(P_+)$ and $f(P_-)$ is enclosed by the singular surface (Σ, f) .*

Proof. To prove the first assertion we use the notation of 8.5: thus the components of $\text{Fix } T_{\xi, \eta}$ are $L_j(\xi, \eta)$ for $j = 1, 2, 3$. We must show that for each j there are exactly two of the six fixed points of $\text{Fix } T_\Sigma$ that are mapped by f into $L_j(\xi, \eta)$. The equivariance condition in the definition of a good map shows that $\tilde{f}(l_j(\alpha, \beta)) = \tilde{f}(\text{Fix } \tau_{\alpha, \beta, j}) \subset \text{Fix } \tau_{\xi, \eta, j} = l_j(\xi, \eta)$. Hence

$f(L_j(\alpha, \beta)) \subset L_j(\xi, \eta)$. In particular, f maps $L_j(\alpha, \beta) \cap \Sigma$ into $L_j(\xi, \eta)$. But $L_j(\alpha, \beta) \cap \Sigma$ consists of fixed points of T_0 and has cardinality 2 by Lemma 8.15. Thus the first assertion is proved. Since $T_{(\xi, \eta)} \circ f = f \circ T_\Sigma$ by 8.20, the second assertion follows from the first by virtue of Proposition 6.10. \square

Proof of Proposition 8.2. It suffices to prove that if ε is a positive number and if (ξ^i, η^i) is a sequence of points in \mathfrak{A} that converges to (α, β) , then for every sufficiently large i there exists an ε -girded surface in $M((\xi, \eta))$ that encloses at least one of the reference points of $M((\xi, \eta))$. According to Lemma 8.21, for each i there exists a (ξ^i, η^i) -good map $\tilde{f}^i: \tilde{\Sigma} \rightarrow \mathbf{H}^3$. Let $f^i: \Sigma \rightarrow M((\alpha, \beta))$ denote the map induced by \tilde{f}^i . By Lemma 8.23, (Σ, f^i, S) is an ε -girded surface for every sufficiently large i . Thus in order to prove the proposition it suffices to show that for every sufficiently large i the hyperbolically simplicial surface (Σ, f^i) encloses some reference point of $M((\xi^i, \eta^i))$.

Recall from 8.17 that S has at least two nonseparating components; by Lemma 8.18, each such component contains at least two fixed points of T_Σ . Thus $\text{card}(S \cap \text{Fix } T_\Sigma) \geq 4$. On the other hand, we have $\text{Fix } T_\Sigma \subset \text{Fix } T_0$, and T_0 has three components. Hence there must be two points $P_+, P_- \in S \cap \text{Fix } T_\Sigma$ that lie in the same component L of $\text{Fix } T_0$. By Lemma 8.15 we must have $L \cap \tilde{\Sigma} = \{P_+, P_-\}$. In the notation of 8.5, we have $L = L_{j_0}(\xi, \eta)$ for some $j_0 \in \{1, 2, 3\}$. Let us set $l = l_{j_0}(\alpha, \beta)$, so that $L = p_0(l)$. Then $\tilde{\Sigma}$ meets l in two points $\tilde{P}_+ \in p_0^{-1}(P_+)$ and $\tilde{P}_- \in p_0^{-1}(P_-)$.

By 8.5, l is the fixed line of $\tau_{\alpha, \beta, j_0} = \hat{\rho}_0(t_{j_0})$. For each $i \geq 1$, we let l^i denote the fixed line of $\tau_{\xi^i, \eta^i, j_0} = \hat{\rho}_{\xi^i, \eta^i}(t_{j_0})$, and set $L^i = p_{\xi^i, \eta^i}(l^i)$. Since \tilde{f}^i is \hat{F} -equivariant for every i , we have $\tilde{f}^i(\tilde{P}_-), \tilde{f}^i(\tilde{P}_+) \in l^i$. Hence $f^i(P_-), f^i(P_+) \in L^i$. If α^i denotes the arc in L^i with end points $f^i(P_-), f^i(P_+)$, it follows from Lemma 8.24 that (Σ, f^i) encloses α^i . We shall complete the proof of the proposition by showing that α^i contains a reference point of $M((\xi^i, \eta^i))$ for sufficiently large i .

We have $\tilde{P}_-, \tilde{P}_+ \in \tilde{S} \subset \tilde{\mathcal{A}}$. Let $\tilde{C}_+ \subset \tilde{A}_+$ and $\tilde{C}_- \subset \tilde{A}_-$ denote the components of \tilde{S} and $\tilde{\mathcal{A}}$ containing \tilde{P}_+ and \tilde{P}_- respectively. Since $\tilde{\mathcal{A}} = p_0^{-1}(N_{\{\varepsilon_0\}}((\alpha, \beta)))$ is invariant under the action of \hat{F} on \mathbf{H}^3 defined by the representation $\hat{\rho}_0$ and since the involution $\tau_{\alpha, \beta, j_0}$ fixes \tilde{P}_- and \tilde{P}_+ , the sets \tilde{A}_- and \tilde{A}_+ are invariant under $\tau_{\alpha, \beta, j_0}$. Hence $\zeta(\tilde{A}_\pm)$, which is the base point of the unique horoball $C(\tilde{A}_\pm)$ containing \tilde{A}_\pm , is fixed by $\tau_{\alpha, \beta, j_0}$. It follows that each of the points $\zeta(\tilde{A}_-)$ and $\zeta(\tilde{A}_+)$ is an end point of the line l . We shall show that $\zeta(\tilde{A}_-) \neq \zeta(\tilde{A}_+)$, so that $\zeta(\tilde{A}_-)$ and $\zeta(\tilde{A}_+)$ are the two end points of l .

Since the horospheres $C(\tilde{A}_-)$ and $C(\tilde{A}_+)$ contain the end points of l , they meet l orthogonally; hence each of these horospheres meets l in at most a single point. Since $P_- \neq P_+$, it follows that $C(\tilde{A}_-) \neq C(\tilde{A}_+)$. Thus the

horospheres $C(\tilde{A}_-)$ and $C(\tilde{A}_+)$ are distinct components of $p_0^{-1}(M_{\{\epsilon_0\}}(\langle \alpha, \beta \rangle))$. This implies that the base points $\zeta(\tilde{A}_-)$ and $\zeta(\tilde{A}_+)$ of $C(\tilde{A}_-)$ and $C(\tilde{A}_+)$ are distinct. Thus $\zeta(\tilde{A}_-)$ and $\zeta(\tilde{A}_+)$ are indeed the two end points of l . For each $i \geq 1$, let \tilde{P}^i denote the point of l^i that is closest to w_0 . By definition, $P^i = p_{\xi^i, \eta^i}(\tilde{P}^i)$ is a reference point of $M(\langle \xi_i, \eta_i \rangle)$. Let \tilde{P} denote the point of l that is closest to w_0 . By 1.9 we have $l^i \rightarrow l$ as $i \rightarrow \infty$, so that $\tilde{P}^i \rightarrow \tilde{P}$. On the other hand, since $\tilde{P}_+ \in \tilde{S}_+$ and $\tilde{P}_- \in \tilde{S}_-$, Lemma 8.22 gives that $\tilde{f}^i(\tilde{P}_+) \rightarrow \zeta(\tilde{A}_+)$ and $\tilde{f}^i(\tilde{P}_-) \rightarrow \zeta(\tilde{A}_-)$ in $\bar{\mathbf{H}}$ as $i \rightarrow \infty$. Thus we have a sequence of triples of points $(\tilde{f}^i(\tilde{P}_-), \tilde{P}^i, \tilde{f}^i(\tilde{P}_+))_{i \geq 1}$; the points in the i th triple all lie in the line l_i ; and in the limit triple $(\zeta(\tilde{A}_-), \tilde{P}, \zeta(\tilde{A}_+))$, the point \tilde{P} lies on the line $l \subset \mathbf{H}^3$, whereas $\zeta(\tilde{A}_-)$ and $\zeta(\tilde{A}_+)$ are the end points of l . It follows that for sufficiently large i the point \tilde{P}^i lies between $\tilde{f}^i(\tilde{P}_-)$ and $\tilde{f}^i(\tilde{P}_+)$.

By Proposition 8.6 this means that for sufficiently large i the point P^i lies between $f^i(P_-)$ and $f^i(P_+)$, so that $P^i \in \alpha^i$ for large i , as required. \square

9. ESTIMATES FOR GEOMETRICALLY FINITE GROUPS

9.1. The following theorem is the main result of this paper.

Theorem. *Let ξ and η be noncommuting isometries of \mathbf{H}^3 . Suppose that ξ and η generate a torsion free discrete group that is topologically tame, is not cocompact, and contains no parabolics. Then for any $z \in \mathbf{H}^3$ we have*

$$\max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) \geq \log 3.$$

The following two results are used in the proof of Theorem 9.1 and put the statement in perspective.

9.2. **Proposition.** *Let ξ and η be noncommuting isometries of \mathbf{H}^3 that generate a torsion free discrete group that is not cocompact and contains no parabolics. Then $\langle \xi, \eta \rangle$ is a free group on the generators ξ and η .*

Proof. Set $\Gamma = \langle \xi, \eta \rangle$. It follows from the hypotheses that the space $M = \mathbf{H}^3/\Gamma$ is a complete, noncompact, orientable hyperbolic 3-manifold without cusps. In particular, the universal cover of M is homeomorphic to \mathbf{R}^3 , so that M is irreducible; and $\Gamma = \pi_1(M)$ has no free abelian subgroup of rank 2.

It follows from [BaS, Proposition 3 and Theorem A] that if M is any noncompact, irreducible, orientable 3-manifold such that $\pi_1(M)$ has no free abelian subgroup of rank 2, then $\pi_1(M)$ is 2-free in the sense that each of its 2-generator subgroups is free of some rank ≤ 2 . (This fact is also implicit in [Tuc; JaS, §VI.4].) Thus Γ is 2-free; since Γ is itself generated by two noncommuting elements, it is a free group of rank 2. Hence by [K, p. 59], Γ is in fact free on the given generators ξ and η . \square

9.3. Proposition. *Suppose that ξ and η satisfy the hypotheses of Theorem 9.1. Then either the group $\langle \xi, \eta \rangle$ is geometrically finite, or we have $M(\langle \xi, \eta \rangle) = N(\langle \xi, \eta \rangle)$.*

Proof. Since $\Gamma = \langle \xi, \eta \rangle$ is not cocompact, it follows from Proposition 9.2 that Γ is a free group. In particular, $H_2(\Gamma; \mathbf{Z}) = 0$.

According to 1.4, the set $N = N(\Gamma) \subset M(\Gamma)$ is a 3-manifold-with-boundary and a deformation retract of $M(\Gamma)$. Thus N is aspherical and $\pi_1(N) \cong \Gamma$, and therefore $H_2(N; \mathbf{Z}) \cong H_2(\Gamma; \mathbf{Z}) = 0$. If ∂N has a compact component, it follows that \tilde{N} is compact (and has connected boundary); by the definition (1.7) this implies that Γ is geometrically finite. On the other hand, if $\partial N = \emptyset$ then $N = M(\Gamma)$. Hence to prove the proposition it suffices to show that ∂N is compact.

Let Ω denote the set of discontinuity of Γ . The surface Ω/Γ inherits a conformal structure from Ω . According to the Ahlfors finiteness theorem [A], Ω/Γ is a hyperbolic Riemann surface that has finite area in the Poincaré metric. Since Γ contains no parabolics, it also follows from Ahlfors's theorem that Ω/Γ has no cusps. Hence Ω/Γ is compact. We shall prove that ∂N is compact by showing that there is a continuous map of Ω/Γ onto ∂N . For this purpose it suffices to produce a Γ -equivariant surjection $\Theta: \Omega \rightarrow \partial \tilde{N}$, where $\tilde{N} = \text{nbhd}_1(\text{hull}(\Lambda_\Gamma))$ is the preimage of N in \mathbf{H}^3 . Recall from 1.4 that \tilde{N} is convex.

For any point $\zeta \in \Omega$ let C_ζ denote the smallest horoball based at ζ that meets the set \tilde{N} . Then C_ζ meets \tilde{N} in a single point; for if z and z' were distinct points of $\tilde{N} \cap C_\zeta$ then, since a horoball is convex, the open line segment in \mathbf{H}^3 joining z to z' would be contained in $\tilde{N} \cap \text{int } C_\zeta$ and the minimality of C_ζ would be contradicted. We define $\Theta(\zeta)$ to be the point of intersection of C_ζ with \tilde{N} . It is clear that $\Theta(\zeta) \in \partial \tilde{N} \cap \partial C_\zeta$ for each $\zeta \in \Omega$.

The function $\Theta: \Omega \rightarrow \partial \tilde{N}$ is obviously Γ -equivariant. To see that it is surjective, we observe that, since \tilde{N} is closed and convex, every point $z \in \partial \tilde{N}$ lies on the boundary of a hyperbolic half space $Q \subset \mathbf{H}^3$ with $\tilde{N} \cap \text{int } Q = \emptyset$. (One deduces the existence of such a "supporting half space" for a closed convex set in \mathbf{H}^3 from the corresponding fact in Euclidean space by using the projective model as in 1.2.) Now let C denote the unique horoball that is contained in Q and tangent to the plane ∂Q , and let ζ denote the base point of C . It is clear that $\zeta \in \Omega$ and that $\Theta(\zeta) = z$. Thus Θ is surjective.

It remains to show that Θ is continuous. For this purpose we identify \mathbf{H}^3 with the upper half-space model $\mathbf{R}^2 \times (0, \infty)$ by an isometry that maps some point of Λ_Γ to ∞ . This guarantees that every vertical ray in $\mathbf{R}^2 \times (0, \infty)$ whose lowest point is in N is entirely contained in N . Now let (ζ_i) be a convergent sequence in Ω with $\zeta_\infty = \lim \zeta_i \in \Omega$. For $1 \leq i \leq \infty$ we set $z_i = \Theta(\zeta_i)$ and $C_i = C_{\zeta_i}$. The closure $\overline{C_i}$ of C_i in $\mathbf{R}^2 \times [0, \infty)$ is a Euclidean ball of some radius r_i , tangent to $\mathbf{R}^2 \times \{0\}$ at ζ_i , and $\overline{C_i} \cap N = \{z_i\}$. Since the vertical ray with lowest point z_i is contained in N , the point z_i must lie on the closed upper hemisphere U_i of the sphere $\partial \overline{C_i}$.

We claim that $r_i \rightarrow r_\infty$ as $i \rightarrow \infty$. If this is false then after passing to a subsequence we can assume that $r_i \rightarrow r$ for some $r \neq r_\infty$ in $[0, \infty]$. If $r > r_\infty$ then for large enough i we have $U_\infty \subset \text{int } C_i$ and, in particular, $z_\infty \in \text{int } C_i$. This is impossible since $\overline{C_i} \cap N = \{z_i\} \subset \partial \overline{C_i}$. Similarly, if $r < r_\infty$ then for large i we have $U_i \subset \text{int } C_\infty$, and again we obtain a contradiction. This proves the claim.

Now suppose that (z_i) does not converge to z_∞ . Then after passing to a subsequence we may assume that $z_i \rightarrow z$ for some point $z \in N - \{z_\infty\}$. Since $z_i \in \overline{C_i}$ for $1 \leq i < \infty$ and since $\zeta_i \rightarrow \zeta$ and $r_i \rightarrow r$, we have $z \in \overline{C_\infty}$. This is impossible since $\overline{C_\infty} \cap N = \{z_\infty\}$. Hence we must have $z_i \rightarrow z_\infty$. This shows that Θ is continuous and completes the proof. \square

9.4. By Proposition 9.3, the proof of Theorem 9.1 breaks up into two disjoint cases, according to whether the group $\Gamma = \langle \xi, \eta \rangle$ —which is free by 9.2—satisfies $N(\Gamma) = M(\Gamma)$ or is geometrically finite. The first case is covered by the following result.

Lemma. *Let Γ be a Kleinian group which contains no parabolics and is free on two generators ξ and η . Suppose that Γ is topologically tame and satisfies $N(\Gamma) = M(\Gamma)$. Then for any $z \in \mathbf{H}^3$ we have*

$$\max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) \geq \log 3.$$

Proof. According to Proposition 6.9 every positive superharmonic function on the manifold $M(\Gamma)$ is constant. Thus by Proposition 3.9, every Γ -invariant conformal density is a constant multiple of the area density. By Proposition 5.2 we therefore have $\max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) \geq \log 3$ for every $z \in \mathbf{H}^3$. \square

9.5. The following result covers the remaining case of Theorem 9.1.

Lemma. *Let Γ be a Kleinian group which contains no parabolics and is free on two generators ξ and η . Suppose that Γ is geometrically finite. Then for any $z \in \mathbf{H}^3$ we have*

$$\max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) \geq \log 3.$$

Proof. We use the notation of §8. Let a point $z \in \mathbf{H}^3$ be given. Let us define a continuous nonnegative-valued function f on V by

$$f(\xi, \eta) = \max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)).$$

The proposition asserts that $f(\xi, \eta) \geq \log 3$ for all $(\xi, \eta) \in \mathfrak{G}\mathfrak{F}$, or, equivalently, that $f(\xi, \eta) \geq \log 3$ for all $(\xi, \eta) \in \mathfrak{G}\mathfrak{F}$.

Every local minimum of the function f on V occurs at a point (ξ, η) such that either ξ or η fixes z . Indeed, if (ξ, η) is a point of V where $\xi(z) \neq z$ and $\eta(z) \neq z$, then there exist a sequence (z_i) of points on the open hyperbolic line segment between z and $\xi(z)$ that converges to $\xi(z)$, and a sequence (w_i) of points on the open segment between z and $\eta(z)$ that converges to $\eta(z)$. If we choose sequences (α_i) and (β_i) of hyperbolic isometries converging to the identity, such that $\alpha_i(z) = z_i$ and $\beta_i(z) = w_i$, then $(\alpha_i \xi, \beta_i \eta)$ converges to (ξ, η) as $i \rightarrow \infty$, and $f(\alpha_i \xi, \beta_i \eta) < f(\xi, \eta)$ for all i . Hence (ξ, η) is not a local minimum of f .

For any point $(\xi, \eta) \in \mathcal{D}$, the elements ξ and η of $\text{PSL}_2(\mathbb{C})$ have infinite order in the discrete group $\langle \xi, \eta \rangle$, and hence have no fixed points in \mathbb{H}^3 . It follows that f has no local minimum on the open subset $\mathcal{G}\mathfrak{F}$ of V .

On the other hand, f is a proper function on V ; indeed, for any $B \geq 0$, the set $f^{-1}([0, B])$ consists of all pairs (ξ, η) such that ξ and η each move the point z a distance at most B , and this set is clearly compact. Hence f must take a minimum value on the closed set $\overline{\mathcal{G}\mathfrak{F}}$. Since it has no local minimum on the open set $\mathcal{G}\mathfrak{F}$, it must take its minimum value on \mathfrak{B} . Hence we need only show that $f(\xi, \eta) \geq \log 3$ for all $(\xi, \eta) \in \mathfrak{B}$.

Let $\mathcal{C} \subset \mathfrak{B}$ denote the dense G_δ given by Theorem 8.2. Since f is continuous and \mathcal{C} is dense in \mathfrak{B} , we need only prove that $f(\xi, \eta) \geq \log 3$ for all $(\xi, \eta) \in \mathcal{C}$. But for any $(\xi, \eta) \in \mathcal{C}$, the Kleinian group $\langle \xi, \eta \rangle$ is girded; hence by Proposition 7.4, every positive super-harmonic function on the manifold $M(\Gamma)$ is constant. Thus according to Proposition 3.9, every $\langle \xi, \eta \rangle$ -invariant conformal density is a constant multiple of the area density. By Proposition 5.2 we therefore have

$$f(\xi, \eta) = \max(\text{dist}(z, \xi \cdot z), \text{dist}(z, \eta \cdot z)) > \log 3. \quad \square$$

Proof of Theorem 9.1. The theorem follows immediately from Propositions 9.2 and 9.3 and Lemmas 9.4 and 9.5. \square

10. APPLICATIONS TO CLOSED HYPERBOLIC MANIFOLDS

10.1. The following result is an immediate consequence of Theorem 9.1 and the definition of a Margulis number given in §1.

Proposition. *Let $M = \mathbb{H}^3/\Gamma$ be a closed, orientable hyperbolic 3-manifold such that every 2-generator subgroup of $\Gamma = \pi_1(M)$ has nonempty set of discontinuity in S_∞ . Then $\log 3$ is a Margulis number for M . \square*

10.2. In this section we shall prove the

Proposition. *Let M be a closed, orientable hyperbolic 3-manifold such that the rank of $H_1(M; \mathbb{Q})$ is at least 3. Then every 2-generator subgroup of $\pi_1(M)$ is noncocompact and topologically tame.*

10.3. By combining Propositions 10.1 and 10.2 one immediately obtains the following result.

Theorem. *Let M be a closed, orientable hyperbolic 3-manifold such that the rank of $H_1(M; \mathbb{Q})$ is at least 3. Then $\log 3$ is a Margulis number for M .*

10.4. **Corollary.** *Let M be a closed, orientable hyperbolic 3-manifold such that the rank of $H_1(M; \mathbb{Q})$ is at least 3. Then M contains an isometric copy of a ball of radius $\frac{1}{2} \log 3$ in \mathbb{H}^3 . Hence the volume of M is greater than 0.92.*

Proof. By Theorem 10.3, the number $\varepsilon = \log 3$ is a Margulis number for M . Thus $M_{(0, \varepsilon]}$ is a disjoint union of tubes in M and, in particular, $M_{(\varepsilon, \infty]} \neq \emptyset$. It follows from the definition of $M_{(\varepsilon, \infty]}$ that the $(\varepsilon/2)$ -neighborhood of any point of $M_{(\varepsilon, \infty]}$ is isometric to a ball of radius $\varepsilon/2$ in \mathbb{H}^3 . This proves the first assertion.

To deduce the second assertion from the first, we use an observation due to Meyerhoff [Me]. Using a result due to Böröczky [Bör], Meyerhoff shows that if a hyperbolic manifold M contains a hyperbolic ball $B(r)$ of a given radius r then the volume of M is at least $2 \text{Vol}(T)/(6\alpha - \pi)$, where $\alpha = \frac{1}{2} \text{arcsec}(2 + \text{sech } 2r)$, and T is a regular tetrahedron in \mathbf{H}^3 whose edges have length $2r$. (This is equivalent to the inequality $\text{Vol}(M) \geq \text{Vol}(B(r))/d(r)$ given on p. 277 of [Me]: the factor $\sinh(2r) - 2r$ in the definition of $d(r)$ on p. 275 of [Me] is equal by [F, p. 209] to $\frac{1}{\pi} \text{Vol}(B(r))$.) Taking $r = \frac{1}{2} \log 3$, and using Lobachevsky functions as in [F, §IX.4] to compute the volume of a regular hyperbolic tetrahedron with edges of length $\log 3$, one obtains the estimate $\text{Vol } M > 0.92$. \square

Proof of Proposition 10.2. Let F be any 2-generator subgroup of $\Gamma = \pi_1(M)$. Since $H_1(M; \mathbf{Q})$ has rank at least 3, there is an epimorphism $\eta: \pi_1(M) \rightarrow \mathbf{Z}$ such that $\eta(F) = \{0\}$. In particular, F has infinite index in Γ and, therefore, is noncocompact. It remains to show that F is topologically tame.

It follows from [He, Lemma 6.5] that η is induced by some map $\phi: M \rightarrow S^1$ such that ϕ is transverse to $1 \in S^1$ and the surface $\phi^{-1}(1)$ is incompressible; that is, for each component Σ of $\phi^{-1}(1)$, we have $\pi_1(\Sigma) \neq \{1\}$, and the inclusion homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(M)$ is injective.

Let (\tilde{M}_1, p_1) denote the covering space of M corresponding to the subgroup $\Gamma_1 = \ker \eta$ of Γ . Then ϕ is covered by a map $\tilde{\phi}$ from \tilde{M}_1 to the universal cover \mathbf{R} of S^1 . Let \mathcal{E} be a graph whose fundamental group is free of rank 2, and let $\psi: \mathcal{E} \rightarrow M$ be a base-point-preserving map such that $\psi_*(\pi_1(\mathcal{E})) = F$. Since $F \subset \ker \eta$, the map ψ admits a lift $\tilde{\psi}: \mathcal{E} \rightarrow \tilde{M}_1$. Since \mathcal{E} is compact, we have $\tilde{\psi}(\mathcal{E}) \subset \tilde{\phi}^{-1}([-D, D])$ for some positive integer D . The component of $\tilde{\phi}^{-1}([-D, D])$ containing $\tilde{\psi}(\mathcal{E})$ is a compact 3-manifold-with-boundary K_1 with $\partial K_1 \subset p_1^{-1}(\phi^{-1}(1))$; hence ∂K_1 is incompressible in \tilde{M}_1 . We define $\Gamma_2 \leq \Gamma_1 \leq \Gamma$ to be the image of the inclusion homomorphism $\pi_1(K_1) \rightarrow \pi_1(\tilde{M}_1)$, so that $F \leq \Gamma_2$.

Let (\tilde{M}_2, p_2) denote the covering space of \tilde{M}_1 corresponding to the subgroup Γ_2 . Then \tilde{M}_2 contains a submanifold K_2 that is mapped homeomorphically onto K_1 by p_2 . Thus if we denote the boundary components of K_2 by $\tilde{\Sigma}^1, \dots, \tilde{\Sigma}^k$, each $\tilde{\Sigma}^j$ is mapped homeomorphically by p_2 onto a component Σ^j of ∂K . It is clear that K_2 is a compact core of \tilde{M}_2 in the sense that the inclusion homomorphism $\pi_1(K_2) \rightarrow \pi_1(\tilde{M}_2)$ is an isomorphism. It is also clear that the $\tilde{\Sigma}^j$ are incompressible.

Thus \tilde{M}_2 is a hyperbolic 3-manifold having a compact core whose boundary is incompressible. It follows from the main theorem of [Bo] that any such hyperbolic manifold is topologically tame. On the other hand, Γ_2 is contained in Γ_1 and, therefore, has infinite index in Γ ; hence \tilde{M}_2 has infinite volume. According to [Ca1, Proposition 3.2], any covering space, with finitely generated fundamental group, of an infinite-volume topologically tame hyperbolic 3-manifold is itself topologically tame. Since $F \leq \Gamma_2$, it follows that F is topologically tame, as required. \square

A stronger version of Proposition 10.2 can be established: namely, that under the same hypothesis, every 2-generator subgroup of $\pi_1(M)$ is in fact geometrically finite. This can be proved by combining results of [Th, Chapter 9; ScS]; see also [Ca2].

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DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, ILLINOIS 60680

E-mail address, Marc Culler: culler@math.uic.edu

E-mail address, Peter B. Shalen: u10123@uicvm.uic.edu