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Lifting Representations to Covering Groups*

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0. INTRODUCTION

The primary purpose of this note is to prove that if a discrete subgroup Γ of $PSL_2(\mathbf{C})$ has no 2-torsion then it lifts to $SL_2(\mathbf{C})$; i.e., there is a homomorphism $\Gamma \to SL_2(\mathbf{C})$ such that the composition with the natural projection $SL_2(\mathbf{C}) \to PSL_2(\mathbf{C})$ is the identity on Γ. (Note that the absence of 2-torsion is necessary, since cyclic subgroups of even order do not lift.) This extends a result of *Kra* [2], for which Γ was required to be a function group. Our methods are simple and straightforward, requiring only elementary covering space theory.

1. LIFTING REPRESENTATIONS

Our question seems to be clarified by viewing it first in a more general setting. Thus we begin by considering a lifting problem for representations of an abstract group Γ into an arbitrary connected topological group G. If $\tilde{G}$ is any covering space of G, with a compatible group structure, and $\rho: \Gamma \to G$ is a representation, then by a lift of ρ to $\tilde{G}$ we mean a representation $\tilde{\rho}: \Gamma \to \tilde{G}$ such that $\pi \circ \tilde{\rho} = \rho$, where $\pi: \tilde{G} \to G$ is the covering projection.

We analyze this lifting problem in terms of a *graph* of Γ. If $\{ g_\alpha \mid \alpha \in I\}$ is any set of generators of G then the associated graph of Γ is the graph $\mathscr{G}$ whose vertices are the elements of Γ, and which has an edge joining γ to $\gamma \cdot g_\alpha$ for each $\alpha \in I$. The left action of Γ on itself by multiplication clearly extends to a properly discontinuous action on $\mathscr{G}$. In fact X can be identified with a covering space whose group of covering translations is Γ. Let R denote a wedge product of circles indexed by I. If $\pi_1(R)$ is identified with the free group F on the set I in the obvious way, then $\mathscr{G}$ is the covering of R corresponding to ker ϕ, where $\phi: F \to \Gamma$ is the homomorphism defined by $\phi(\alpha) = g_\alpha$ for $\alpha \in I$. In particular, the fundamental group of $\mathscr{G}$ is the relation subgroup for a presentation of Γ in terms of the generators g_α, $\alpha \in I$.

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1.1. LEMMA. *Let X be a path-connected topological space upon which the group Γ acts. Let $\mathcal{G}$ be a graph of Γ. Then any Γ-equivariant map from Γ to X extends to a Γ-equivariant map from $\mathcal{G}$ to X. In fact, if $\mathcal{G}$ is associated with the generating set $\{ g_\alpha \mid \alpha \in I\}$ then the set of homotopy classes relative to Γ of such extensions is in one-to-one correspondence with $\pi_1(x)^I$.*

Proof. Let $f\colon \Gamma \to X$ be Γ-equivariant. We will construct an extension $\bar{f}\colon \mathcal{G} \to X$. Let $\mathcal{G}$ be associated with the generating set $\{ g_\alpha \mid \alpha \in I\}$ of Γ. For each $\alpha \in I$, define $\bar{f}$ on the edge e_α from 1 to g_α to be a path from $f(1)$ to $f(g_\alpha)$. Define $\bar{f}$ on the rest of $\mathcal{G}$ by the formula $f(\gamma \cdot f(x)$ for $x \in \bigcup_{\alpha \in I} e_\alpha$. Clearly this construction produces all Γ-equivariant extensions of f, and the homotopy class of such an extension relative to Γ is determined by the homotopy classes of the paths $\bar{f}\,|_{e_\alpha}$. ∎

Thus, in particular, any representation $\rho\colon \Gamma \to G$ of a group Γ into a path-connected topological group G extends to a Γ-equivariant map $\bar{\rho}\colon \mathcal{G} \to G$ from a graph of Γ into G.

1.2. PROPOSITION. *Let $\rho\colon \Gamma \to G$ be a representation of a group Γ into a path-connected topological group G. Let $\tilde{G}$ be a covering group of G, and $\mathcal{G}$ a graph of Γ. Then ρ lifts to $\tilde{G}$ if and only if there exists a Γ-equivariant extension $\bar{\rho}\colon \mathcal{G} \to G$ of ρ such that $\bar{\rho}$ lifts to $\tilde{G}$.*

Proof. If $\tilde{\rho}$ is a lift of ρ then for any graph $\mathcal{G}$ of Γ we may extend $\tilde{\rho}$ to $\tilde{\bar{\rho}}\colon \mathcal{G} \to \tilde{G}$, and define $\bar{\rho}$ to be $\pi \circ \tilde{\bar{\rho}}$, where $\pi\colon \tilde{G} \to G$ is the covering projection.

To prove the converse, let $\tilde{\bar{\rho}}$ be a lift of $\bar{\rho}$ to $\tilde{G}$. Observe that, since $\pi_1(G)$ is abelian, we may assume that $\tilde{\bar{\rho}}(1) = 1 \in \tilde{G}$. We will show that $\tilde{\rho} = \tilde{\bar{\rho}}\,|_\Gamma$ is a representation; it is necessarily a lift of ρ. That $\tilde{\rho}$ is a representation follows immediately from uniqueness of path lifting. If σ is a path in $\mathcal{G}$ from 1 to $\gamma \in F$, and if $\delta \in \Gamma$, then $\delta \cdot \sigma$ is a path from δ to $\delta \cdot \gamma$. Both $\tilde{\rho}(\delta) \cdot (\tilde{\bar{\rho}} \circ \sigma)$ and $\tilde{\bar{\rho}} \circ (\delta \cdot \sigma)$ are lifts of $\delta \cdot \sigma$ to $\tilde{\rho}(\delta)$. Hence these paths are equal, so

$$\tilde{\rho}(\delta) \cdot \tilde{\rho}(\gamma) = \tilde{\bar{\rho}}(\delta \cdot \gamma) = \tilde{\rho}(\delta \cdot \gamma). \quad ∎$$

In summary, the lifting problem for representations reduces to the problem of lifting Γ-equivariant maps of graphs of Γ to the covering space $\tilde{G}$. The difficulty in using this approach to determine whether a given representation lifts lies in the fact that there are so many Γ-equivariant exensions of ρ to a graph of Γ. One needs some guidance in choosing the extension. In the cases where Γ is the fundamental group of a differentiable manifold, and the representation is related to the geometry of the manifold, this guidance is available.

2. Lie Groups and Holonomy Representations

We specialize somewhat to the case where G is a connected Lie group. Then G has a maximal compact subgroup K and the homogeneous space G/K is analytically homeomorphic to $\mathbf{R}^n$ for some n. By a G/K-structure on an n-manifold M we mean a differentiable structure for which the charts map open sets in M to G/K, and for which the pseudo-group consists of homeomorphisms of G/K arising from the action of G on G/K. Associated with a G/K-structure on M is an immersion of the universal cover $\tilde{M}$ of M into G/K, defined by analytic continuation of the charts, and a representation $\rho: \pi_1(M) \to G$ with respect to which the immersion is $\pi_1(M)$-equivariant. This representation is called the *holonomy* of the G/K-structure. Also associated with a G/K-structure on M is a K-bundle over M defined as follows. Let E be the K-bundle over $\tilde{M}$ which is the pull-back under the immersion $\tilde{M} \to G/K$ of the K-bundle G over K. Then E inherits a fiber-preserving properly discontinuous action by $\pi_1(M)$. The quotient under this action is a K-bundle over M.

2.1. Proposition. *Suppose that the n-manifold M admits a G/K-structure with holonomy $\rho: \pi_1(M) \to G$. Suppose in addition that the associated K-bundle over M admits a section. Then ρ lifts to any covering group of G.*

Proof. Let $\tilde{M}$ be the universal cover of M and let $\mathscr{G}$ be a graph of $\pi_1(M)$. There is a $\pi_1(M)$-equivariant map $f: \pi_1(M) \to \tilde{M}$ taking the identity element to the base point in $\tilde{M}$, and a $\pi_1(M)$-equivariant section s for the K-bundle E over $\tilde{M}$ which is induced by the immersion $\iota: \tilde{M} \to G/K$. Let $\bar{f}: \mathscr{G} \to \tilde{M}$ be a $\pi_1(M)$-equivariant extension of f to a graph $\mathscr{G}$ of $\pi_1(M)$. Then we have the following commutative diagram in which the vertical maps are bundle projections and the horizontal maps are $\pi_1(M)$-equivariant:

$$\begin{array}{ccccc} & & E & \xrightarrow{\bar{\iota}} & G \\ & & {\scriptstyle s}\big\downarrow & & \big\downarrow \\ \mathscr{G} & \xrightarrow{\bar{f}} & \tilde{M} & \xrightarrow{\iota} & G/K \end{array}$$

The composition $\bar{\rho} = \bar{\iota} \circ s \circ \bar{f}$ is $\pi_1(M)$-equivariant with respect to the holonomy ρ. By multiplying by an element of K we may assume that $\bar{\iota} \circ s$ maps the base-point in $\tilde{M}$ to $1 \in G$. Thus $\bar{\rho}$ is actually a $\pi_1(M)$-equivariant extension of ρ. Moreover $\bar{\rho}$ lifts to any covering space of G since it factors through the simply-connected manifold $\tilde{M}$. ∎

A *hyperbolic structure* on a 3-manifold M is a $PSL_2(\mathbf{C})/SO_3$-structure.

For these structures it turns out that the hypotheses of Proposition 2.1 are always satisfied. Thus we have

2.2. COROLLARY. *The holonomy of a hyperbolic structure on a* 3*-manifold lifts to* $SL_2(\mathbf{C})$.

Proof. In this case the SO_3-bundle determined by the $PSL_2(\mathbf{C})/SO_3$-structure is just the bundle of orthonormal frames in the tangent bundle of the 3-manifold. This admits a section since 3-manifolds are parallelizable (In the compact case this is an exercise in [3]. The extension to the non-compact case is not difficult.) ∎

The following special case of our theorem is proved in [1, 4].

2.3. COROLLARY. *Any discrete torsion-free subgroup of* $PSL_2(\mathbf{C})$ *lifts to* $SL_2(\mathbf{C})$.

Proof. If Γ is discrete and torsion-free then Γ is the (faithful) image of the holonomy of the obvious hyperbolic structure on $M = \Gamma\backslash PSL_2(\mathbf{C})/SO_3 = \Gamma\backslash \mathbf{H}^3$. (Here the immersion of M in $PSL_2(\mathbf{C})/SO_3$ is a homeomorphism). ∎

3. DISCRETE SUBGROUPS OF $PSL_2(\mathbf{C})$

We now turn to the theorem mentioned in the introduction. Let Γ be a discrete subgroup of $PSL_2(\mathbf{C})$ with no 2-torsion. Consider the action of Γ on $PSL_2(\mathbf{C})/SO_3 \equiv \mathbf{H}^3$. The stabilizers of points under this action are conjugate to discete subgroups of SO_3, and hence are isomorphic to symmetry groups of Platonic solids. Since Γ has no 2-torsion, these stabilizers must all be cyclic groups. Let $S \subset \mathbf{H}^3$ be the set of points whose stabilizers are non-trivial. Since a cyclic subgroup of $PSL_2(\mathbf{C})$ actually fixes an entire geodesic line in $\mathbf{H}^3$, S is a discrete set of lines. (The quotient $\Gamma\backslash\mathbf{H}^3$ is an orientable 3-manifold M and the image L of S is a 1-dimensional submanifold of M.) The action of Γ on $\mathbf{H}^3 - S$ is properly discontinuous with quotient $M - L$.

3.1. THEOREM. *Any discrete subgroup of* $PSL_2(\mathbf{C})$ *which has no* 2*-torsion lifts to* $SL_2(\mathbf{C})$.

Proof. We mimic the proof of Proposition 2.1 with $\tilde{M}$ replaced by $\mathbf{H}^3 - S$. Thus we get a Γ-equivariant map $\tilde{f}: \mathscr{G} \to \mathbf{H}^3 - S$ where $\mathscr{G}$ is a graph of Γ. We get a Γ-equivariant map from $\mathbf{H}^3 - S$ to $PSL_2(\mathbf{C})$ as follows. The manifold $M = \Gamma\backslash\mathbf{H}^3$ is parallelizable so there exists a section s of its tangent frame bundle. We restrict this section to $M - L$ and lift to get a Γ-

equivariant section $\tilde{s}$ of the tangent frame bundle of $\mathbf{H}^3 - S$. The total space of this bundle is a subset of the total space of the tangent frame bundle of $\mathbf{H}^3$, which we identify with $PSL_2(\mathbf{C})$. Let $\bar{\imath}: E \to PSL_2(\mathbf{C})$ denote the inclusion and set $\bar{\rho} = \bar{\imath} \circ s \circ \bar{f}$. This gives a Γ-equivariant map from $\mathscr{G}$ to $PSL_2(\mathbf{C})$ which factors through $\mathbf{H}^3 - S$. As before we may assume that $\bar{\rho}(1) = 1 \in PSL_2(\mathbf{C})$, and hence that $\bar{\rho}$ is an extension of the inclusion representation of Γ into $PSL_2(\mathbf{C})$.

Of course $\pi_1(\mathbf{H}^3 - S)$ is not trivial. However, we will show that $\bar{\imath} \circ \tilde{s}$ induces the trivial map from $\pi_1(\mathbf{H}^3 - S)$ to $\pi_1(PSL_2(\mathbf{C})) \cong \mathbf{Z}_2$, so we can still conclude that $\bar{\rho}$ lifts to $SL_2(\mathbf{C})$. Since $\pi_1(\mathbf{H}^3 - S)$ is normally generated by the elements represented by the meridians of the lines in S, it suffices to show that $\bar{\rho}$ maps each of these to a null-homotopic loop in $PSL_2(\mathbf{C})$. This will be done by making an explicit local calculation. The idea is this. The restriction of $\bar{\rho}$ to a meridian $\tilde{\mu}$ in $\mathbf{H}^3 - S$ is obtained by lifting the restriction of s to a meridian μ of $M - L$. Since s extends over the disk spanned by μ in M, it must twist an odd number of times relative to the radial framing of μ. But the radial framing of μ lifts to the radial framing of $\tilde{\mu}$, and $\tilde{\mu}$ covers μ an odd number of times. Thus $\bar{\rho}$ twists an odd number of times with respect to the radial framing of $\tilde{\mu}$, and consequently extends over the disk bounded by $\tilde{\mu}$ in $\mathbf{H}^3$.

Specifically, we choose local coordinates in a neighborhood $\tilde{U}$ of a point on a component S_0 of S so that S_0 meets $\tilde{U}$ in the z-axis and so that the stabilizer of S_0 acts by rotation through an angle $2\pi/k$, where k is odd. The image, U, of this neighborhood in M has a compatible coordinate system in which the z-axis represents a segment of a component L_0 of L and the quotient map is given by

$$(r\cos(\theta), r\sin(\theta), z) \to (r\cos(k\theta), r\sin(k\theta), z)$$

for $0 \leqslant \theta < 2\pi$ and $r \geqslant 0$. The meridians μ and $\tilde{\mu}$ will be the unit circles in these coordinate systems. We will show that if a section of the frame bundle over μ extends over the unit disk in U then its lift to $\tilde{\mu}$ will extend over the unit disk in $\tilde{U}$. The sections of the frame bundle over the unit circle can be identified with maps from S^1 to SO_3 by choosing one section and expressing the others in terms of it. In order to make this correspondence natural we must choose our base section to be equivariant with respect to rotation. Thus we use the radial section, which is given in terms of the standard section $\{\partial x, \partial y, \partial z\}$ as

$$(\cos\theta, \sin\theta, 0) \to \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is well known that this section does not extend over the unit disk. Thus we have the following situation. Up to fiber-preserving homotopy, the sections over μ are identified with homotopy classes in $[S^1, SO_3] \cong \text{Hom}(\mathbf{Z}, \mathbf{Z}_2)$ and the sections that extend over the disk are those that correspond to the non-trivial homomorphism. (The radial section corresponds to the trivial homomoprhism.) The sections over $\tilde{\mu}$ are similarly described, and the correspondence between a section over μ and its lift to $\tilde{\mu}$ is given by the map $\text{Hom}(\mathbf{Z}, \mathbf{Z}_2) \to \text{Hom}(\mathbf{Z}, \mathbf{Z}_2)$ induced by the map $n \to kn$ from $\mathbf{Z}$ to $\mathbf{Z}$. If k is odd this induces an isomorphism, otherwise it induces the trivial homomorphism. Thus when k is odd, a section over μ extends over the disk if and only if its lift to $\tilde{\mu}$ extends as well. This completes the proof of our main theorem. ∎

4. Continuity of the Lifting Criterion

Given an abstract group Γ generated by $\{g_\alpha \mid \alpha \in I\}$, and a topological group G the set $\text{Hom}(\Gamma, G)$ may be topologized as a subspace of G^I. This topology is independent of the choice of generators. We close with the remark that the property of lifting to a covering group $\tilde{G}$ depends only on the path-component of the representation in $\text{Hom}(\Gamma, G)$.

4.1. Theorem. *Let Γ be an abstract group and G a path-connected topological group. Let $\tilde{G}$ be a covering group of G. A representation $\rho: \Gamma \to G$ lifts to $\tilde{G}$ if and only if every representation in the path-component of ρ in* $\text{Hom}(\Gamma, G)$ *lifts to $\tilde{G}$.*

Proof. Let ρ_0 and ρ_1 be representations of Γ in G which are joined by a path in $\text{Hom}(\Gamma, G)$. We can view the path as a Γ-equivariant homotopy between ρ_0 and ρ_1. Let $\mathscr{G}$ be a graph of Γ, and let $\bar{\rho}_0$ and $\bar{\rho}_1$ be Γ-equivariant extensions of ρ_0 and ρ_1 to $\mathscr{G}$. We will construct a Γ-equivariant homotopy $\bar{\rho}_t$ between $\bar{\rho}_0$ and $\bar{\rho}_1$. Thus by Proposition 1.2, ρ_0 lifts to $\tilde{G}$ if and only if ρ_1 does.

This homotopy is easy to construct. If $\mathscr{G}$ is associated with the generating set $\{g_\alpha \mid \alpha \in I\}$, then $\bar{\rho}_t$ is defined on the edge e_α from 1 to g_α by using the homotopy extension property for graphs. The definition of ρ_t on the rest of $\mathscr{G}$ is given by the formula $\bar{\rho}_t(\gamma \cdot x) = \gamma \cdot \bar{\rho}_t(x)$ for $x \in \bigcup_{\alpha \in I} e_\alpha$. ∎

Consider, for example, the situation discussed in Theorem 3.2, where Γ is a discrete subgroup of $G = PSL_2(\mathbf{C})$ with no 2-torsion. Then $\text{Hom}(\Gamma, G)$ is an affine algebraic set and the entire topological component of the inclusion representation consists of representations which lift to $SL_2(\mathbf{C})$. A dense open subset of this component consists of representations which are faithful; their images are thus subgroups of $PSL_2(\mathbf{C})$ which are isomorphic to Γ and are, in a sense, deformations of Γ. These groups are generally not discrete, but by the discussion above they still lift to $SL_2(\mathbf{C})$.

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