PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 1, October 1978

SHORTER NOTES

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HOMOLOGY EQUIVALENT FINITE GROUPS ARE ISOMORPHIC

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ABSTRACT. We prove that if a homomorphism between two finite groups induces an isomorphism between their integral homology groups then it is an isomorphism.

An unpublished result of Mazur asserts that no finite group has the homology of a point. The proof is by embedding the finite group π into a large unitary group U and examining the spectral sequence of the covering $U \to U/\pi$. In [4] Swan used similar techniques to generalize Mazur's result by showing that for any nontrivial subgroup $\rho \subset \pi$ the map i_*: $H_*(\rho) \to H_*(\pi)$ induced by inclusion is nontrivial in infinitely many dimensions. The purpose of this note is to point out that Swan's theorem implies that a homomorphism between finite groups which induces an isomorphism between their integral homology groups is itself an isomorphism. We will use the notation $H_*(\pi)$ to denote the homology of the group π with integral coefficients.

I learned from the referee that this result was announced in [1] by L. Evens, who found the proof given here but did not include it because the method is incongruous with those used in [1].

The author is grateful to John Stallings for suggesting this problem and for helpful comments about this note.

THEOREM. *Let f: $\rho \to \pi$ be a homomorphism between finite groups. If f_*: $H_*(\rho) \to H_*(\pi)$ is an isomorphism, then f is an isomorphism.*

PROOF. We first show that f is injective. Taking η to be the kernel of f, we have the following commutative triangle:

$$\begin{array}{ccccc} & & H_*(\rho) & & \\ & {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle f_*} & \\ H_*(\eta) & & \xrightarrow{(f\circ i)_*} & & H_*(\pi) \end{array}$$

The map f_* is assumed to be an isomorphism, and $(f \circ i)_*$ is zero since it is

Received by the editors October 20, 1977.
AMS (MOS) subject classifications (1970). Primary 20J05; Secondary 18H10, 55H10.

induced by the trivial homomorphism. Therefore i_* is the zero homomorphism, and by Swan's theorem $\eta = \{1\}$.

Since f is injective we may assume that ρ is a subgroup of π, and that f is the inclusion $\rho \subset \pi$. We then have the following commutative square of covering projections (after embedding π in a unitary group U):

$$\begin{array}{ccc} U & = & U \\ \alpha \downarrow & & \downarrow \beta \\ U/\rho & \xrightarrow[\gamma]{} & U/\pi \end{array}$$

Note that α and β have degrees $|\rho|$ and $|\pi|$ respectively. Thus to show $\rho = \pi$ we need only show that α and β have the same degree, that γ has degree 1.

Corresponding to the covering projections α and β there are spectral sequences E and F with

$$E^2_{p,q} = H_p(\rho; H_q(U)) \underset{p}{\Rightarrow} H(U/\rho)$$

and

$$F^2_{p,q} = H_p(\pi; H_q(U)) \underset{p}{\Rightarrow} H(U/\pi).$$

The map γ induces a homomorphism $\gamma_*: E \to F$. The coefficients $H_q(U)$ are trivial in both cases since U is connected. Therefore $\gamma^2_*: E^2_{p,q} \to F^2_{p,q}$ is an isomorphism [2, p. 172], and the comparison theorem implies that $\gamma^\infty_*: E^\infty_{p,q} \to F^\infty_{p,q}$ is an isomorphism. It follows that $\gamma_*: H_*(U/\rho) \to H_*(U/\pi)$ is an isomorphism and hence has degree 1. □

We remark that this proof remains valid under the weaker hypothesis that $f_i: H_i(\rho) \to H_i(\pi)$ is an isomorphism for $i = 0, 1, \ldots, \max(2r, p^2 + 1)$ where r and p are the smallest dimensions of faithful unitary representations of ρ and π respectively.

Work of Stallings [3] shows that the above theorem is true when ρ and π are assumed to be nilpotent. It is natural to ask whether the theorem can be generalized to some larger class of groups such as solvable or nilpotent by finite groups. However, the following example shows that the inclusion of a proper subgroup into a solvable extension of an abelian group by a finite group can induce a homology isomorphism.

EXAMPLE. Let π be the fundamental group of the Klein bottle, $\pi = |a, b: bab^{-1} = a^{-1}|$. Let ρ be the (nonnormal) subgroup generated by a^3 and b. The homology groups of π (and ρ since ρ and π are isomorphic) vanish in dimensions larger than 1. However ρ clearly contains a complete set of coset representatives for $[\pi, \pi]$ in π. Thus the inclusion i of ρ into π induces an isomorphism $i_*: H_*(\rho) \to H_*(\pi)$.

This example also shows that a nontrivial irregular covering projection with finitely many sheets can induce an isomorphism on homology. That this cannot happen for regular coverings follows from Mazur's result. For if p: $E \to B$ is a regular cover with a finite group π of covering translations, then p embeds in the fibration $E \xrightarrow{p} B \to K(\pi, 1)$. If p_* is an isomorphism, then $K(\pi, 1)$ has the homology of a point, so $\pi = \{1\}$.

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