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Volumes of hyperbolic Haken manifolds, I

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Introduction

In [14] a program was initiated for using the topological theory of 3-manifolds to obtain lower bounds for volumes of hyperbolic 3-manifolds. In [1], by a combination of new geometric ideas with relatively standard (but specifically 3-dimensional) topological techniques, we showed that every closed, orientable hyperbolic 3-manifold whose first Betti number is at least 3 has volume exceeding 0.92. By contrast, the best known lower bound [10, 5] for the volume of an arbitrary closed hyperbolic 3-manifold is approximately 0.0012.

In [3] we showed that every closed, orientable hyperbolic 3-manifold whose first Betti number is 2 has volume exceeding 0.34. The proof depended on supplementing the results and techniques of [1] with ingenious elementary arguments due to Zagier [11] and numerical computations.

In the present paper we shall show that if one excludes certain special manifolds, such as fiber bundles over S^1, then the lower bound of 0.34 also holds for hyperbolic 3-manifolds with Betti number 1. The proof depends heavily on the results of [1] and [3], but it involves much deeper topological ideas than these papers. The new topological results needed for the proof occupy most of the present paper. To some extent these results have the flavor of general topology, but the proofs make use of such specifically low-dimensional techniques as the characteristic submanifold theory [9, 8], the interaction between trees and incompressible surfaces, and Scott's theorem [12] that surface groups are locally extended residually finite.

Before giving a precise statement of our main result we must review a few elementary notions from 3-manifold theory.

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Let M be a connected, closed, orientable topological 3-manifold. By an *incompressible surface* in M we mean a closed, connected, orientable, locally flat 2-manifold $S \subset M$ such that genus $S > 0$ and $\pi_1(S) \to \pi_1(M)$ is injective. Recall from [6, p. 62] that M contains a non-separating incompressible surface if and only if $H_1(M; \mathbf{Q}) \neq 0$.

A topological 3-manifold-with-boundary N is said to be *boundary-irreducible* if for every component C of ∂M the natural homomorphism $\pi_1(C) \to \pi_1(M)$, induced by inclusion, is injective. It is plain that if S is an incompressible surface in a connected, closed, orientable 3-manifold M, then each component of the manifold obtained by splitting M along S is boundary-irreducible.

Throughout this paper we will let I denote the unit interval $[0, 1] \subset \mathbf{R}$. An *I-bundle* over a topological 2-manifold B is a locally trivial fiber bundle over B with fiber I.

Definition. *Let N be a compact, connected, orientable topological 3-manifold with boundary. We shall say that N is a* book of I-bundles *if it has the form $N = E \cup V$, where*

(i) *E is an I-bundle over a non-empty compact 2-manifold-with-boundary B;*
(ii) *each component of V is homeomorphic to $D^2 \times S^1$;*
(iii) *the set $A = E \cap V$ is the inverse image of ∂B under the bundle projection $E \to B$; and*
(iv) *each component of A is an annulus in ∂V which is homotopically non-trivial in V.*

It follows from the above conditions that $\operatorname{int} E \cap \operatorname{int} V = \emptyset$.

(The reason for calling N a "book of I-bundles" is that we may regard N as the regular neighborhood of a complex $X \subset N$ which, up to homeomorphism, is obtained from the disjoint union of the 2-manifold B with a 1-manifold C (the core of V) by attaching ∂B to C via a covering map. Thus X is a "book of surfaces": B consists of its "pages" and C is its "binding." One obtains N from X by thickening up the pages to form I-bundles and the binding to form solid tori.)

Note that in the definition of a book of I-bundles we do not require B to be orientable or connected. Note also that we do allow V to be empty, in which case N is an I-bundle over B. It is not hard to show that a book of I-bundles is boundary-reducible if and only if there is a component V_0 of V which intersects E in a single annulus and this annulus carries the fundamental group of V_0. In particular, books of I-bundles of this exceptional type do not arise from splitting closed manifolds along incompressible surfaces.

Definition. *Let S be an incompressible surface in a closed, orientable topological 3-manifold M. Let N denote the manifold-with-boundary obtained by splitting M along S. We shall say that S is a* fibroid *if each component of N is a book of* I-bundles.

Note that in particular S is a fibroid if each component of N is an I-bundle. In this case S is either a fiber in a fibration of M over S^1, or the common boundary of two twisted I-bundles whose union is M. Incompressible surfaces of these types have been extensively studied and are known to appear as exceptions in the statements or proofs of many theorems about incompressible surfaces. Fibroids constitute a somewhat larger but still very special class of incompressible surfaces. It is this class that appears as exceptional in the main theorem of this paper.

Main theorem. *Let M be a closed, orientable hyperbolic 3-manifold. Suppose that M contains a non-separating incompressible surface which is not a fibroid. Then* vol $M > 0.34$.

There is no evidence that the conclusion of the above theorem is false when the given surface is a fibroid. However, if it remains true it will require a different proof in this case.

In the sequel to this paper we will investigate the case where M contains a separating incompressible surface which is not a fibroid.

The above theorem will be proved in Section 11 by combining the results of [3] with Theorem A below, a result which is of independent interest.

By a *closed curve* in a topological space X we mean a map $\alpha: S^1 \to X$.

Theorem A. *Let M be a closed, orientable hyperbolic 3-manifold. Suppose that M contains an incompressible surface S which is not a fibroid. Let λ be a positive number less than* $\log 3$. *Then either* (i) *M contains a non-trivial closed geodesic of length $<\lambda$ which is homotopic in M to a closed curve $M-S$, or* (ii) *M contains a hyperbolic ball of radius $\lambda/2$.*

The proof of Theorem A, which is given in Section 10, depends on a geometric observation made in [2]. We may regard M as the quotient of hyperbolic space $\mathbf{H}^3$ by a discrete torsion-free group of isometries. For each maximal cyclic subgroup X of Γ, the elements of X have a common axis $A_{\mathbf{H}^3}(X)$; and if X is "short" in the sense that it has a generator of length $<\lambda$, then there is a neighborhood $Z_\lambda(X)$ of $A_{\mathbf{H}^3}(X)$ in $\mathbf{H}^3$ consisting of all those points that are displaced a distance less than λ by some element of X. According to [2, Proposition 3.2], if conclusion (ii) of Theorem A does not hold then $\mathbf{H}^3$ is the union of the sets $Z_\lambda(X)$ where X ranges over all short maximal cyclic subgroups of Γ. In particular, the closures of the $Z_\lambda(X)$ form a locally finite family of closed sets covering $\mathbf{H}^3$. In this paper we call this family a "plating" of $\mathbf{H}^3$ rather than a "covering" in order to avoid confusion with the notion of a covering space.

By an elementary construction that is reviewed in Section 3, any incompressible surface S in M defines an action of $\Gamma \cong \pi_1(M)$ on a tree T. If conclusion (i) of Theorem A does not hold then every short maximal cyclic subgroup X of Γ has an axis $A_T(X)$ in T, i.e. a line which is invariant under X and on which X acts by translations. A key ingredient in the proof of

Theorem A is Proposition 10.3, which implies—under the hypothesis $\lambda < \log 3$—that if X and Y are short maximal cyclic subgroups of Γ such that $Z_\lambda(X) \cap Z_\lambda(Y) \neq \emptyset$, then $A_T(X)$ and $A_T(Y)$ have a common edge in T. Proposition 10.3 is proved by combining the geometrical results of [1] with an elementary fact about group actions on trees ([4, Proposition 4.2], reproduced below as Proposition 2.6), and a theorem of Simon's on compactification of covering spaces of 3-manifolds (a special case of [15, Theorem 3.1], re-interpreted in terms of trees in Proposition 3.6 below).

It follows from all this that a counterexample to Theorem A would give a plating $\mathscr{Z}$ of the universal covering space $\tilde{M} = \mathbf{H}^3$ of M, indexed by certain maximal cyclic subgroups of Γ, such that if the sets in the plating corresponding to two maximal cyclic subgroups X and Y meet each other then the axes of X and Y in T overlap at least in an edge. Furthermore, the plating would be Γ-equivariant in a natural sense, and the set corresponding to each maximal cyclic subgroup X would be X-invariant and would have a compact quotient by X. One completes the proof of Theorem A by showing that the existence of a plating with these properties implies that the surface S from which the tree T is constructed is a fibroid. This step is embodied in Theorem 9.1. The latter theorem is purely topological in statement and proof. It depends on the hypothesis that M is simple (see 1.6), which is a topological consequence of hyperbolicity. (Thurston's uniformization theorem implies that any simple 3-manifold which contains an incompressible surface has a hyperbolic structure, but the hyperbolic structure is not used in the proof of 9.1.)

Section 9 is devoted to the proof of Theorem 9.1. The proof involves an intriguing set-theoretical manipulation that constructs from the given plating $\mathscr{Z}$ a new plating $\mathscr{W}$, indexed by edges of T. The plating $\mathscr{W}$ has order at most 2, in the sense that every point lies in at most $3 = 2 + 1$ sets of the plating. The proof that $\mathscr{W}$ has order at most 2 ultimately depends on a simple combinatorial fact about trees which we prove as Proposition 2.4. The plating $\mathscr{W}$ has formal properties, such as equivariance, somewhat similar to those of $\mathscr{Z}$; on the other hand, because it is indexed by edges, $\mathscr{W}$ is related to the tree T in a simpler way than $\mathscr{Z}$ is. This, together with the fact that $\mathscr{W}$ has order at most 2, allows one to use 3-manifold arguments to prove that S is a fibroid. These arguments are presented in Section 8, and are summed up in Proposition 8.2. It is here that one uses the characteristic submanifold theory and Scott's theorem.

In Section 1 we establish some conventions that are used in the paper. In Sections 2 and 3 we prove the facts about trees and incompressible surfaces that will be needed. In Section 4 we review a somewhat weak and special form of the characteristic submanifold theory that is sufficient for the applications that we need and less technical than the general theory. In Sections 5 and 6 we develop a systematic theory of platings of spaces that behave equivariantly under group actions; this facilitates the manipulations in Sections 8 and 9. In Section 7 we prove some more specialized results about equivariant platings in the case where the group is locally extended residually finite, and then deduce from Scott's theorem the result about platings of 2-manifolds that is needed in Section 8.

We are indebted to P. Papasoglou for pointing out that a finiteness hypothesis is needed for some of the results in Section 2. We would also like to thank Peter Scott for useful and encouraging discussions of the material.

1. Conventions

1.1. Since most of the work in the paper is topological, we have generally followed the standard conventions of topology. In particular a *manifold* may have a boundary unless we specify otherwise. (In Sections 10 and 11, where we consider a hyperbolic manifold M, it is explicitly assumed that M is closed. We do not have occasion to consider a hyperbolic manifolds with boundary.)

1.2. The outer category. The following conventions will be useful for dealing with fundamental groups. Let Γ_1 and Γ_2 be groups. By an *outer homomorphism* from Γ_1 to Γ_2 we mean an equivalence class of homomorphisms from Γ_1 to Γ_2, where two homomorphisms h and h' are said to be equivalent if there is an inner automorphism i of Γ_1 such that $h' = h \circ i$. This equivalence relation is respected by composition of homomorphisms; hence if $f_1 : \Gamma_1 \to \Gamma_2$ and $f_2 : \Gamma_2 \to \Gamma_3$ are outer homomorphisms between groups, there is a well-defined outer homomorphism $f_2 \circ f_1 : \Gamma_1 \to \Gamma_3$. This defines a category in which the objects are groups and the morphisms are outer homomorphisms. The automorphism group of an object in this category is the familiar outer automorphism group of a group.

An *outer subgroup* of a group Γ is a conjugacy class of subgroups of Γ. A group in the conjugacy class will be said to *realize* the given outer subgroup. If A and B are outer subgroups of Γ and if some representative of A is contained in a representative of B, we shall say that *A is contained in B*. The *index* of an outer subgroup A is the index of an arbitrary subgroup realizing A. If X is a genuine subgroup of Γ, the outer subgroup realized by X will often be referred to simply as "the outer subgroup X".

The image of an outer homomorphism $f : \Gamma_1 \to \Gamma_2$ is well-defined as an outer subgroup of Γ_2; this outer subgroup will be denoted by $f(\Gamma_1)$. If $f(\Gamma_1)$ is the outer subgroup Γ_2 we shall say that f is *surjective*. The kernel of an outer homomorphism is a well-defined normal subgroup of Γ_1. An outer homomorphism is *injective* if its kernel is trivial.

1.3. The fundamental group. If Ω is a path-connected space, we shall often write $\pi_1(\Omega)$ for the fundamental group of Ω with an unspecified base point. We regard $\pi_1(\Omega)$ as an object in the category described above. Any continuous map $F : \Omega_1 \to \Omega_2$ between path-connected spaces induces a well-defined outer homomorphism $f_\# : \pi_1(\Omega_1) \to \pi_1(\Omega_2)$. For example, the "homomorphisms" appearing in the definitions of incompressibility and boundary-irreducibility given in the introduction are in reality outer homomorphisms.

If $\alpha: S^1 \to \Omega$ is a closed curve in a path connected space Ω, the outer homomorphism $\alpha_\#$ assigns to the clockwise generator of $\pi_1(S^1)$ a conjugacy class in $\pi_1(\Omega)$. This gives a bijective correspondence between conjugacy classes in $\pi_1(\Omega)$ and free homotopy classes of closed curves in Ω.

1.4. Γ-spaces. Let Γ be a group. Recall that a *Γ-set* is a set equipped with an action of Γ. If S is a Γ-set then for each $s \in S$ we shall denote the stabilizer of s in S by Γ_s.

A *Γ-space* is a topological space equipped with an action of Γ by homeomorphisms.

A Γ-space C will be termed *free* if the action of Γ on C is free.

We shall say that a Γ-space C is *uniform* if there is a compact set $R \subset C$ such that $\Gamma \cdot R = C$. If C is uniform then any closed, invariant subset C' of C is uniform. (Indeed, if $R \subset C$ is compact and $\Gamma \cdot R = C$, then $R' = R \cap C'$ is compact and $\Gamma \cdot R' = C'$.)

Note also that if C is a uniform Γ-space and $\Gamma_0 \leqq \Gamma$ is a subgroup of finite index, then C is a uniform Γ_0-space. Indeed, if $\Phi \subset \Gamma$ is a finite set such that $\Gamma_0 \Phi = \Gamma$, and if $R \subset C$ is a compact set such that $\Gamma \cdot R = C$, then $R_0 = \Phi \cdot R$ is compact and $\Gamma_0 \cdot R_0 = C$.

1.5. Piecewise linear Γ-manifolds. Let Γ be a group and n a positive integer. By a *piecewise linear Γ-manifold* of dimension n we shall mean a PL n-manifold equipped with an action of Γ which is simplicial with respect to some triangulation that defines the given PL structure of M. In particular a PL Γ-manifold has the structure of a Γ-space.

If $\tilde{M}$ is a PL Γ-manifold which is free (as a Γ-space), then the quotient space $\tilde{M}/\Gamma$ is a PL manifold of the same dimension as $\tilde{M}$, the manifold $\tilde{M}$ is a covering space of M via the quotient map $p: \tilde{M} \to M$, and the group of deck transformations is Γ. Conversely, the universal covering space of a PL manifold is a free PL Γ-manifold in a natural way, where Γ denotes the group of deck transformations. In this paper the language of free PL Γ-manifolds will sometimes be more convenient than the language of covering space theory.

If $\tilde{M}$ is a free PL Γ-manifold then there is a natural outer homomorphism $v: \pi_1(\tilde{M}/\Gamma) \to \Gamma$. The kernel of v is isomorphic to $\pi_1(\tilde{M})$; in particular, v is an outer isomorphism if $\tilde{M}$ is 1-connected.

A PL Γ-manifold $\tilde{M}$ will be termed *Γ-orientable* if the action of Γ on $\tilde{M}$ preserves orientation. Note that a free PL Γ-manifold $\tilde{M}$ is orientable if and only if $\tilde{M}/\Gamma$ is orientable.

1.6. A topological 3-manifold M is termed *irreducible* if M is connected and every locally flat 2-sphere in M bounds a 3-ball. In this paper we shall say that a PL 3-manifold M is *simple* if it has the following properties:

(i) M is irreducible;

(ii) M is boundary-irreducible (see introduction); and

(iii) $\pi_1(M)$ contains no free abelian subgroup of rank 2.

This definition is more restrictive than the one used, for example, in [8].

Note that if an orientable, closed PL 3-manifold M is simple then M contains no incompressible tori.

2. Trees

2.1. Graphs. By a *graph* we shall mean a CW complex of dimension $\leqq 1$. The 0-cells and 1-cells of a graph Ψ are called *vertices* and *edges*. In particular, an edge is a subset of Ψ and is homeomorphic to an open interval. The boundary of an edge is a subset of the 0-skeleton of Ψ consisting of one or two points. An edge whose boundary is a single point is called a *loop*. A vertex v of Ψ is called an *end point* if v is in the boundary of a unique edge e, and e is not a loop.

2.2. Trees. A *tree* is a 1-connected graph. In particular a tree contains no loops, and may therefore be regarded as a simplicial complex. If e is an edge of a tree T then its closure $\bar{e}$ is homeomorphic to a closed interval.

If T is any tree we shall denote by $\mathscr{E}(T)$ the set of all edges of T.

We may regard the real line $\mathbf{R}$ as a tree by triangulating it so that the vertices are precisely the integer points. A *line* is a tree L which is simplicially isomorphic to $\mathbf{R}$. A *translation* of L is a simplicial automorphism which is simplicially conjugate to an integer translation of $\mathbf{R}$.

A *segment* is a tree σ which is simplicially isomorphic to a (connected) subcomplex of $\mathbf{R}$; we may take the subcomplex to be one of the intervals $(-\infty, \infty)$, $[0, \infty)$ or $[0, n]$, where n is a non-negative integer. In the latter case we have $n = \operatorname{card}(\mathscr{E}(\sigma)) < \infty$; we shall refer to n as the *length* of the finite segment σ. A segment of length 0 consists of a single vertex, and will be termed *degenerate*.

A *triod* is a tree which is simplicially isomorphic to a cone on three vertices. The cone point will be called the *center* of the triod. Note that a triod contains exactly three segments of length 2.

A subcomplex of a tree T which is a segment (or a triod) will be called simply a segment (or a triod) in T.

2.3. **Proposition.** *Let $\mathscr{V}$ be a finite, non-empty collection of subtrees of a tree T. Suppose that for all $V, V' \in \mathscr{V}$ we have $V \cap V' \neq \emptyset$. Then $\bigcap \mathscr{V}$ is a subtree of T. In particular $\bigcap \mathscr{V} \neq \emptyset$.*

Proof. It suffices to show that $\bigcap \mathscr{V}$ has exactly one connected component. First we show that it has at most one. Let s and t be vertices of $\bigcap \mathscr{V}$. Since T is a tree, there is a unique segment $\alpha \subset T$ with endpoints s and t. Since each $V \in \mathscr{V}$ is connected we must have $\alpha \subset V$ for every $V \in \mathscr{V}$. Hence $\alpha \subset \bigcap \mathscr{V}$. This shows that $\bigcap \mathscr{V}$ has at most one connected component.

It remains to show that $\bigcap \mathscr{V} \neq \emptyset$. If card $\mathscr{V} \leqq 2$ this is trivial. Suppose that card $\mathscr{V} = 3$ and write $\mathscr{V} = \{V, V', V''\}$. Let us choose vertices $s \in V \cap V'$ and

$t \in V \cap V''$. There is a unique segment $\sigma \subset T$ with endpoints s and t. Since V is connected we have $\sigma \subset V$. Since V' and V'' are connected and $V' \cap V'' \neq \emptyset$, the set $V' \cup V''$ is connected and hence $\sigma \subset V' \cup V''$. Thus we may write σ as the union of the two closed subsets $\sigma \cap V'$ and $\sigma \cap V''$. Since σ is connected, these two subsets must intersect, i.e. $\emptyset \neq \sigma \cap V' \cap V'' \subset V \cap V' \cap V''$. This proves the assertion when card $\mathscr{V} = 3$.

Finally, suppose that card $\mathscr{V} = n > 3$ and that the assertion is true for sets of cardinality $< n$. Write $\mathscr{V} = \{V_1, \ldots, V_n\}$, and set $W_i = V_i \cap V_n$ for $i = 1, \ldots, n-1$. Since we have proved the proposition for $n = 2$, each W_i is a subtree of T. Since we have also proved the proposition for $n = 3$ we have $W_i \cap W_j = V_i \cap V_j \cap V(n) \neq \emptyset$ for all $i, j \in \{1, \ldots, n-1\}$. Hence by the induction hypothesis we have $\emptyset \neq W_1 \cap \cdots \cap W_{n-1} = V_1 \cap \cdots \cap V_n$. □

2.3.1. **Corollary.** *Let $\mathscr{L}$ be a finite, non-empty collection of lines in a tree T. Suppose that for all $L, L' \in \mathscr{L}$ we have $L \cap L' \neq \emptyset$. Then $\bigcap \mathscr{L}$ is a (possibly degenerate and possibly infinite) segment in T. In particular, $\bigcap \mathscr{L} \neq \emptyset$.* □

2.4. In Section 9 we shall need to apply Corollary 2.3.1 under the more restrictive condition that $\mathscr{E}(L) \cap \mathscr{E}(L') \neq \emptyset$ for all $L, L' \in \mathscr{L}$. In this case we will need to know exactly how the segment $\bigcap \mathscr{L}$ can degenerate. This question is answered by the following result.

Proposition. *Let $\mathscr{L}$ be a finite, non-empty collection of lines in a tree T. Suppose that for all $L, L' \in \mathscr{L}$ we have $\mathscr{E}(L) \cap \mathscr{E}(L') \neq \emptyset$. Let v be a vertex of T. Then we have $\bigcap \mathscr{L} = \{v\}$ if and only if there is a triod Y centered at the vertex v such that every segment of length 2 in Y is contained in some line in $\mathscr{L}$. If such a triod Y does exist, it is uniquely determined by $\mathscr{L}$, and it has the property that every line in $\mathscr{L}$ intersects Y in a segment of length 2.*

Proof. First suppose that there is a triod Y centered at $\{v\}$ such that every length-2 segment in Y is contained in a line in $\mathscr{L}$. Let e_0, e_1, e_2 be the edges of Y. Then there is a line $L_0 \in \mathscr{L}$ containing $e_1 \cup e_2$. By Corollary 2.3.1, $\bigcap \mathscr{L}$ is a segment; clearly $v \in \bigcap \mathscr{L} \subset L_0$. Hence if $\bigcap \mathscr{L}$ is non-degenerate it must contain e_1 or e_2. But there is a line in $\mathscr{L}$ which contains e_0 and e_2, and hence cannot contain e_1; thus $e_1 \not\subset \bigcap \mathscr{L}$. Similarly $e_2 \not\subset \bigcap \mathscr{L}$. Hence $\bigcap \mathscr{L} = \{v\}$.

For the rest of the proof we assume that $\bigcap \mathscr{L} = \{v\}$. We shall construct a triod with the asserted properties, and prove that it is unique. We begin by choosing an arbitrary line $L_0 \in \mathscr{L}$. Then L_0 contains exactly two edges of T incident to v, say f_1 and f_2. Now for $i = 1, 2$ there must exist a line $L_i \in \mathscr{L}$ which does not contain the edge f_i; otherwise we would have $f_i \subset \bigcap \mathscr{L}$, contradicting the hypothesis.

Let us set $\sigma_0 = L_1 \cap L_2$, $\sigma_1 = L_0 \cap L_2$ and $\sigma_2 = L_0 \cap L_1$. By Corollary 2.3.1, each σ_i is a segment. Each segment σ_i contains the vertex v, and is non-degenerate by the hypothesis of the proposition. Hence each σ_i contains an edge e_i incident to v. The edge e_1 is contained in L_0, is incident to v, and is distinct from f_2 since $f_2 \not\subset L_2$. Hence $e_1 = f_1$. Similarly $e_2 = f_2$.

For $i=1,2$ we have $e_0 \subset L_i$ and $e_i = f_i \not\subset L_i$. Hence e_0 is distinct from e_1 and e_2. Thus $Y = \bar{e}_0 \cup \bar{e}_1 \cup \bar{e}_2$ is a triod. Note that $\bar{e}_0 \cup \bar{e}_1 \subset L_2$, $\bar{e}_0 \cup \bar{e}_2 \subset L_1$ and $\bar{e}_1 \cup \bar{e}_2 \subset L_0$. Thus every segment of length 2 in Y is contained in L_i, and is in fact equal to $Y \cap L_i$, for some $i \in \{0, 1, 2\}$.

Now let L be any line in $\mathscr{L}$. We shall show that L meets Y in a length-2 segment. Assume this is false. Then there are at least two edges of Y that are not contained in L. Hence there is a length-2 segment in Y that has no edges in common with L; we may write this length-2 segment in the form $Y \cap L_i$ for some $i \in \{0, 1, 2\}$. By Corollary 2.3.1, the set $\sigma = L_i \cap L$ is a segment in T. By the hypothesis of the Proposition, σ is non-degenerate and contains v. But the two edges of L_i that are incident to v are both contained in Y, and hence neither of them is an edge of L. This is a contradiction.

It remains to prove the uniqueness assertion. Let Y' be any triod in T, centered at v, such that every length-2 segment in Y' is contained in a line in $\mathscr{L}$. Since every line in $\mathscr{L}$ meets the closed star of v in a length-2 segment contained in Y, it follows that every length-2 segment in the triod Y' is contained in the triod Y. This implies that $Y = Y'$. □

2.5. Γ-trees. Let Γ be a group. By a *Γ-tree* we mean a tree T equipped with an action of Γ by simplicial automorphisms. Recall from [13] that Γ is said to act *without inversions* on T if for every $x \in \Gamma$ and every edge e of T which is x-invariant, x fixes the vertices of T. In this case we shall say that the Γ-tree T is *non-inversive.* Recall that if Γ acts without inversion on T, then for every $x \in \Gamma$ exactly one of the following alternatives holds:

(i) $\mathrm{Fix}(x) \neq \emptyset$; or

(ii) x is *T-hyperbolic*, i.e. there is a unique x-invariant line $A \subset T$, called the *axis* of x in T, and $x|A$ is a non-trivial translation.

When x is T-hyperbolic, we shall denote the axis of x in T by $A_T(x)$.

A cyclic subgroup X of Γ will be termed *T-hyperbolic* if it has a T-hyperbolic generator. Note that if X is a T-hyperbolic cyclic subgroup then X is infinite cyclic. Furthermore, it is clear that all non-trivial elements of X are T-hyperbolic and have the same axis in T. If a cyclic subgroup X is T-hyperbolic we shall write $A_T(X)$ for the common axis of the non-trivial elements of X.

2.6. **Proposition.** *Let Θ be a group generated by two elements ξ and η. Let T be a non-inversive Θ-tree, and suppose that ξ and η are both T-hyperbolic. Suppose also that $\mathscr{E}(A_T(\xi)) \cap \mathscr{E}(A_T(\eta)) = \emptyset$. Then Θ is free on the generators ξ and η, and the action of Θ on T is free.*

Proof. This is a special case of [4, Proposition 4.2]. □

3. Surfaces in 3-manifolds

3.1. Bi-collared surfaces and dual graphs. Let M be an orientable 3-manifold without boundary. By a *bi-collared surface* in M we shall mean a map

$c: \mathscr{S} \times [-1, 1] \to M$, where $\mathscr{S}$ is a 2-manifold without boundary and c maps $\mathscr{S} \times [-1, 1]$ homeomorphically onto a closed subset of M. In the case that M has a PL structure, we shall say that c is a PL *bi-collared surface* if $\mathscr{S}$ has a PL structure such that c is a PL map. The set $c(\mathscr{S} \times \{0\})$ is called the *core* of c. Any locally flat 2-manifold without boundary embedded as a closed subset of M is the core of a bi-collared surface c, and c is determined up to isotopy by the given 2-manifold. Studying bi-collared surfaces in M is therefore essentially equivalent to studying locally flat orientable 2-manifolds without boundary in M. However, for the constructions in this paper, including the construction of the dual tree to an incompressible surface, it is more convenient to take bi-collared surfaces as the basic objects.

A bi-collared surface in a closed, orientable 3-manifold M is said to be *incompressible* if its core S is incompressible in the sense defined in the Introduction. (In particular S is then connected and is not a 2-sphere.)

If M is an orientable 3-manifold without boundary and $c: \mathscr{S} \times [-1, 1] \to M$ is a bi-collared surface in M, we shall write $|c| = c(\mathscr{S} \times [-1, 1])$ and $\mathrm{Split}(c) = M - c(\mathscr{S} \times (-1, 1))$. The 3-manifold $\mathrm{Split}(c)$ may be thought of as being obtained by splitting M along the core of c.

We shall denote by $\mathrm{Dual}(c)$ the quotient space of M obtained by identifying each component of the closure of $M - |c|$ to a point, and identifying $c(\mathscr{S} \times \{t\})$ to a point for each component $\mathscr{S}$ of $\mathscr{S}$ and each $t \in [-1, 1]$. Then $\mathrm{Dual}(c)$ has the structure of a graph, of which the vertices are in natural one-one correspondence with the components of $M - |c|$, and the edges are in natural one-one correspondence with the components of $|c|$. We shall denote by q_c the quotient map from M to $\mathrm{Dual}(c)$.

3.2. The tree associated to an incompressible surface. Now let c be an incompressible bi-collared surface in a closed, orientable 3-manifold M. Let $\tilde{M}$ denote the universal covering space of M, let $p: \tilde{M} \to M$ denote the covering projection, and let $\Gamma \cong \pi_1(M)$ denote the group of deck transformations. Let $\tilde{c}$ denote the bi-collared surface in $\tilde{M}$ lying above c. Since $\tilde{M}$ is 1-connected, $\mathrm{Dual}(\tilde{c})$ is also 1-connected; that is, it is a tree. We shall write $T(c) = \mathrm{Dual}(\tilde{c})$. The action of Γ on $\tilde{M}$ induces a natural simplicial action without inversion on $T(c)$. We shall always regard $T(c)$ as a non-inversive Γ-tree by equipping it with this action.

For any vertex v of $T = T(c)$ we shall denote by $\tilde{M}_v$ the set $q_{\tilde{c}}^{-1}(\{v\})$. Note that $\tilde{M}_v$ is a component of $\mathrm{Split}(\tilde{c})$, and covers a component of $\mathrm{Split}(c)$. This component of $\mathrm{Split}(c)$ will be denoted by M_v. For each edge s of T we shall denote by $\tilde{M}_s$ the set $q_{\tilde{c}}^{-1}(\bar{x})$; this set is a component of $|\tilde{c}|$, and covers a component of $|c|$, which will be denoted by M_s.

Let v be any vertex of T. The components of $\partial \tilde{M}_v$ are the sets of the form $\tilde{M}_v \cap \tilde{M}_s$, where s ranges over the edges incident to v. We shall denote the set $\tilde{M}_v \cap \tilde{M}_s$ by $\partial_s \tilde{M}_v$. Each $\partial_s \tilde{M}_v$ covers a component of $\partial \mathrm{Split}(c) = \partial |c|$, which will be denoted by $\partial_s M_v$. Since c is incompressible, the components of $\partial \tilde{M}_v$ are simply connected. As $\tilde{M}$ is also simply connected, it follows from van Kampen's theorem that $\tilde{M}_v$ is simply connected.

The group of deck transformations of the simply connected covering space $\tilde{M}_v$ of M_v is Γ_v. Likewise, the group of deck transformations of the simply connected covering space $\partial_s\tilde{M}_v$ of $\partial_s M_v$ is Γ_s. Thus we have a natural outer isomorphism from $\pi_1(M_v)$ to Γ_v, which will be denoted by v_v. Similarly, there is a natural outer isomorphism $v_{s,v}:\pi_1(\partial_s M_v)\to\Gamma_s$ for every edge s incident to v. We have a commutative diagram of groups and outer homomorphisms.

$$\begin{array}{ccccc} \Gamma_s & \to & \Gamma_v & \to & \Gamma \\ \uparrow^{v_{s,v}} & & \uparrow^{v_v} & & \uparrow^{v} \\ \pi_1(\partial_s M_v) & \to & \pi_1(M_v) & \to & \pi_1(M) \end{array}$$

where v is the natural outer isomorphism between $\pi_1(M)$ and Γ, and where the horizontal arrows in the bottom row represent outer homomorphisms induced by inclusion and those in the top row represent inclusions of groups.

It follows from the commutativity of the right-hand half of the above diagram that the outer subgroup $\mathrm{im}(\pi_1(M_v)\to\pi_1(M))$ of $\pi_1(M)$ is mapped to the outer subgroup Γ_v of Γ by the outer isomorphism v. In particular an element x of Γ has a fixed point in T if and only if the conjugacy class in $\pi_1(M)$ corresponding to the conjugacy class of x in Γ is represented by a closed curve in Split(c).

3.3. Let Γ be a group. A Γ-tree is said to be *minimal* if it has no proper Γ-invariant subtree.

Proposition. *Let c be an incompressible bi-collared surface in a closed, orientable PL 3-manifold M. Let Γ denote the group of deck transformations of the universal covering space of M. Then the Γ-tree $T(c)$ is minimal.*

Proof. Let S denote the core of c. If S does not separate the given closed 3-manifold M then $T=T(c)$ contains only one Γ-orbit of vertices. In this case T is obviously minimal. Now suppose that S separates M. Since S is connected, there is only one Γ-orbit of edges in T. Hence a proper Γ-invariant subtree would consist of a single vertex v, fixed by the entire group Γ. Choose an edge e incident to v, and let w be the second vertex of e. We have $\Gamma_w=\Gamma_e$. In view of the commutative diagram in 3.2, this implies that the natural outer homomorphism $\pi_1(\partial_e M_w)\to\pi_1(M_w)$, induced by inclusion, is an isomorphism. In particular the inclusion homomorphism $H_1(\partial_e M_w)\to H_1(M_w)$ is an isomorphism. According to Poincaré–Lefschetz duality, as M_w is a compact 3-manifold whose boundary is $\partial_e M_w$, the groups $H_1(\partial_e M_w)$ *and* $H_1(M_w)$ can be isomorphic only if S has genus 0; but this contradicts the incompressibility of c. □

3.4. It is a standard observation that if an irreducible 3-manifold M contains an incompressible surface S, there is a covering space of M which deform-retracts to some surface that maps homeomorphically to S under the covering projection. The following result is a somewhat more precise formulation of this observation from the arboreal point of view.

Proposition. *Let $c: S \times [-1,1] \to M$ be an incompressible PL bi-collared surface in a closed, irreducible, orientable PL 3-manifold M. Let $\tilde{M}$ denote the universal covering space of M, and let Γ denote the group of deck transformations. Let s be any edge of $T(c)$. Then $\tilde{M}_s/\Gamma_s$ is a deformation retract of $\tilde{M}/\Gamma_s$, and $\tilde{M}_s/\Gamma_s$ is mapped homeomorphically onto $|c|$ under the induced covering map $\tilde{M}/\Gamma_s \to M$. In particular, $\tilde{M}/\Gamma_s$ is homotopy equivalent to S.*

Proof. The group Γ_s is the stabilizer of $\tilde{M}_s \subset \tilde{M}$ in Γ. The manifold $\tilde{M}$ is connected, and since c is incompressible, $\tilde{M}_s$ is also simply connected. By elementary covering space theory it follows that $\pi_1(\tilde{M}_s/\Gamma_s) \to \pi_1(\tilde{M}/\Gamma_s)$ is an isomorphism, and that $\tilde{M}_s/\Gamma_s$ is mapped homeomorphically onto $|c|$ by the induced covering map. But the incompressibility of c also implies that S has positive genus; hence S is aspherical and $\pi_1(S)$ is infinite. Since $c: S \times [-1,1] \to M$ induces a monomorphism of fundamental groups, $\pi_1(M)$ is also infinite. It follows from the sphere theorem [6] that an irreducible, orientable PL 3-manifold with infinite fundamental group is aspherical. The covering space $\tilde{M}/\Gamma_s$ of the aspherical manifold M is automatically aspherical. Thus $\tilde{M}_s/\Gamma_s \hookrightarrow \tilde{M}/\Gamma_s$ is a map between aspherical triangulable spaces which induces an isomorphism of fundamental groups, and is therefore a homotopy equivalence. □

3.4.1. In view of Proposition 3.4, 3-manifolds of the homotopy type of closed surfaces arise naturally as covering spaces of manifolds containing incompressible surfaces. The following result gives a property of such manifolds that will be useful in Section 8.

Proposition. *Let M be an orientable PL 3-manifold, let S be a closed, connected, orientable surface, and suppose that $\eta: M \to S$ is a homotopy equivalence. Let $F \subset M$ be a closed, connected, orientable PL surface. Then either*
(a) F bounds a compact PL 3-manifold in M, or
(b) $\eta|F: F \to S$ has degree one.
If (b) holds then the inclusion $F \hookrightarrow M$ induces an epimorphism of fundamental groups.

Proof. Suppose that (a) does not hold. Then the image u of the fundamental class of F in $H_2(M; \mathbf{Z})$ is primitive, i.e. it is not of the form nu_0 with $u_0 \in H_2(M; \mathbf{Z})$ and $n > 1$. Now since η is a homotopy equivalence we have $H_2(M; \mathbf{Z}) \cong \mathbf{Z}$, and hence u is a generator of $H_2(M; \mathbf{Z})$. This shows that (b) holds.

Now assume that (b) holds but that $F \hookrightarrow M$ does not induce an epimorphism of fundamental groups. Then $\eta|F: F \to S$ does not induce an epimorphism of fundamental groups. Hence $\eta|F$ lifts to some non-trivial covering space $\tilde{S}$ of S. If $\tilde{S}$ is compact then the covering projection $\tilde{S} \to S$ has a finite degree $d>1$; it follows that d divides the degree of $\eta|F\to S$, which contradicts (b). If $\tilde{S}$ is non-compact then $H_2(\tilde{S})=0$ and hence $\eta|F\to S$ has degree 0, and again (b) is contradicted. □

3.5. The following interesting property of the tree associated to an incompressible bicollared surface will also be needed in Section 8.

Proposition. *Let $c: S\times[-1,1]\to M$ be an incompressible bi-collared surface in a closed, irreducible, orientable PL 3-manifold M. Let Γ denote the group of deck transformations of the universal covering space of M. Let s be any edge of $T=T(c)$. Then the graph T/Γ_s is 1-connected and has no end points.*

Proof. Let $\tilde{M}$ denote the universal covering space of M. Let $\tilde{c}/\Gamma_s$ denote the bi-collared surface in $\tilde{M}/\Gamma_s$ lying over c. We may identify T/Γ_s canonically with $\Psi=\mathrm{Dual}(\tilde{c}/\Gamma_s)$.

To show that Ψ is 1-connected it suffices to show that every edge separates. Let S denote the core of c and $\tilde{S}$ the core of $\tilde{c}$. The edges of Ψ are in canonical one-one correspondence with the components of $\tilde{S}/\Gamma_s$. To show that every edge e separates Ψ, it suffices to show that every component of $\tilde{S}/\Gamma_s$ separates $\tilde{M}/\Gamma_s$.

Let $\tilde{F}_0$ denote the component of $\tilde{S}$ contained in $\tilde{M}_s$ and set $F_0=\tilde{F}_0/\Gamma_s$. By Proposition 3.4, F_0 is a deformation retract of $\tilde{M}/\Gamma_s$. Now let F denote an arbitrary component of $\tilde{S}/\Gamma_s$. Then either $F=F_0$ or $F\cap F_0=\emptyset$. In either case, F is isotopic to a surface disjoint from F_0. Since F_0 carries $H_1(\tilde{M}/\Gamma_s;\mathbf{Z}_2)$, the mod 2 intersection number of F with any class in $H_1(\tilde{M}/\Gamma_s;\mathbf{Z}_2)$ is zero. This implies that F separates $\tilde{M}/\Gamma_s$.

It remains to show that Ψ has no end points. Suppose that Ψ does have an endpoint and let v be a vertex of T which maps to it under the quotient map $T\to\Psi=T/\Gamma_s$. Then $\tilde{M}_v/\Gamma_s\subset\tilde{M}/\Gamma_s$ is a component of $\mathrm{Split}(\tilde{c}/\Gamma_s)$ whose boundary consists of a single component Φ of $\partial|\tilde{c}/\Gamma_s|$. We may write $\tilde{M}/\Gamma_s=(\tilde{M}_v/\Gamma_s)\cup X$, where X is a PL 3-manifold such that $X\cap(\tilde{M}_v/\Gamma_s)=\partial X=\Phi$. Since c is incompressible, $\pi_1(\Phi)\to\pi_1(M/\Gamma_s)$ is injective. Thus if $*$ is a base point in Φ we may identify $\pi_1(\tilde{M}/\Gamma_s,*)$ with an amalgamated free product $G=A*_C B$, where $A=\pi_1(X,*)$, $B=\pi_1(\tilde{M}_v/\Gamma_s,*)$ and $C=\pi_1(\Phi,*)$. In particular we have $A\cap B=C$. But since $\tilde{M}_s/\Gamma_s\subset|\tilde{c}/\Gamma_s|\subset X$, and since $\tilde{M}_s/\Gamma_s$ is a deformation retract of $\tilde{M}/\Gamma_s$, we have $A=G$. Hence $B=C$, i.e. the inclusion homomorphism $\pi_1(\Phi,*)\to\pi_1(\tilde{M}_v/\Gamma_s,*)$ is an isomorphism. It follows that every path in $\tilde{M}_v/\Gamma_s$ with end points in Φ is fixed-endpoint homotopic to a path in Φ.

Now $\tilde{M}_v/\Gamma_s$ covers M_v. Since $\tilde{M}_v/\Gamma_s$ has connected boundary, so does M_v. By the covering homotopy property, every path in M_v with endpoints in ∂M_v is fixed-endpoint homotopic to a path in ∂M_v. In particular the inclusion homomorphism $\pi_1(\partial M_v)\to\pi_1(M_v)$ is surjective. It is injective since c is incompressible. Thus $\pi_1(\partial M_v)\to\pi_1(M_v)$ is an isomorphism, and hence so is $H_1(\partial M_v)\to H_1(M_v)$. According to Poincaré–Lefschetz duality, as M_v is compact, $H_1(\partial M_v)$ and $H_1(M_v)$ can be isomorphic only if ∂M_v has genus 0; but this contradicts the incompressibility of c. □

3.6. The tree $T(c)$ associated with an incompressible bi-collared surface c in a closed PL 3-manifold M gives a useful way of formulating certain results about covering spaces of M. This is illustrated by the following proposition, which will be used in Section 10.

By a *handlebody* we mean a compact 3-manifold which is homeomorphic to the regular neighborhood of a graph in $\mathbf{R}^3$.

Proposition. *Let c be an incompressible bi-collared surface in a closed, orientable 3-manifold M. Let Θ be a finitely generated subgroup of the group Γ of deck transformations of the universal cover $\tilde{M}$ of M, and suppose that the natural action of Γ on $T=T(c)$ restricts to a free action of Θ on T. Then the covering space $\tilde{M}/\Theta$ of M is homeomorphic to the interior of a handlebody.*

Proof. Let us fix a PL structure in which the bi-collared surface c is PL. We can write M as a union $M_1 \cup M_2$, where $M_1 = |c|$ and $M_2 = \text{Split}(c)$.

Theorem 3.1 of [15] deals with a compact PL 3-manifold written in the form $M = M_1 \cup M_2$, where the M_i are compact PL 3-manifolds. The theorem gives sufficient conditions for a covering space of such a manifold M, given in the form $\tilde{M}/\Theta$ where Θ is a finitely generated subgroup of the group Γ of the universal covering space $\tilde{M}$, to be homeomorphic to the interior of a compact 3-manifold. We shall show that the conditions of [15, Theorem 3.1] hold under the hypotheses of Proposition 3.6 if M_1 and M_2 are defined as above and if Θ acts freely on T. Condition (i) of [15, Theorem 3.1] asserts, in the orientable case, that M_1 and M_2 are irreducible. This holds in our situation because M is irreducible and c is incompressible.

Condition (ii), that the components of $M_1 \cap M_2$ are incompressible, also follows from the incompressibility of c. Condition (iii) is equivalent to the condition that if $\tilde{s}$ is any component of the preimage of $M_1 \cap M_2$ in $\tilde{M}/\Theta$, the intersection of Θ with the stabilizer of $\tilde{S}$ in Γ is finitely generated. In the notation of 3.2 we have $\tilde{S} = \partial_s M_v$ for some edge s of T and some vertex v of s. The intersection of Θ with the stabilizer of $\tilde{S}$ is Θ_s. Since Θ is assumed to act freely on T, the group Θ_s is trivial. Finally, condition (iv) is equivalent to the condition that if A is a component of the preimage of M_1 or M_2 in $\tilde{M}/\Theta$, such that the intersection Θ_A of Θ with the stabilizer of A is finitely generated, then A/Θ_A is PL homeomorphic to a manifold of the form $Q-K$, where Q is a compact manifold and K is a closed subset of ∂Q. We have either $A = M_s$, where s is an edge of T, or $A = M_v$, where v is a vertex of T. Hence Θ_A is the stabilizer of s or v respectively, and is therefore trivial since Γ acts freely on T. Thus $A = A/\Theta_A$ is the universal cover of a compact, irreducible manifold with non-empty boundary. By [16, Theorem 8.1], A is PL homeomorphic to $Q-K$, where Q is a closed 3-ball and K is a closed subset of ∂Q.

Thus $\tilde{M}/\Theta$ is PL homeomorphic to the interior of some compact PL 3-manifold J. But by [16, Theorem 8.1], $\tilde{M}$ is homeomorphic to $\mathbf{R}^3$. It follows that $\tilde{M}/\Theta$ is irreducible, and hence that J is irreducible. Furthermore, $\pi_1(J)$ is isomorphic to the group Θ, which acts freely on the tree T and is therefore a free group. It follows from [6, Theorem 5.2] that if a compact, orientable, irreducible 3-manifold J has a free fundamental group then J is a handlebody or a 3-sphere. Since M contains an incompressible surface, $\pi_1(M)$ is infinite, and thus J cannot be a 3-sphere. □

4. The characteristic submanifold

In 3.2 we considered an incompressible bi-collared surface c in a closed, orientable 3-manifold M. For each edge s incident to a vertex v of $T = T(c)$, we defined a natural outer isomorphism $v_{s,v}: \pi_1(\partial_s M_v) \to \Gamma_s$. If s and e are two distinct edges incident to a vertex v of T, we will need a topological interpretation of the outer subgroup $v_{s,v}^{-1}(\Gamma_s \cap \Gamma_e)$ of $\pi_1(\partial_s M_v)$. This requires the use of the theory of the characteristic submanifold [8.9]. The relevant material is reviewed below.

4.1. The following relatively weak and special version of the characteristic submanifold theorem of [9] or [8] contains the information that we need.

Proposition. *Let N be a simple, compact, orientable PL 3-manifold with non-empty boundary. Then there exists a (possibly empty) compact PL 3-manifold $\Sigma \subset N$ having the following properties:*

(i) *$\Phi = \Sigma \cap \partial N$ is a compact PL 2-manifold;*

(ii) *each component of the frontier $\partial\Sigma - \operatorname{int} \Phi$ of Σ is an annulus A such that the natural outer homomorphism $\pi_1(A) \to \pi_1(N)$ is injective;*

(iii) *each component of Σ is either (a) a solid torus Σ_i such that $\Theta_i = \Sigma_i \cap \Phi$ is a non-empty disjoint union of annuli, all homotopically non-trivial in Σ_i, or (b) an I-bundle Σ_i over a non-simply connected, compact, possibly non-orientable PL 2-manifold, such that $\Phi_i = \Sigma_i \cap \Phi$ is the associated ∂I-bundle;*

(iv) *if C is any compact, connected, non-simply-connected, orientable 2-manifold, and $F: (C \times I, C \times \partial I) \to (M, \partial M)$ is any map of pairs such that the outer homomorphism $F_{\#}: \pi_1(C \times I) \to \pi_1(M)$ is non-trivial, and if F is not homotopic (as a map of pairs) to a map from $(C \times I, C \times \partial I)$ to $(\partial M, \partial M)$, then F is homotopic (as a map of pairs) to a map from $(C \times I, C \times \partial I)$ to (Σ, Φ).*

Proof. This is included, for example, in the Characteristic Pair Theorem stated in the introduction to Chapter V of [8]. In this theorem one is given a pair (M, T), where M is a sufficiently large, compact, irreducible, orientable PL 3-manifold and $T \subset \partial M$ is a 2-manifold with boundary such that $\pi_1(T_i) \to \pi_1(M)$ is injective for every component T_i of T. Here we take $(M, T) = (N, \partial N)$. Since the compact, irreducible, orientable PL 3-manifold N has non-empty boundary, it is automatically sufficiently large. The injectivity condition is included in our definition 1.6 of a simple manifold. The conclusion of the Characteristic Pair Theorem of [8] gives a pair (Σ, Φ) such that every component of Σ is either an I-bundle or a Seifert fibered space. For every component Σ_i of Σ the natural outer homomorphism $\pi_1(\Sigma_i)\to\pi_1(M)$ is injective. In our case, since $M=N$ is simple, $\pi_1(\Sigma_i)$ has no free abelian subgroup of rank 2. But the only Seifert fiber space whose fundamental group has no free abelian subgroup of rank 2 is a solid torus. Now that it has been established that the components of (Σ, Φ) are I-bundles and solid tori, the information about how they intersect ∂N is included in the statement of the Characteristic Pair Theorem of [8]. Conclusion (iv) is also included in the latter statement. □

If Σ is any submanifold of N having the properties stated in the Proposition above, we shall refer to the pair (Σ, Φ) as a *characteristic pair* of the simple manifold pair $(N, \partial N)$. (This is a weaker definition than the one given in [8] or [9]: we do not require Φ to be "perfectly embedded" in the sense of [8]. In particular, with the present definition, (Σ, Φ) is unique up to ambient isotopy. However, this definition is well-adapted to the applications in the present paper.)

4.2. **Proposition.** *Let c be an incompressible bi-collared surface in a closed, orientable, irreducible PL 3-manifold M. Let v be any vertex of $T=T(c)$. Let (Σ_v, Φ_v) be a characteristic pair of $(M_v, \partial M_v)$. Let s and e be any two distinct edges incident to v. Then $\Gamma_s\cap\Gamma_e$ is finitely generated. Furthermore, if $\Gamma_s\cap\Gamma_e\neq\{1\}$ then there is a component $\Phi_e\subset\partial_s M_v$ of Φ_v such that the outer subgroup $v_{s,v}^{-1}(\Gamma_s\cap\Gamma_e)$ of $\pi_1(\partial_s M_v)$ is contained in the outer subgroup $\mathrm{im}(\pi_1(\Phi_e)\to\pi_1(\partial_s M_v))$.*

Proof. According to the commutative diagram in 3.2, the outer isomorphism $v_v^{-1}:\Gamma_v\to\pi_1(M_v)$ carries the outer subgroup Γ_s onto the outer subgroup $\mathrm{im}(\pi_1(\partial_s M_v)\to\pi_1(M_v))$. As $\partial_s M_v$ is a boundary component of the compact PL 3-manifold M_v and the natural outer homomorphism $\pi_1(\partial_s M_v)\to\pi_1(M_v)$ is injective, it follows from [7, Proposition 1.4] that a subgroup realizing the outer subgroup $\mathrm{im}(\pi_1(\partial_s M_v)\to\pi_1(M_v))$ has finitely generated intersection with every finitely generated subgroup of $\pi_1(M_v)$. Thus Γ_s has finitely generated intersection with every finitely generated subgroup of Γ_v. But $\Gamma_e\leqq\Gamma_v$ is isomorphic to $\pi_1(\partial_e M_v)$ and is therefore finitely generated. Thus $\Gamma_s\cap\Gamma_e$ is finitely generated.

Now suppose that $\Gamma_s\cap\Gamma_e$ is non-trivial. Consider the PL 3-manifold $J=\tilde{M}_v/(\Gamma_s\cap\Gamma_e)$, which is a covering space of M_v. The universal covering space of J is $\tilde{M}_v$, and the group of covering transformations obviously leaves both $\partial_s\tilde{M}_v$ and $\partial_e\tilde{M}_v$ invariant. It follows that $\partial_s\tilde{M}_v$ and $\partial_e\tilde{M}_v$ project to distinct components Q_s and Q_e of ∂J, and that the inclusion-induced outer homomorphisms $\pi_1(Q_s)\to\pi_1(J)$ and $\pi_1(Q_e)\to\pi_1(J)$ are both surjective. It follows from the incompressibility of c that these outer homomorphisms are in fact both outer isomorphisms.

Since $\pi_1(Q_s)\cong\pi_1(J)\cong\Gamma_s\cap\Gamma_e$ is finitely generated, there exists a compact connected PL 2-manifold $C\subset Q_s$ such that $\pi_1(C)\to\pi_1(J)$ is surjective. We may take C to have non-empty boundary, so that C is homotopy-equivalent to a 1-complex. Since $\pi_1(Q_e) \to \pi_1(J)$ is surjective, the inclusion $C \subsetneq Q_s$ is homotopic in J to a map of C into Q_e. Let $\tilde{F}: C \times [0,1] \to J$ be a homotopy such that $\tilde{F}_0$ is the inclusion and $\tilde{F}_1(C) \subset Q_e$. Then $\tilde{F}$ projects via the covering map $p: J \to M_v$ to a homotopy $F: C \times [0,1] \to M_v$.

Since the lift $\tilde{F}$ of F maps $C \times \{0\}$ and $C \times \{1\}$ to distinct components Q_s and Q_e of ∂J, the map of pairs $F: (C \times [0,1], C \times \{0,1\}) \to (M_v, \partial M_v)$ is not homotopic to a map into $(\partial M_v, \partial M_v)$. Furthermore, since we have assumed that $\Gamma_s \cap \Gamma_e$ is non-trivial, the outer homomorphism $F_{\#}: \pi_1(C \times [0,1]) \to \pi_1(M)$ is non-trivial. By Property (iv) of the characteristic pair from 4.1, it follows that $F = p \circ \tilde{F}$ is homotopic as a map of pairs to a map into (Σ_v, Φ_v). In particular, $p|C$ is homotopic in $\partial_s M_v$ to a map of C into some component $\Phi_e \subset \partial_s M_v$ of Φ_v. Hence the outer subgroup $v_{s,v}^{-1}(\Gamma_s \cap \Gamma_e) = (p|C)_{\#}(\pi_1(C))$ of $\pi_1(\partial_s M_v)$ is contained in the outer subgroup $\mathrm{im}(\pi_1(\Phi_e) \to \pi_1(\partial_s M_v))$, as asserted. □

4.3. The following characterization of books of I-bundles in terms of the characteristic submanifold theory will be used in Section 8.

Proposition. *Let N be a compact, simple, orientable, PL 3-manifold with non-empty boundary. Let (Σ, Φ) be a characteristic pair of $(N, \partial N)$, and suppose that every component of $\partial N - \mathrm{int}\,\Phi$ is an annulus. Then N is a book of I-bundles.*

Proof. Let W denote the union of all components of Σ which are solid tori, and let E denote the union of all other components of Σ. Then according to 4.1, each component of E is an I-bundle over a compact PL 2-manifold-with-boundary, and intersects ∂N in the associated ∂I-bundle. Hence we may regard E as an I-bundle over a possibly disconnected PL 2-manifold-with-boundary B, and the frontier of B is the preimage of ∂B under the bundle projection $E \to B$.

Now let V denote the closure of $N - E$. The boundary of V is the union of the three sets $X = W \cap \partial N$, $Y = \partial N - \mathrm{int}\,\Phi$, and the frontier Z of E in N. By 4.1, the components of X are annuli. The hypothesis of the proposition says that the components of Y are annuli. The components of Z are in particular components of the frontier of Σ, and are therefore annuli by 4.1. Thus ∂V is a union of annuli in a natural way, and it is clear that any component of intersection of two of these annuli is a common boundary component. Thus each component of ∂V has Euler characteristic 0. Since N is orientable it follows that ∂V is a disjoint union of tori.

Let V_i be any component of V. It is clear from 4.1 that $\partial V_i \neq \emptyset$. Choose any component S of ∂V_i. Then S is a torus. On the other hand, by 4.1, each component of the frontier of V_i is an annulus A such that the natural outer homomorphism $\pi_1(A) \to \pi_1(N)$ is injective. This implies that the natural outer homomorphism $\pi_1(V_i) \to \pi_1(N)$ is injective. Since N is simple it follows that $\pi_1(V_i)$ has no free abelian subgroup of rank 2. Hence the natural outer homomorphism $\pi_1(S) \to \pi_1(V_i)$ has a non-trivial kernel. It then follows from the loop theorem (see [6]) that there is a PL disk $D \subset V_i$ such that $\partial D = D \cap \partial V_i$ is a homotopically non-trivial simple closed curve in S. If R is a regular neighborhood of D in V_i, the closure of $V_i - R$ is a PL 3-manifold whose boundary contains a 2-sphere; using the irreducibility of N we conclude that $\overline{V_i - R}$ is a ball, and hence that V_i is a solid torus. In view of the definition it is now clear that N is a book of I-bundles. □

5. Platings

5.1. Let Ω be a topological space. Recall that a family of closed subsets of Ω is said to be *locally finite* if every point of Ω has a neighborhood which meets only finitely many sets in the family. By a *plating* of Ω we shall mean a locally finite family $\mathscr{C} = (C_i)_{i \in \mathscr{I}}$ of non-empty closed subsets of Ω, indexed by some set $\mathscr{I}$, such that $\bigcup_{i \in \mathscr{I}} C_i = \Omega$. (We have avoided the word "covering" in this context because in the applications Ω will often arise as a covering space.)

A plating $\mathscr{C} = (C_i)_{i \in \mathscr{I}}$ of a space Ω will be said to have *finite order* if there is an integer m such that for any point $P \in \Omega$ there are at most $m+1$ distinct indices $i \in \mathscr{I}$ for which $P \in C_i$. The smallest such integer m will then be called the *order* of $\mathscr{C}$.

5.2. *Γ-platings.* Now suppose that Γ is a group and that Ω is a Γ-space. By a *Γ-plating* of Ω we shall mean a triple $(\mathscr{I}, \mathscr{G}, \mathscr{C})$, where $\mathscr{I}$ is a Γ-set, $\mathscr{G} = (G_i)_{i \in \mathscr{I}}$ is a family of subgroups of Γ and $\mathscr{C} = (C_i)_{i \in \mathscr{I}}$ is a plating of Ω, and the following conditions are satisfied:

(i) G_i is contained in the stabilizer Γ_i of i for every $i \in \mathscr{I}$;
(ii) $\gamma G_i \gamma^{-1} = G_{\gamma \cdot i}$ for every $i \in \mathscr{I}$ and every $\gamma \in \Gamma$; and
(iii) $\gamma \cdot C_i = C_{\gamma \cdot i}$ for every $i \in \mathscr{I}$ and every $\gamma \in \Gamma$.

Recall that for any i in the Γ-set $\mathscr{I}$, $\Gamma_i \leqq \Gamma$ denotes the stabilizer of $i \in \mathscr{I}$. It is clear from the definition of a Γ-plating that C_i is invariant under G_i.

5.3. **Proposition.** *Let Γ be a group and let Ω be a uniform Γ-space. Suppose that $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is a Γ-plating of Ω. Then $\mathscr{C}$ has finite order.*

Proof. Let $R \subset \Omega$ be a compact set such that $\Gamma \cdot R = \Omega$. Write $\mathscr{C} = (C_i)_{i \in \mathscr{I}}$. Let $\mathscr{I}_0$ denote the set of all indices $i \in \mathscr{I}$ such that $C_i \cap R \neq \emptyset$. Since the plating $\mathscr{C}$ is by definition locally finite, the cardinality k of $\mathscr{I}_0$ is finite. In particular, for every $P \in R$ there are at most k indices i such that $P \in C_i$. It now follows from the equivariance condition (iii) in the definition of a Γ-plating that for any $P \in \Omega$ there are at most k indices i such that $P \in C_i$. □

The *order* of a Γ-plating $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ of a uniform Γ-space Ω is defined to be the order of $\mathscr{C}$.

5.4. **Proposition.** *Let Γ be a group and let $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ be a Γ-plating of a uniform Γ-space Ω, where $\mathcal{C}=(C_i)_{i\in\mathcal{I}}$. Then the Γ-set $\mathcal{I}$ contains only finitely many Γ-orbits.*

More generally, for any positive integer d, let $\mathcal{J}_d$ denote the subset of the Cartesian power $\mathcal{I}^d$ consisting of all d-tuples $(i_1, \ldots, i_d)$ such that $C_{i_1} \cap \cdots \cap C_{i_d} \neq \emptyset$. Then $\mathcal{J}_d$ is a union of finitely many Γ-orbits under the diagonal action of Γ on $\mathcal{I}^d$.

Proof. Let $R \subset \Omega$ be a compact set such that $\Gamma \cdot R = \Omega$. Let Φ_d denote the subset of $\mathcal{J}_d$ consisting of all d-tuples $(i_1, \ldots, i_d)$ such that $C_{i_1} \cap \cdots \cap C_{i_d} \cap R \neq \emptyset$. Since $\mathcal{C}$ is locally finite. Φ_d is finite. But for any $(i_1, \ldots, i_d) \in \mathcal{J}_d$ there is a point $P \in C_{i_1} \cap \cdots \cap C_{i_d}$, and there is an element γ of Γ such that $\gamma \cdot P \in R$. It follows that $(\gamma \cdot i_1, \ldots, \gamma \cdot i_d) \in \Phi_d$. This proves the final assertion of the proposition. Since the C_i are non-empty according to the definition of a plating, we have $\mathcal{J}_1 = \mathcal{I}$, and so the first assertion is a special case of the final assertion. □

5.5. *Uniform Γ-platings.* We shall say that the Γ-plating $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ of Ω is *uniform* if for each $i \in \mathcal{I}$ the G_i-set C_i is uniform.

5.6. *Induced platings of invariant subsets.* Suppose that $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ is a Γ-plating of a uniform Γ-space Ω, where $\mathcal{C}=(C_i)_{i\in\mathcal{I}}$ and $\mathcal{G}=(G_i)_{i\in\mathcal{I}}$. Suppose that Ω' is a closed Γ-invariant subspace of Ω. Let $\mathcal{I}'$ denote the set of all indices $i \in \mathcal{I}$ such that $C_i \cap \Omega' \neq \emptyset$, and set $\mathcal{C}'=(C_i \cap \Omega')_{i\in\mathcal{I}_\Omega}$ and $\mathcal{G}'=(G_i)_{i\in\Omega'}$. Then it is clear that $(\mathcal{I}', \mathcal{G}', \mathcal{C}')$ is a Γ-plating of Ω' and that its order is at most the order of $(\mathcal{I}, \mathcal{G}, \mathcal{C})$. If $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ is uniform then it follows from 1.4 that $(\mathcal{I}', \mathcal{G}', \mathcal{C}')$ is uniform.

5.7. *Boundary platings.* Let Γ be a group, let Ω be a uniform Γ-space, and let $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ be a Γ-plating of Ω, where $\mathcal{C}=(C_i)_{i\in\mathcal{I}}$ and $\mathcal{G}=(G_i)_{i\in\mathcal{I}}$. Let s be an index in $\mathcal{I}$, and let Q_s denote the frontier of C_s. Since C_s is G_s-invariant by 5.2, Q_s is also G_s-invariant. It follows that the set $\mathcal{I}^s \subset \mathcal{I}$, consisting of all indices $i \neq s$ such that $C_i \cap Q_s \neq \emptyset$, is also G_s-invariant, and therefore has the structure of a G_s-set. Let us set $\mathcal{G}^s=(G_i \cap G_s)_{i\in\mathcal{I}^s}$ and $\mathcal{C}^s=(C_i \cap Q_s)_{i\in\mathcal{I}^s}$.

Proposition. *With the above notation, $(\mathcal{I}^s, \mathcal{G}^s, \mathcal{C}^s)$ is a G_s-plating of Q_s, and its order is strictly less than the order of $(\mathcal{I}, \mathcal{G}, \mathcal{C})$. If $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ is uniform then Q_s is a uniform G_s-space.*

Proof. Since the plating $\mathcal{C}$ is by definition a locally finite family of closed subsets of Ω, it is clear that $\mathcal{C}^s$ is a locally finite family of closed subsets of Q_s. To show that $\mathcal{C}^s$ is a plating of Q_s we must show that $Q_s \subset \bigcup_{i\in\mathcal{I}_s} C_i$. For this purpose, note that the set $D_s = \bigcup_{s\neq i\in\mathcal{I}} C_s$ is the union of a locally finite family of closed subsets of Ω and is therefore closed. Since $\Omega = C_s \cup D_s$, it follows that the frontier Q_s of C_s is contained in D_s. This immediately implies that $Q_s \subset \bigcup_{i\in\mathcal{I}_s} C_i$, so that $\mathcal{C}^s$ is a plating of Q_s. Since the triple $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ satisfies conditions (i)–(iii) of Definition 5.2, the triple $(\mathscr{I}^s, \mathscr{G}^s, \mathscr{C}^s)$ satisfies the same conditions with G_s in place of Γ. Thus $(\mathscr{I}^s, \mathscr{G}^s, \mathscr{C}^s)$ is a G_s-plating.

Now let $m < \infty$ denote the order of $\mathscr{C}$, and let P be any point of Q_s. There are at most $m+1$ indices $i \in \mathscr{I}$ such that $P \in C_i$. But s is one such index, since $P \in Q_s \subset C_s$. Hence there are at most m indices $i \neq s$ such that $P \in Q_i$. It follows at once that $\mathscr{C}^s$ has order at most m.

If $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is uniform then by definition C_s is a uniform G_s-space. Since Q_s is a closed, G_s-invariant subset of C_s, it follows from 1.4 that Q_s is also a uniform G_s-space. □

The G_s-plating $(\mathscr{I}^s, \mathscr{G}^s, \mathscr{C}^s)$ will be called the *boundary plating* of Q_s induced by the Γ-plating $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ of Ω.

5.7.1. Doubly uniform platings. It will be convenient to formulate a sufficient condition for a boundary plating to be uniform.

Let $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ be a Γ-plating of a Γ-space Ω, where $\mathscr{G} = (G_i)_{i \in \mathscr{I}}$ and $\mathscr{C} = (C_i)_{i \in \mathscr{I}}$. Note that for any two indices $i, i' \in \mathscr{I}$ the set $C_i \cap C_{i'}$ is $G_i \cap G_{i'}$-invariant. We shall say that the Γ-plating $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is *doubly uniform* if for all $i, i' \in \mathscr{I}$ the set $C_i \cap C_{i'}$ is a uniform $G_i \cap G_{i'}$-space.

Proposition. *Let Γ be a group, and let $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ be a doubly uniform Γ-plating of a Γ-space Ω, where $\mathscr{C} = (C_i)_{i \in \mathscr{I}}$ and $\mathscr{G} = (G_i)_{i \in \mathscr{I}}$. Then $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is uniform. Furthermore, for any index $s \in \mathscr{I}$, the boundary-plating of the frontier of C_s induced by $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is a uniform G_s-plating.*

Proof. By applying the above definition when $i = i'$, we see that C_i is a uniform G_i-space for every $i \in \mathscr{I}$. This proves that $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is uniform.

Now let s be any index in $\mathscr{I}$, let Q_s denote the frontier of C_s and let $\mathscr{I}^s$ be defined as in 5.7. Since $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is doubly uniform, $C_i \cap C_s$ is a uniform $(G_i \cap G_s)$-space for every $i \in \mathscr{I}^s$. Hence by 1.4, the $G_i \cap G_s$-invariant closed subset $C_i \cap Q_s$ of $C_i \cap C_s$ is also a uniform $(G_i \cap G_s)$-space. □

6. Neighborhoods of platings

6.1. Let $\mathscr{C} = (C_i)_{i \in \mathscr{I}}$ be a plating of a topological space Ω. A *neighborhood* of $\mathscr{C}$ is a family $\mathscr{C}' = (C'_i)_{i \in \mathscr{I}}$ of subsets of Ω, indexed by the same set $\mathscr{I}$, such that $C_i \subset \operatorname{int} C'_i$ for every $i \in \mathscr{I}$. A neighborhood $\mathscr{C}' = (C'_i)_{i \in \mathscr{I}}$ of $\mathscr{C}$ will be termed *closed* if C'_i is a closed subset of Ω for every $i \in \mathscr{I}$. Note that if a neighborhood C' of a plating C is closed and is a locally finite family, then it is itself a plating of Ω.

6.2. Now let Γ be a group and let $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ be a Γ-plating of a Γ-space Ω. A *neighborhood* of $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is a triple of the form $(\mathscr{I}, \mathscr{G}, \mathscr{C}')$, where

(i) $\mathscr{C}' = (C'_i)_{i \in \mathscr{I}}$ is a neighborhood of C, and

(ii) $\gamma \cdot C_i = C_{\gamma \cdot i}$ for every $i \in \mathscr{I}$.

Note that by definition a neighborhood of a Γ-plating involves the same underlying Γ-set and the same indexed family of subgroups $\mathcal{G}$ as the given Γ-plating.

If $(\mathcal{I}, \mathcal{G}, \mathcal{C}')$ is a neighborhood of the Γ-plating $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ such that $\mathcal{C}'$ is a locally finite family, we shall say that $(\mathcal{I}, \mathcal{G}, \mathcal{C}')$ is locally finite. Similarly, if $\mathcal{C}'$ is closed we shall say that $(\mathcal{I}, \mathcal{G}, \mathcal{C}')$ is closed. Note that any locally finite closed neighborhood of a Γ-plating of Ω is itself a Γ-plating of Ω.

Any Γ-plating $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ of a Γ-space Ω has the neighborhood $(\mathcal{I}, \mathcal{G}, \mathcal{U})$ given by $\mathcal{U} = (\Omega)_{\iota \in \mathcal{I}}$. This neighborhood is never locally finite unless $\mathcal{I}$ is finite.

6.3. **Proposition.** *Let Γ be a group, let $\tilde{M}$ be a uniform PL Γ-manifold of some dimension $n > 0$, and let $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ be a Γ-plating of $\tilde{M}$, where $\mathcal{C} = (C_i)_{\iota \in \mathcal{I}}$. Let $(\mathcal{I}, \mathcal{G}, \mathcal{C}^+)$ be any neighborhood of $(\mathcal{I}, \mathcal{G}, \mathcal{C})$. Then there is a closed, locally finite neighborhood $(\mathcal{I}, \mathcal{G}, \mathcal{C}')$ of $(\mathcal{I}, \mathcal{G}, \mathcal{C})$, where $\mathcal{C}' = (C'_i)_{\iota \in \mathcal{I}}$, such that*

(i) *$(\mathcal{I}, \mathcal{G}, \mathcal{C}^+)$ is a neighborhood of $(\mathcal{I}, \mathcal{G}, \mathcal{C}')$;*

(ii) *C'_i is a PL n-manifold for every $i \in \mathcal{I}$;*

(iii) *there is a Γ-invariant triangulation of $\tilde{M}$ in which all the C_i are subcomplexes; and*

(iv) *if $i_1, \dots, i_r$ are indices in $\mathcal{I}$ such that $C_{i_1} \cap \cdots \cap C_{i_r} = \emptyset$, then $C'_{i_1} \cap \cdots \cap C'_{i_r} = \emptyset$. Furthermore, if $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ is uniform (or doubly uniform), we can take $(\mathcal{I}, \mathcal{G}, \mathcal{C}')$ to be uniform (or, respectively, doubly uniform).*

6.3.1. *Remark.* It follows from condition (iv) that if $(\mathcal{I}, \mathcal{G}, \mathcal{C})$ has order $m < \infty$ then $(\mathcal{I}, \mathcal{G}, \mathcal{C}')$ also has order m. This will be important when we apply Proposition 6.3 and its corollaries 6.4 and 6.5.

6.3.2. The next four lemmas will be used in the proof of Proposition 6.3. In these lemmas, Γ will denote a group, $\tilde{M}$ will denote a PL Γ-manifold of some dimension $n > 0$, and $\mathcal{T}$ will denote a Γ-invariant triangulation of $\tilde{M}$ which defines its given PL structure. For each integer $d \geqq 0$ we shall denote by $\mathcal{T}^{(d)}$ the d-th barycentric subdivision of $\mathcal{T}$. For each closed set $C \subset \tilde{M}$ and each $d \geqq 0$ we shall let $N_d(C)$ denote the union of all closed simplices in $\mathcal{T}^{(d)}$ that meet C_i, and we shall let $N_d^*(C)$ denote the union of all closed simplices of $\mathcal{T}^{(d+2)}$ that meet $N_d(C)$. Then $N_d^*(C)$ is a regular neighborhood of $N_d(C)$; in particular, it is a PL n-manifold and a neighborhood of C. It is clear that $N_d^*(C) \subset N_d(C)$ for every $d \geqq 0$, and that $\bigcap_{d \geqq 0} N_d(C) = C$. Hence $\bigcap_{d \geqq 0} N_d^*(C) = C$. It is also clear that $N_d(\gamma \cdot C) = \gamma \cdot N_d(C)$ and $N_d^*(\gamma \cdot C) = \gamma \cdot N_d^*(C)$ for every $\gamma \in \Gamma$ and every $d \geqq 0$. In particular, if the closed set $C \subset \tilde{M}$ is invariant under a subgroup G of Γ then $N_d(C)$ and $N_d^*(C)$ are G-invariant.

6.3.3. **Lemma.** *If C is a closed subset of $\tilde{M}$ invariant under a subgroup G of Γ, and if C is a uniform Γ-space, then $N_d(C)$ and $N_d^*(C)$ are uniform for every $d \geqq 0$.*

Proof. Let $R \subset C$ be a compact set such that $G \cdot R = C$. Then $N_d(R)$ and $N_d^*(R)$ are compact, and we have $G \cdot N_d(R) = N_d(C)$ and $G \cdot N_d^*(C) = N_d^*(C)$. □

6.3.4. **Lemma.** *Suppose that $(C_i)_{i\in\mathcal{I}}$ is a locally finite indexed family of closed subsets of $\tilde{M}$. Then $(N_0(C_i))_{i\in\mathcal{I}}$ is also a locally finite family.*

Proof. It is enough to show that for every open simplex σ of $\mathcal{T}$ there are only finitely many indices $i\in\mathcal{I}$ such that $C_i\cap\sigma\neq\emptyset$. But if $C_i\cap\sigma\neq\emptyset$ then C_i meets some closed simplex having σ as a face; that is, $C_i\cap K\neq\emptyset$, where K is the closure of the star of σ in $\mathcal{T}$. Since K is compact, the assertion follows. □

6.3.5. **Lemma.** *Let $(\mathcal{I},\mathcal{G},\mathcal{C})$ be a Γ-plating of $\tilde{M}$, and let $t\geqq 0$ be an integer. Then there is an integer $d\geqq 0$ such that for any set $\Phi\subset\mathcal{I}$ we have $\bigcap_{i\in\Phi}N_d(C_i)\subset N_t(\bigcap_{i\in\Phi}C_i)$.*

Proof. Suppose that t is an integer for which there exists no d with the stated property. Then for every $d\geqq 0$ we can find a set $\Phi_d\subset\mathcal{I}$ and a point $P_d\in\bigcap_{i\in\Phi_d}N_d(C_i)$ which does not lie in $N_t(\bigcap_{i\in\Phi_d}C_i)$. Now since $\tilde{M}$ is Γ-uniform, there is a compact set $R\subset\tilde{M}$ such that $\Gamma\cdot R=\tilde{M}$. For each $d\geqq 0$ there is an element γ_d of Γ such that $\gamma_d\cdot P_d\in R$. We have

$$\gamma_d\cdot P_d\in\bigcap_{i\in\Phi_d}\gamma_d\cdot N_d(C_i)=\bigcap_{i\in\Phi_d}N_d(C_{\gamma_d\cdot i})=\bigcap_{i\in\gamma_d^{-1}\cdot\Phi_d}N_d(C_i).$$

Similarly

$$\gamma_d\cdot P_d\notin N_t\Big(\bigcap_{i\in\gamma_d^{-1}\cdot\Phi_d}N_d C_i\Big).$$

Thus after replacing the P_d by the $\gamma_d\cdot P_d$ and the Φ_d by the $\gamma_d^{-1}(\Phi_d)$ we may assume that all the P_d lie in R.

In particular, R meets $N_d(C_i)\subset N_0(C_i)$ for every $i\in\Phi_d$. Since the family $(N_0(C_i))_{i\in\mathcal{I}}$ is locally finite by Lemma 6.3.4, it follows that $\bigcup_{d\geqq 0}\Phi_d$ is a finite subset of $\mathcal{I}$. Hence the sequence Φ_0, $\Phi_1,\ldots$ contains only finitely many distinct sets; in particular there is a sequence $d_1<d_2<\ldots$ such that all the Φ_{d_i} are equal to the same set Φ. The sequence $(P_{d_k})_{k\geqq 0}$ lies in the compact set R. Hence after replacing the sequence $(d_k)_{k\geqq 0}$ by a subsequence we may assume that $(P_{d_k})_{k\geqq 0}$ converges to some point $P\in\tilde{M}$.

We have $P_{d_k}\in N_{d_k}(C_i)$ for every $i\in\Phi$ and every $k\geqq 1$. Since d_k tends to infinity with k it follows that $P\in C_i$ for every $i\in\Phi$, i.e. $P\in\bigcap_{i\in\Phi}C_i$. But we have $P_{d_k}\notin N_t(\bigcap_{i\in\Phi}C_i)$. This is a contradiction. □

6.3.6. **Lemma.** *Let $(\mathcal{I},\mathcal{G},\mathcal{C})$ be a Γ-plating of $\tilde{M}$, where $\mathcal{C}=(C_i)_{i\in\mathcal{I}}$. Let $(\mathcal{I},\mathcal{G},\mathcal{C}^+)$ be a neighborhood of $(\mathcal{I},\mathcal{G},\mathcal{C})$, and write $\mathcal{C}^+=(C_i^+)_{i\in\mathcal{I}}$. Then there is an integer $d\geqq 0$ such that $N_d(C_i)\subset\operatorname{int}C_i^+$ for every $i\in\mathcal{I}$.*

Proof. We proceed in the same spirit as in the proof of Lemma 6.3.5. Assume that there is not integer d with the stated property. Then for every $d\geqq 0$ there exist an index $i_d\in\mathcal{I}$ and a point P_d which lies in $N_d(C_{i_d})$ but not in $C_{i_d}^+$. Let R be a compact subset of $\tilde{M}$ such that $\Gamma\cdot R=\tilde{M}$, and choose $\gamma_d\in\Gamma$ such that $\gamma_d\cdot P_d\in R$. Then $\gamma_d\cdot P_d$ lies in $N_d(C_{\gamma_d})$ but not in $C_{\gamma_d\cdot i_d}^+$. Thus after replacing the

P_d by $\gamma_d \cdot P_d$ and the i_d by $\gamma_d \cdot i_d$ we may assume that all the P_i lie in R. Now for each d we have $P_d \in N_d(C_{i_d}) \subset N_0(C_{i_d})$, and by Lemma 6.3.4 the family $(N_0(C_i))_{i \in \mathscr{I}}$ is locally finite. Since R is compact it follows that the sequence $(i_d)_{d \geqq 0}$ contains only finitely many distinct terms. Hence there is a sequence $d_1 < d_2 < \ldots$ such that all the i_{d_k} are equal to a single index s. After replacing $(d_k)_{k \geqq 0}$ by a subsequence we may assume that (P_{d_k}) converges to a point P. Since $P_{d_k} \in N_{d_k}(C_s)$ and d_k tends to infinity with k, we have $P \in C_s$. Since the P_{d_k} lie outside the neighborhood C_s^+ of C_s, we have a contradiction. □

6.3.7. *Proof of Proposition* 6.3. We fix a Γ-invariant triangulation $\mathscr{T}$ of $\tilde{M}$ and use the notation of 6.3.2. For each integer $d \geqq 0$ we have a family $\mathscr{C}^d = (N_d^*(C_i))_{i \in \mathscr{I}}$ of closed subsets of $\tilde{M}$. For every i and every d the set $N_d^*(C_i)$ is a neighborhood of C_i. By the remarks in 6.3.2 we have $\gamma \cdot N_d^*(C_i) = N_d^*(\gamma \cdot C_i) = N_d^*(C_{\gamma \cdot i})$ for all i, d and γ. Thus $\mathscr{C}^d$ is a neighborhood of $\mathscr{C}$ for every d.

We obviously have $N_d^*(C_i) \subset N_0(N_0 C_i))$ for every i. By two successive applications of Lemma 6.3.4 the family $(N_0(N_0(C)))_{i \in \mathscr{I}}$ is locally finite. Hence $\mathscr{C}^d$ is locally finite for every d. Furthermore, it is clear that for every d conditions (ii) and (iii) hold for $\mathscr{C}' = \mathscr{C}^d$. We shall show that for sufficiently large d the other conclusions of the proposition hold as well.

Lemma 6.3.6 shows that if d is sufficiently large then for every index $i \in \mathscr{I}$ we have $N_d(C_i) \subset \operatorname{int} C_i^+$. Since $N_{d+2}^*(C_i) \subset N_d(C_i)$ for all d and i, it follows that if d is sufficiently large then for every index $i \in \mathscr{I}$ we have $N_d^*(C_i) \subset \operatorname{int} C_i^+$. This shows that (i) holds for large d.

Lemma 6.3.5 implies that if d is sufficiently large, then for any indices $i_1, \ldots, i_r$ in $\mathscr{I}$ we have $N_d(C_{i_1}) \cap \cdots \cap N_d(C_{i_r}) \subset N_1(C_{i_1} \cap \cdots \cap C_{i_r})$. In particular if $C_{i_1} \cap \cdots \cap C_{i_r} = \emptyset$ then $N_d(C_{i_1}) \cap \cdots \cap N_d(C_{i_r}) = \emptyset$. Again using that $N_{d+2}^*(C_i) \subset N_d(C_i)$ for all d and i, we conclude that conclusion (iv) holds for large d.

Now suppose that $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is uniform. Then for each $i \in \mathscr{I}$ the G_i-space C_i is uniform. By Lemma 6.3.3, $N_d^*(C_i)$ is uniform for every d. This shows that $\mathscr{C}^d$ is uniform for every d.

Finally, suppose that $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is doubly uniform. By Lemma 6.3.5, if d is sufficiently large then for all indices $i, i' \in \mathscr{I}$ we have $N_d(C_i) \cap N_d(C_{i'}) \subset N_0(C_i \cap C_{i'}) \subset N_0(C_i \cap C_{i'})$. Since $N_{d+2}^*(C_i) \subset N_d(C_i)$ for all d and i, it follows that if d is sufficiently large then for all $i, i' \in \mathscr{I}$ we have $N_d^*(C_i) \cap N_d^*(C_{i'}) \subset N_0(C_i \cap C_{i'})$. *Since* $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is doubly uniform the $G_i \cap G_{i'}$-space $C_i \cap C_{i'}$ is uniform. Hence by Proposition 6.3.3, the $G_i \cap G_{i'}$-space $N_0(C_d \cap C_{i'})$ is uniform; by 1.4, its closed invariant subspace $N_d^*(C_{i'}) \cap N_d^*(C_{i'})$ is also uniform. This proves that $\mathscr{C}^d$ is doubly uniform for large d. □

6.4. We can apply Proposition 6.3 to any Γ-plating $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ of $\tilde{M}$ by taking $(\mathscr{I}, \mathscr{G}, \mathscr{C}^+)$ to be the neighborhood $(\mathscr{I}, \mathscr{G}, \mathscr{U})$ of $(\mathscr{I}, \mathscr{G}, \mathscr{C})$, where $\mathscr{U} = (\tilde{M})_{i \in \mathscr{I}}$ (cf. 6.2). We record the special case in which $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is uniform, which will be needed in Section 9.

Corollary. *Let Γ be a group, let $\tilde{M}$ be a PL Γ-manifold of some dimension $n>0$, and let $(\mathcal{I},\mathcal{G},\mathcal{C})$ be a uniform Γ-plating of $\tilde{M}$, where $\mathcal{C}=(C_i)_{i\in\mathcal{I}}$. Then there is a closed, locally finite neighborhood $(\mathcal{I},\mathcal{G},\mathcal{C}')$ of $(\mathcal{I},\mathcal{G},\mathcal{C})$, where $\mathcal{C}'=(C'_i)_{i\in\mathcal{I}}$, such that*

(i) *C'_i is a PL n-manifold for every $i\in\mathcal{I}$;*

(ii) *there is a Γ-invariant triangulation of $\tilde{M}$ in which all the C_i are subcomplexes;*

(iii) *if $i_1,\dots,i_r$ are indices in $\mathcal{I}$ such that $C_{i_1}\cap\cdots\cap C_{i_r}=\emptyset$, then $C'_{i_1}\cap\cdots\cap C'_{i_r}=\emptyset$; and*

(iv) *$(\mathcal{I},\mathcal{G},\mathcal{C}')$ is uniform.* □

6.5. Another special case of Proposition 6.3 is the case in which Γ is the trivial group.

Corollary. *Let M be a compact PL n-manifold, and let $\mathcal{C}=(C_i)_{i\in\mathcal{I}}$ be a finite plating of M. Let $\mathcal{C}^+$ be a neighborhood of $\mathcal{C}$. Then there is a closed neighborhood $\mathcal{C}'=(C_i)_{i\in\mathcal{I}}$ of $\mathcal{C}$ such that*

(i) *$\mathcal{C}^+$ is a neighborhood of $\mathcal{C}'$;*

(ii) *C'_i is a PL n-manifold for every $i\in\mathcal{I}$;*

and

(iii) *if $i_1,\dots,i_r$ are indices in $\mathcal{I}$ such that $C_{i_1}\cap\cdots\cap C_{i_r}=\emptyset$, then $C'_{i_1}\cap\cdots\cap C'_{i_r}=\emptyset$.* □

7. Precise platings, LERF groups and 2-manifolds

7.1. Precise Γ-platings. Let $(\mathcal{I},\mathcal{G},\mathcal{C})$ be a Γ-plating of a Γ-space Ω, where $\mathcal{G}=(G_i)_{i\in\mathcal{I}}$ and $\mathcal{C}=(C_i)_{i\in\mathcal{I}}$. We will say that $(\mathcal{I},\mathcal{G},\mathcal{C})$ is *precise* if for every $i\in\mathcal{I}$ and every $\gamma\in\Gamma-G_i$ the set $C_{\gamma,i}=\gamma\cdot C_i$ is disjoint from C_i.

7.1.1. Observe that if $(\mathcal{I},\mathcal{G},\mathcal{C})$ is precise then for every $i\in\mathcal{I}$ and every component K of C_i, the stabilizer of K in Γ is a subgroup of G_i.

7.2. **Proposition.** *Let Γ be a group and let $\tilde{M}$ be a free, uniform PL Γ-manifold of dimension n. Set $M=\tilde{M}/\Gamma$, and let $p:\tilde{M}\to M$ denote the quotient map. Suppose that $\tilde{M}$ has a precise, uniform Γ-plating $(\mathcal{I},\mathcal{G},\mathcal{C})$ where $\mathcal{C}$ is a family of PL subsets and $\mathcal{G}=(G_i)_{i\in\mathcal{I}}$. Then M has a finite plating $(B_1,\dots,B_r)$, having the same order as $(\mathcal{I},\mathcal{G},\mathcal{C})$, such that for each $j\in\{1,\dots,r\}$ the set B_j is a compact, connected PL subset of M and each component of $p^{-1}(B_j)$ is a component of C_i for some $i\in\mathcal{I}$.*

Proof. Let m denote the order of $(\mathcal{I},\mathcal{G},\mathcal{C})$. First note that it suffices to construct a finite plating $(A_j)_{j\in\mathcal{J}}$ of M, with order m, such that for each $j\in\mathcal{J}$ the set A_j is a compact (but possibly disconnected) PL subset of M, and each component of $p^{-1}(A_j)$ is a component of C_i for some $i\in\mathcal{I}$. Indeed, if $(A_j)_{j\in\mathcal{J}}$ is such a plating, the connected components of the A_j form a plating with the required properties.

Since $\tilde{M}$ is a uniform Γ-space, it follows from Proposition 5.4 that $\mathscr{I}$ contains only finitely many Γ-orbits. Let $\mathscr{J} \subset \mathscr{I}$ be a complete set of orbit representatives, in the sense that each Γ-orbit in $\mathscr{I}$ contains a unique element of $\mathscr{J}$. Set $A_j = p(C_j)$ for each $j \in \mathscr{J}$. We have

$$p^{-1}(A_j) = \bigcup_{x \in \Gamma} (x \cdot C_j) = \bigcup_{i \in \Gamma \cdot j} C_i.$$

Since the plating $\mathscr{C}$ is by definition a locally finite family, $p^{-1}(A_j)$ is the union of a locally finite family of closed PL subsets of $\tilde{M}$, and is therefore itself a closed PL subset of $\tilde{M}$. Hence A_j is a closed PL subset of M. Note also that since $\mathscr{I} = \bigcup_{j \in \mathscr{J}} \Gamma \cdot j$, we have

$$\bigcup_{j \in \mathscr{J}} p^{-1}(A_j) = \bigcup_{i \in \mathscr{I}} C_i = \tilde{M},$$

and hence $\bigcup_{j \in \mathscr{J}} A_j = M$. Thus $(A_j)_{j \in \mathscr{J}}$ is a plating of M.

Since $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is precise we have $C_i \cap C_{i'} = \emptyset$ for any two indices $i \neq i'$ in the orbit $\Gamma \cdot j$. Thus $p^{-1}(A_j)$ is the union of a *disjoint* locally finite family of sets C_i, where i ranges over $\Gamma \cdot j$. Hence every component of $p^{-1}(A_j)$ is a component of some C_i.

Now let τ be any point of M. Then there is a point $\tilde{\tau} \in \tilde{M}$ such that $p(\tilde{\tau}) = \tau$. For any $j \in \mathscr{J}$ such that $\tau \in A_j$, we have $\tilde{\tau} \in p^{-1}(A_j)$, and hence $\tilde{\tau} \in C_i$ for some $i \in \Gamma \cdot j$. But since $\mathscr{C}$ has order m, there are at most $m+1$ indices $i \in \mathscr{I}$ such that $\tau \in C_i$. Since the indices in $\mathscr{J}$ are in distinct orbits, it follows that there are at most $m+1$ indices $j \in \mathscr{J}$ such that $\tau \in A_j$. This shows that $(A_j)_{j \in \mathscr{J}}$ has order at most m. (Notice that this step does not depend on the assumption that $\mathscr{C}$ is precise.)

It remains to show that the order of $(A_j)_{j \in \mathscr{J}}$ is at least m. Since $\mathscr{C}$ has order m, there is a point $\tilde{\tau} \in \tilde{M}$ which lies in C_i for $m+1$ distinct values of i, say for $i = i_0, \ldots, i_m$. For any two distinct integers $s, t \in \{0, \ldots, m\}$ we have $i_s \neq i_t$ and $C_{i_s} \cap C_{i_t} \neq \emptyset$. Since $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is precise it follows that $i_0, \ldots, i_m$ lie in distinct Γ-orbits. Hence we may write $i_t = \gamma_t \cdot j_t$ for $t = 0, \ldots, m$, where $j_0, \ldots, j_m$ are distinct elements of $\mathscr{J}$. We have $p(\tilde{\tau}) \in \bigcap_{t=0}^{m} C_{j_t}$; this shows that $(A_j)_{j \in \mathscr{J}}$ has order at least m, and completes the proof of the proposition. □

7.3. LERF groups. Recall that a group Γ is said to be *locally extended residually finite* if every finitely generated subgroup of Γ is an intersection of subgroups of finite index. According to [12], the fundamental group of any connected 2-manifold is locally extended residually finite.

Proposition. *Let Γ be a locally extended residually finite group, and let $\tilde{M}$ be a free PL Γ-manifold. Suppose that $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ is a uniform Γ-plating of $\tilde{M}$, where $\mathscr{G} = (G_i)_{i \in \mathscr{I}}$. Suppose that G_i is finitely generated for every $i \in \mathscr{I}$. Then there is a subgroup Γ_0 of finite index in Γ such that the triple $(\mathscr{I}, \mathscr{G}^0, \mathscr{C})$, where $\mathscr{G}^0 = (G_i \cap \Gamma_0)_{i \in \mathscr{I}}$, is a precise, uniform Γ_0-plating of $\tilde{M}$.*

Remark. The hypothesis that the G_i are finitely generated is easily seen to hold automatically if each set in the family $\mathscr{C}$ has a finite number of connected components. In general, however, this may not be the case, and so the hypothesis of finite generation is needed.

Proof of Proposition 7.3. Let us write $\mathscr{C}=(C_i)_{i\in\mathscr{I}}$. By 6.4, there is a closed, locally finite neighborhood $(\mathscr{I},\mathscr{G},\mathscr{C}')$ of $(\mathscr{I},\mathscr{G},\mathscr{C})$, where $\mathscr{C}'=(C'_i)_{i\in\mathscr{I}}$, such that $(\mathscr{I},\mathscr{G},\mathscr{C}')$ is a uniform plating of $\tilde{M}$ and all the C'_i are subcomplexes in a fixed Γ-invariant triangulation of $\tilde{M}$. If Γ_0 is a finite-index subgroup of Γ, and if the Γ_0-plating $(\mathscr{I},\mathscr{G}^0,\mathscr{C}')$, where $\mathscr{G}^0=(G_i\cap\Gamma_0)_{i\in\mathscr{I}}$, is precise and uniform, then it follows immediately that $(\mathscr{I},\mathscr{G}^0,\mathscr{C})$ is precise and uniform. Hence we may assume without loss of generality that the C_i are all subcomplexes in a fixed triangulation $\mathscr{T}$ of $\tilde{M}$.

By Proposition 5.4, the Γ-set $\mathscr{I}$ contains only finitely many Γ-orbits. Thus we may write $\mathscr{I}=\Gamma\cdot S$, for some finite subset S of $\mathscr{I}$. Since $(\mathscr{I},\mathscr{G},\mathscr{C})$ is uniform, C_s is a uniform G_s-space. For every $s\in S$ we choose a finite subcomplex R_s of C_s (in the triangulation $\mathscr{T}$) such that $G_s\cdot R_s=C_s$. For each $s\in S$ we define Φ_s to be the set of all elements $y\in\Gamma-G_s$ such that $y\cdot R_s\cap R_s\neq\emptyset$. For any $y\in\Phi_s$ there is a vertex v of R_s such that $y\cdot v$ is also a vertex of R_s. Since R_s has finitely many vertices and Γ acts freely on $\tilde{M}$ it follows that Φ_s is a finite subset of Γ for every $s\in S$.

For each $s\in S$ and each $y\in\Phi_s$ we have $y\in\Gamma-G_s$, where G_s is a finitely generated subgroup of the locally extended residually finite group Γ. Hence there is a subgroup $\Theta_{s,y}$ of finite index in Γ such that $G_s\leqq\Theta_{s,y}$ but $y\notin\Theta_{s,y}$. For each $s\in S$ we set $\Theta_s=\bigcap_{y\in\Phi_s}\Theta_{s,y}$. Note that Θ_s has finite index in Γ, that $G_s\leqq\Theta_s$ and that $\Theta_s\cap\Phi_s=\emptyset$.

Now set $\Gamma_0=\bigcap_{s\in S}\Theta_s$. Then Γ_0 has finite index in Γ.

We set $\mathscr{G}^0=(G_i\cap\Gamma_0)_{i\in\mathscr{I}}$. It is clear that $(\mathscr{I},\mathscr{G}^0,\mathscr{C})$ is a Γ_0-plating of $\tilde{M}$. Since $G_i\cap\Gamma_0$ has finite index in G_i for every $i\in\mathscr{I}$, it follows from 1.4 that $(\mathscr{I},\mathscr{G}^0,\mathscr{C})$ is uniform. We must show that $(\mathscr{I},\mathscr{G}^0,\mathscr{C})$ is precise. Thus for every $i\in\mathscr{I}$ and every $x\in\Gamma_0$ such that C_i meets $C_{x\cdot i}=x\cdot C_i$, we must show that $x\in G_i$.

We first consider the case where $i=s\in S$. Assume that $C_s\cap x\cdot C_s\neq\emptyset$, and let v be a vertex of C_s such that $x\cdot v\in C_s$. We may write $v=g_1\cdot w_1$ and $x\cdot v=g_2\cdot w_2$, where w_1, $w_2\in R_s$ and $g_1,g_2\in G_s$. Then $g_2^{-1}xg_1\cdot w_1=w_2$, and so $R_s\cap g_2^{-1}xg_1\cdot R_s\neq\emptyset$. Hence by the definition of Φ_s we have either $g_2^{-1}xg_1\in G_s$ or $g_2^{-1}xg_1\in\Phi_s$. But we have $g_1,g_2\in G_s\leqq\Theta$ and $x\in\Gamma_0\leqq\Theta_s$. Hence $g_2^{-1}xg_1\in\Theta_s$. Since $\Theta_s\cap\Phi_s=\emptyset$ we cannot have $g_2^{-1}xg_1\in\Phi_s$. Hence we must have $g_2^{-1}xg_1\in G_s$, Since $g_1,g_2\in G_s$. it follows that $x\in G_s$, as required.

Now consider an arbitrary index $i\in\mathscr{I}$. Suppose that for a given $x\in\Gamma_0$ we have $C_i\cap C_{x\cdot i}\neq\emptyset$. We may write $i=\gamma\cdot s$ for some $s\in S$ and some $\gamma\in\Gamma$. Then we have $C_s\cap C_{\gamma^{-1}x\gamma\cdot s}=\gamma^{-1}(C_i\cap C_{x\cdot i})\neq\emptyset$, and hence $\gamma^{-1}x\gamma\in G_s$, by the case of the assertion already proved. Now by condition (ii) in the definition of a Γ-plating (5.2) it follows that $x\in G_i$. $\square$

7.4. The following elementary lemma about 2-manifolds will be combined with Propositions 7.2 and 7.3 to prove Proposition 7.4.2 below, which will find a crucial application in Section 8.

7.4.1. **Lemma.** *Let $f: F \to S$ be a map between closed, connected orientable, PL 2-manifolds. Let $\Xi \subset S$ be a compact, possibly disconnected PL 2-manifold; assume that no component of $\partial \Xi$ is a homotopically trivial curve in S. Let $\mathscr{B} = (B_j)_{j \in \mathscr{J}}$ be a finite plating of F by PL sets, having order $\leqq 1$. Assume that for each $j \in \mathscr{J}$ the map $f|B_j$ is homotopic in S to a map whose image is contained in Ξ. Then either f has degree 0, or every component of $S - \operatorname{int} \Xi$ is an annulus.*

Proof. Let N_j be a regular neighborhood of B_j in F for each $j \in \mathscr{J}$. Then $(N_j)_{j \in \mathscr{J}}$ is a neighborhood of $\mathscr{B}$. By Corollary 6.5 and Remark 6.3.1, there is a closed neighborhood $\mathscr{B}' = (B'_j)_{j \in \mathscr{J}}$ of $\mathscr{B}$ such that (i) $B'_j \subset N_j$ for every $j \in \mathscr{J}$, (ii) B'_j is a PL 2-manifold for every $j \in \mathscr{J}$ and (iii) $\mathscr{B}'$ has order $\leqq 1$. Since N_j is a regular neighborhood of B_j, the map $f|B'_j$ is homotopic to a map with image contained in Ξ for every $j \in \mathscr{J}$. Thus the hypotheses all continue to hold if $\mathscr{B}$ is replaced by $\mathscr{B}'$, and we may therefore assume without loss of generality that each B_j is a PL 2-manifold and that $F = \bigcup_{j \in \mathscr{J}} \operatorname{int} B_j$.

Since the plating $\mathscr{B} = (B_j)_{j \in \mathscr{J}}$ of F by PL 2-manifolds has order at most 1 and $F = \bigcup_{j \in \mathscr{J}} \operatorname{int} B_j$, the closed PL 1-manifolds ∂B_j are pairwise disjoint. Hence $L = \bigcup \partial B_j$ is a closed PL 1-manifold. Thus if A is a regular neighborhood of L in F, the components of A are annuli. Set $R = F - \operatorname{int} A$. Then each component of R is contained in some B_j. Hence after modifying f within its homotopy class we may assume that $f(R) \subset \operatorname{int} \Xi$.

Suppose that there is a component Q of $S - \Xi$ which is not an annulus. After modifying f by a small homotopy we may assume that it is transverse to ∂Q. Set $P = f^{-1}(Q) \subset \operatorname{int} A$, let $P_1, \ldots, P_k$ denote the components of P, and set $f_i = f|P_i: P_i \to Q$. The f_i are boundary-preserving maps. If we assign to each P_i the orientation induced from F and to Q the orientation induced from S, we have $\deg f = \sum_{i=1}^{k} \deg f_i$. Now each P_i is contained in an annulus component A_i of A. If we choose consistent base points in $P_i \subset A_i$ and $Q \subset S$, we have a commutative diagram of groups and outer homomorphisms

$$\begin{array}{ccc} \pi_1(P_i) & \xrightarrow{(f_i)_\#} & \pi_1(Q) \\ \downarrow & & \downarrow \\ \pi_1(A_i) & \xrightarrow[(f|A_i)_\#]{} & \pi_1(S) \end{array}$$

in which the vertical arrows represent outer homomorphisms induced by inclusion. The outer homomorphism $\pi_1(Q) \to \pi_1(S)$ is injective since no component of $\partial Q \subset \partial \Xi$ is homotopically trivial in S. Since $\pi_1(A_i)$ is cyclic it follows that the outer group $Z = (f_i)_\#(\pi_1(P_i)) \leqq \pi_1(Q)$ is cyclic. Since Q is not an annulus, Z has infinite index in $\pi_1(Q)$. Hence f_i lifts to an infinite-sheeted covering space $\tilde{Q}$ of Q. We have $H_2(\tilde{Q}, \partial \tilde{Q}) = 0$, and thus $\deg f_i = 0$. Summing over i we get $\deg f = 0$. □

7.4.2. **Proposition.** *Let Γ be a group, let $\tilde{F}$ and $\tilde{S}$ be free, uniform, Γ-orientable, connected 2-dimensional PL Γ-manifolds without boundary, and let $\tilde{f}: \tilde{F} \to \tilde{S}$ be a Γ-equivariant continuous map. Set $S = \tilde{S}/\Gamma$ and $F = \tilde{F}/\Gamma$. Assume that genus $S > 0$. Suppose that $\tilde{S}$ is simply connected, and let $v_S: \pi_1(S) \to \Gamma$ denote the natural outer isomorphism. Suppose that the map $f: F \to S$ induced by $\tilde{f}$ has non-zero degree. Let $\Xi \subset S$ be a compact PL 2-manifold. Suppose that no component of $\partial\Xi$ is homotopically trivial in S. Suppose that there is a uniform Γ-plating $(\mathscr{I}, \mathscr{G}, \mathscr{C})$ of $\tilde{F}$, where $\mathscr{G} = (G_i)_{i \in \mathscr{I}}$, such that*

(i) *$\mathscr{C}$ has order at most 1,*

(ii) *G_i is finitely generated for every $i \in \mathscr{I}$, and*

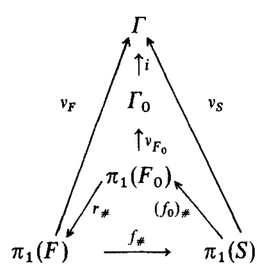
(iii) *for each $i \in \mathscr{I}$ the outer subgroup $v_S^{-1}(G_i)$ of $\pi_1(S)$ is contained in the outer subgroup $\mathrm{im}(\pi_1(\Xi_i) \to \pi_1(S))$ for some component Ξ_i of Ξ.*

Then every component of $S - \mathrm{int}\,\Xi$ is an annulus.

Proof. We have $\Gamma \cong \pi_1(S)$. Hence by [12], Γ is locally extended residually finite. By 7.3, there is a subgroup Γ_0 of finite index in Γ such that the triple $(\mathscr{I}, \mathscr{G}_0, \mathscr{C})$, where $\mathscr{G}^0 = (G_i \cap \Gamma_0)_{i \in \mathscr{I}}$, is a precise, uniform Γ-plating of $\tilde{F}$.

Set $F_0 = \tilde{F}/\Gamma_0$. Then F_0 is a finite-sheeted regular covering space of F. In particular F_0 is compact and the covering projection $r: F_0 \to F$ has non-zero degree. Hence $f_0 = f \circ r$ also has non-zero degree.

Let $v_F: \pi_1(F) \to \Gamma$ and $v_{F_0}: \pi_1(F_0) \to \Gamma_0$ denote the natural outer homomorphisms. Letting $\iota: \Gamma_0 \to \Gamma$ denote the inclusion, we then have a diagram of groups and outer homomorphisms



which we may think of as a triangle subdivided into left-hand, right-hand and bottom subtriangles. This is a commutative diagram of outer homomorphisms. Indeed, the full triangle is commutative by virtue of the Γ-equivariance of $\tilde{f}$; the left-hand triangle is commutative by elementary covering space theory, and the bottom triangle is commutative by the definition of f_0. It follows that the right-hand triangle of outer homomorphisms also commutes.

Let $p_0: \tilde{F} \to F_0$ denote the covering projection. By Proposition 7.2, the closed 2-manifold F_0 has a finite plating $(B_1, \dots, B_r)$ of order 1, where for each $j \in \{1, \dots, r\}$ the set B_j is a compact, connected, PL subset of F_0, and each component of $p_0^{-1}(B_j)$ is a component of C_i for some $i \in \mathscr{I}$. For each j let Θ_j denote the outer subgroup $\mathrm{im}(\pi_1(B_j) \to \pi_1(F))$, a subgroup of $\pi_1(F_0)$. For each j the outer subgroup $\nu_{F_0}(\Theta_j)$ of Γ_0 is realized by the stabilizer of a component R_j of $p_0^{-1}(B_j)$. Since R_j is a component of C_i for some i, the observation 7.1.1, applied to the precise Γ_0-plating $(\mathscr{I}, \mathscr{G}_0, \mathscr{C})$, gives that $\nu_{F_0}(\Theta_j)$ is contained in the outer subgroup G_i of Γ_0 for some $i \in \mathscr{I}$. In particular the outer subgroup $\iota\nu_{F_0}(\Theta_j)$ of Γ is contained in the outer subgroup G_i of Γ for some i.

By the commutativity of the right-hand subtriangle in the diagram above, we have $(f_0)_{\#}(\Theta_j) = \nu_S^{-1}\iota\nu_{F_0}(\Theta_j)$. Thus the outer subgroup $(f_0)_{\#}(\Xi_j)$ of $\pi_1(S)$ is contained in the outer subgroup $\nu_S^{-1}(G_i)$ for some i. By condition (ii) in the hypothesis of the proposition it follows that the outer subgroup $(f_0)_{\#}(\Theta_j)$ of $\pi_1(S)$ is contained in the outer subgroup $\operatorname{im}(\pi_1(\Xi_j) \to \pi_1(S))$ for some component Ξ_j of Ξ. Now S is aspherical since genus $S > 0$, and $\pi_1(\Xi_j) \to \pi_1(S)$ is injective since no component of $\partial\Xi$ is homotopically trivial. We can therefore conclude that for each j the map $f_0 | B_j$ is homotopic in S to a map whose image is contained in Ξ. Since f_0 has non-zero degree, it now follows from Lemma 7.4.1 that every component of $S - \Xi$ is an annulus. □

8. Fibroids and platings, I

8.1. If Γ is a group and T is a Γ-tree, the action of Γ on T induces an action on the set $\mathscr{E}(T)$ of edges of T. We shall regard $\mathscr{E}(T)$ as a Γ-set by equipping it with this action.

8.2. This section is devoted to the proof of the following topological result, which gives a sufficient condition for an incompressible surface to be a fibroid.

Proposition. *Let M be a simple, closed, orientable 3-manifold containing an incompressible bi-collared surface c. Let $\Gamma \cong \pi_1(M)$ denote the group of deck transformations of the universal covering space $\tilde{M}$ of M, and let T denote the Γ-tree $T(c)$. Suppose that there is a doubly uniform Γ-plating $(\mathscr{E}(T), \mathscr{G}, \mathscr{W})$ of the Γ-space $\tilde{M}$, where $\mathscr{G} = (\Gamma_e)_{e \in \mathscr{E}(T)}$ and $\mathscr{W} = (W_e)_{\in \mathscr{E}(T)}$, which satisfies the following conditions.*

(i) *The order of $\mathscr{W}$ is at most 2.*

(ii) *There is a Γ-equivariant map $f: \tilde{M} \to T$ such that for every $e \in \mathscr{E}(T)$ we have $f(W_e) \subset \mathring{e}$.*

Then the core of c is a fibroid.

8.3. Throughout the section, M will denote a manifold, and c a bi-collared surface, satisfying the hypotheses of Proposition 8.2. We shall fix a PL structure on M in which c is a PL bi-collared surface. We shall denote the core of c by S. As in the statement of the proposition, $\tilde{M}$ will denote the universal covering space of M and Γ its group of deck transformations, and T will denote the Γ-tree $T(c)$. We shall set $q = q_c : \tilde{M} \to T$. We shall write $p: \tilde{M} \to M$ for the covering projection. The bi-collared surface in $\tilde{M}$ lying over c (3.1) will be denoted by $\tilde{c}$. We shall write $\mathscr{E} = \mathscr{E}(T)$, and we shall fix a doubly uniform

Γ-plating $\mathscr{W}=(W_e)_{e\in\mathscr{E}}$ which is indexed by the Γ-set $\mathscr{E}$ and satisfies conditions (i) and (ii) of proposition 8.2.

8.4. **Lemma.** *We have* $f(\tilde{M})=T$.

Proof. Since f is equivariant, $f(\tilde{M})$ is a Γ-invariant subtree of T. But T is a minimal Γ-tree by 3.3. □

8.5. **Lemma.** *For any* $e\in\mathscr{E}$ *we have* $f^{-1}(e)\subset W_e\subset f^{-1}(\bar{e})$.

Proof. By the hypothesis of Proposition 8.2 we have $f(W_e)\subset\bar{e}$ and hence $W_e\subset f^{-1}(\bar{e})$. Now let P be any point of $f^{-1}(e)$. Since $\mathscr{W}$ is a plating we have $P\in W_{e'}$ for some $e'\in\mathscr{E}$. It follows that $f(P)\in\bar{e}'$ and hence that $e'=e$. This proves that $f^{-1}(e)\subset W_e$. □

8.6. For any subcomplex Λ of T we shall denote by $\mathrm{nbhd}_1\ \Lambda$ the union of Λ with all (open) edges of the first barycentric subdivision of T that share a vertex with Λ.

Lemma. *There is a locally finite closed neighborhood* $(\mathscr{E},\mathscr{G},\mathscr{W}')$ *of* $(\mathscr{E},\mathscr{G},\mathscr{W})$, *where* $\mathscr{W}'=(W'_e)_{e\in\mathscr{E}}$, *having the following properties*:

(i) $f(W'_e)\subset\mathrm{nbhd}_1\ \bar{e}$ *for every* $e\in\mathscr{E}$;

(ii) $(\mathscr{E},\mathscr{G},\mathscr{W}')$ *is doubly uniform*;

(iii) W'_e *is a PL 3-manifold for every* $e\in\mathscr{E}$;

and

(iv) $\mathscr{W}'$ *has order at most* 2.

Proof. For every $e\in\mathscr{E}(T)$, the set $\mathrm{nbhd}_1(\bar{e})$ is a neighborhood of e in T, and for any $\gamma\in\Gamma$ we have $\gamma\cdot\mathrm{nbhd}_1\ \bar{e}=\mathrm{nbhd}_1(\gamma\cdot\bar{e})$. This naturality property, and the properties of the map f given in hypothesis (ii) of Proposition 8.2, imply that $(\mathscr{E},\mathscr{G},(f^{-1}(\mathrm{nbhd}_1\bar{e}))_{e\in\mathscr{E}})$ is a neighborhood of $(\mathscr{E},\mathscr{G},\mathscr{W}')$. We can now apply Proposition 6.3, with $(\mathscr{E},\mathscr{G},(f^{-1}(\mathrm{nbhd}_1\bar{e}))_{e\in\mathscr{E}})$ playing the rôle of $(\mathscr{I},\mathscr{G},\mathscr{C}^+)$. The resulting locally finite closed neighborhood $(\mathscr{E},\mathscr{G},\mathscr{W}')$ of $(\mathscr{E},\mathscr{G},\mathscr{W})$ has the property (i) stated in the conclusion of the proposition because it has $(\mathscr{E},\mathscr{G},(f^{-1}(\mathrm{nbhd}_1\bar{e}))_{e\in\mathscr{E}})$ as a neighborhood. Property (iv) follows from Remark 6.3.1, and the others are immediate from Proposition 6.3. □

For the rest of the section we fix a neighborhood $(\mathscr{E},\mathscr{G},\mathscr{W}')$ of $(\mathscr{E},\mathscr{G},\mathscr{W})$ having the properties stated in the above lemma.

8.7. If s is any edge of T, we shall denote by $p/\Gamma_s:\tilde{M}/\Gamma_s\to M$ the covering map induced by p. We shall let $\tilde{c}/\Gamma_s$ denote the bi-collared surface in $\tilde{M}/\Gamma_s$ lying over c. According to Proposition 3.4, the component $\tilde{M}_s/\Gamma_s$ of $|\tilde{c}/\Gamma_s|$ is a deformation retract of $\tilde{M}/\Gamma_s$, and p/Γ_s maps $\tilde{M}_s/\Gamma_s$ homeomorphically onto $|c|$. In particular, $\tilde{M}/\Gamma_s$ is homotopy equivalent to S.

If v is any vertex of T incident to s, it follows that $\partial_s\tilde{M}_v/\Gamma_s$ is a deformation retract of $\tilde{M}/\Gamma_s$, and that p/Γ_s maps $\partial_s\tilde{M}_v/\Gamma_s$ homeomorphically onto $\partial_s M_v$. We shall let $r_{s,v}$ denote a deformation retraction (chosen arbitrarily) of $\tilde{M}_v/\Gamma_s$ to $\partial_s\tilde{M}_v/\Gamma_s$.

8.8. **Lemma.** *Let s be any edge of T, and let v be any vertex of T incident to s. Then there is a component $\tilde{F}$ of the boundary of W'_s/Γ_s such that*

(i) *$\tilde{F}$ is invariant under Γ_s;*

(ii) *$\tilde{F}$ is a uniform, Γ_s-orientable PL Γ_s-manifold;*

(iii) *the map $r_{s,v}|(\tilde{F}/\Gamma_s)$: $\tilde{F}/\Gamma_s \to \partial_s\tilde{M}_v/\Gamma_s$ has degree one;*

and

(iv) *for every edge e of T not incident to v, we have $\tilde{F}\cap W'_e=\emptyset$.*

Proof. Since $\mathscr{W}'$ is doubly uniform according to Lemma 8.6, it is uniform by Proposition 5.7.1. Thus W'_s is a uniform Γ_s-space, i.e. $W'_s \subset \tilde{M}/\Gamma_s$ is compact. Hence the PL 2-manifold $\partial(W'_s/\Gamma_s)$ is closed. Note also that since $\tilde{M}/\Gamma_s$ is orientable, $\partial(W'_s/\Gamma_s)$ is orientable.

The map f induces a map $f_s:\tilde{M}/\Gamma_s \to T/\Gamma_s$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{f} & T \\ \downarrow & & \downarrow \\ \tilde{M}/\Gamma_s & \xrightarrow{f_s} & T/\Gamma_s \end{array}$$

in which the vertical arrows represent quotient maps. Since Γ_s fixes s, there is an edge s_0 of the graph T/Γ_s whose preimage in T is precisely s. One vertex of s_0 is the image of v in T/Γ_s, which we shall denote by v_0; let us denote the other vertex of s_0 by w_0. By Lemma 8.5, we have $f^{-1}(s) \subset W_s \subset W'_s$. From the above commutative diagram it follows that $f_s^{-1}(s_0) \subset W'_s/\Gamma_s$.

By Proposition 3.5, T/Γ_s is a tree without end points. In particular, $(T/\Gamma_s)-s_0$ has two connected components, each of which is a tree with infinitely many vertices. Let Λ_v and Λ_w denote the components of $(T/\Gamma_s)-s_0$ containing v_0 and w_0 respectively. Since $f_s^{-1}(s_0) \subset W'_s/\Gamma_s$, each component of $\tilde{M}/\Gamma_s - W'_s/\Gamma_s$ is mapped by f_s into either Λ_v or Λ_w.

By Lemma 8.4, f maps $\tilde{M}$ onto T. Hence f_s maps $\tilde{M}/\Gamma_s$ onto T/Γ_s. Since each of the trees Λ_v and Λ_w has infinitely many vertices, $f^{-1}(\Lambda_v)$ and $f^{-1}(\Lambda_w)$ are non-compact. But since W'_s/Γ_s is compact and $\tilde{M}/\Gamma_s$ is connected, $\tilde{M}/\Gamma_s - \operatorname{int} W'_s/\Gamma_s$ has only finitely many components. Since $f^{-1}(\Lambda_v)$ is non-compact, f must map some non-compact component N of $\tilde{M}/\Gamma_s - \operatorname{int} W'_s/\Gamma_s$ into Λ_v.

We claim that there is a component F of ∂N which does not bound a compact 3-manifold in $\tilde{M}/\Gamma_s$. Assume to the contrary that every component C of ∂N bounds a compact 3-manifold J_C. No J_C can contain N since N is non-compact. Since $\tilde{M}_e$ is connected it follows that $\tilde{M}_e = N \cup \bigcup_C J_C$, where C ranges over the components of ∂N. This means that $\tilde{M} - \operatorname{int} N$ is compact, which is impossible since $f^{-1}(\Lambda_w)$ is non-compact. This proves the claim.

Since $\partial_s \tilde{M}_v/\Gamma_s$ is a deformation retract of $\tilde{M}/\Gamma_s$ via $r=r_{s,v}$, we may apply Proposition 3.4.1 to conclude that $\pi_1(F)\to\pi_1(\tilde{M}/\Gamma_s)$ is surjective and that $r|F: F\to\partial_s M_v/\Gamma_s$ has degree 1. Since $\pi_1(F)\to\pi_1(\tilde{M}/\Gamma_s)$ is surjective, the preimage $\tilde{F}$ of F in $\tilde{M}$ is connected and is therefore a component of $\partial W'_s$. We have $F=\tilde{F}/\Gamma_s$. Since W'_s is a uniform Γ_s-space, its Γ_s-invariant closed subset $\tilde{F}$ is also a uniform Γ_s-space by 1.4. Furthermore, since $\tilde{M}$ is Γ-orientable, W'_s is Γ_s-invariant and hence its Γ_s-invariant boundary component $\tilde{F}_s$ is Γ_s-orientable as well. It remains to check conclusion (iv) of the lemma.

By Lemma 8.6, we have $f(W'_e)\subset \mathrm{nbhd}_1 \bar{e}$ for every edge e of T. By our choice of the surface $F\subset\partial W'_s$ we have $f_s(F)\subset\Lambda_v$; hence $f(\tilde{F})\subset\tilde{\Lambda}_v$, where $\tilde{\Lambda}_v$ is the component of $T-s$ containing v. It follows that $f(\tilde{F})\subset\tilde{\Lambda}_v\cap\mathrm{nbhd}_1\bar{s}\subset\mathrm{nbhd}_1\{v\}$. Hence if e is any edge of T not incident to v, we have $f(\tilde{F}\cap W'_e)\subset\mathrm{nbhd}_1\{v\}\cap\mathrm{nbhd}_1\bar{e}=\emptyset$, and hence $\tilde{F}\cap W'_e=\emptyset$. □

8.9. Proof of Proposition 8.2. We are required to prove that the core of c is a fibroid. According to the definitions, this is equivalent to showing that every component of Split(c) is a book of I-bundles.

Consider an arbitrary component of Split(c), which we may write in the form M_v for some vertex v of $T=T(c)$. Since M is simple and c is incompressible, it follows from the definitions that M_v is simple. Let (Σ, Φ) denote the characteristic pair of $(M_v, \partial M_v)$. According to Proposition 4.3, we need only show that every component of $\partial M_v-\mathrm{int}\,\Phi$ is an annulus.

Consider an arbitrary component of ∂M_v, which we may write in the form $\partial_s M_v$ for some edge s incident to v. Set $\Xi_0=\partial_s M_v\cap\Phi$. We are required to show that every component of $\partial_s M_v-\mathrm{int}\,\Xi_0$ is an annulus. For technical reasons that will become apparent we consider a PL 2-manifold $\Xi\subset\partial M_v$ constructed as follows. If $\Xi_0\neq\emptyset$ we set $\Xi=\Xi_0$. If $\Xi=\emptyset$ we take Ξ to be an arbitrarily chosen homotopically non-trivial annulus in $\partial_s M_v$; such an annulus exists because the incompressibility of c guarantees that $\partial_s M_v$ is not a 2-sphere. Note that since M is simple, $\partial_s M_v$ cannot be a torus. Hence if we can show that every component of $\partial_s M_v-\mathrm{int}\,\Xi$ is an annulus, it will follow that $\Xi=\Xi_0$ and hence that every component of $\partial_s M_v-\mathrm{int}\,\Xi_0$ is an annulus, as required. The rest of the argument will be devoted to proving that every component of $\partial_s M_v-\mathrm{int}\,\Xi$ is an annulus.

Let us choose a component $\tilde{F}_{s,v}$, of the boundary of W'_s which satisfies conditions (i)–(iv) of Lemma 8.8. Set $r=r_{s,v}$, and let $\tilde{r}:\tilde{M}\to\tilde{F}$ be a retraction covering r. We have a Γ_s-equivariant map $\tilde{f}=\tilde{r}|\tilde{F}:\tilde{F}\to\partial_s\tilde{M}_v$. We shall show that $\partial_s\tilde{M}_v, \tilde{F}, \tilde{f}$ and $\Xi\subset\partial_s\tilde{M}_v$ satisfy the hypotheses of Proposition 7.4.2, with the group Γ_s playing the rôle of Γ. It will then follow from Proposition 7.4.2 that every component of $\partial_s\tilde{M}_v-\mathrm{int}\,\Xi$ is an annulus, as required.

By Lemma 8.8, $\tilde{F}$ is a uniform, Γ_s-orientable Γ_s-manifold. Since $\partial_s\tilde{M}_v$ is a closed, orientable 2-manifold, the PL Γ-manifold $\partial_s\tilde{M}_v$ is also uniform and Γ_s-orientable. Furthermore, since $\partial_s M_v$ is incompressible, $\partial_s\tilde{M}_v$ is simply connected and $\partial_s M_v$ is not a 2-sphere. We have observed that $\tilde{f}:\tilde{F}\to\partial_s\tilde{M}_v$ is Γ_s-equivariant. The induced map $f: F\to\partial_s M_v$ is the map $r_{s,v}|(\tilde{F}/\Gamma_s)$, which by Lemma 8.8 has degree 1.

It remains to construct a uniform Γ-plating of $\tilde{F}$ satisfying conditions (i)–(iii) of Proposition 7.4.2.

Set $Q_s = \partial W'_s$. According to Proposition 5.7, the Γ-plating $(\mathscr{E}, \mathscr{G}, \mathscr{W}')$ of $\tilde{M}$ induces a boundary plating of Q_s. This boundary plating is a Γ_s-plating and will be denoted $(\mathscr{E}^s, \mathscr{G}^s, (\mathscr{W}')^s)$. Here $\mathscr{E}^s$ is the set of all edges $e \neq s$ of T such that $W'_e \cap Q_s \neq \emptyset$; and we have $\mathscr{G}^s = (\Gamma_e \cap \Gamma_s)_{e \in \mathscr{E}^s}$ and $(\mathscr{W}') = (W'_e \cap Q_s)_{e \in \mathscr{E}^s}$. Since $(\mathscr{E}, \mathscr{G}, \mathscr{W})$ has order at most 2 by Lemma 8.6, it follows from Proposition 5.7 that $(\mathscr{E}^s, \mathscr{G}^s, (\mathscr{W}')^s)$ has order at most 1. Moreover, since $(\mathscr{E}, \mathscr{G}, \mathscr{W})$ is doubly uniform by Lemma 8.6, it follows from Proposition 5.7.1 that the plating $(\mathscr{E}^s, \mathscr{G}^s, (\mathscr{W}')^s)$ is uniform.

It now follows from 5.6 that the Γ_s-space $\tilde{F}$ inherits a Γ_s-plating $(\mathscr{E}_{\tilde{F}}, \mathscr{G}_{\tilde{F}}, \mathscr{W}'_{\tilde{F}})$ from the Γ_s-plating $(\mathscr{E}^s, \mathscr{G}^s, (\mathscr{W}')^s)$ of Q_s. Hence $\mathscr{E}_{\tilde{F}}$ consists of all edges $e \neq s$ of T such that $W'_e \cap F \neq \emptyset$; and we have $\mathscr{G}_{\tilde{F}} = (\Gamma_e \cap \Gamma_s)_{e \in \mathscr{E}_{\tilde{F}}}$ and $(\mathscr{W}')_{\tilde{\mathscr{F}}} = (W'_e \cap F)_{e \in \mathscr{E}_{\tilde{F}}}$. Since $(\mathscr{E}^s, \mathscr{G}^s, \mathscr{W}')^s)$ is uniform and has order at most 1, $(\mathscr{E}_{\tilde{F}}, \mathscr{G}_{\tilde{F}}, \mathscr{W}'_{\tilde{F}})$ has the same properties. That $(\mathscr{E}_{\tilde{F}}, \mathscr{G}_{\tilde{F}}, \mathscr{W}'_{\tilde{F}})$ has order at most 1 is condition (i) of Proposition 7.4.2. We shall complete the proof by showing that $(\mathscr{E}_{\tilde{F}}, \mathscr{G}_{\tilde{F}}, \mathscr{W}'_{\tilde{F}})$ satisfies conditions (ii) and (iii) of Proposition 7.4.2.

Consider any element e of the index set $\mathscr{E}_{\tilde{F}}$. The group in the family $\mathscr{G}_{\tilde{F}}$ indexed by e is $\Gamma_s \cap \Gamma_e$.

The definition of $\mathscr{E}_{\tilde{F}}$ implies that e is an edge distinct from s. Furthermore, we have $W'_e \cap F \neq \emptyset$; this implies that e is incident to v, since by condition (iv) of Lemma 8.8 we have $\tilde{F} \cap W'_s = \emptyset$ for every edge e of T not incident to v. It now follows from Proposition 4.2 that $\Gamma_s \cap \Gamma_e$ is finitely generated. This is condition (ii) of 7.4.2.

Finally we must verify condition (iii), that there is a component Ξ_e of Ξ such that the outer subgroup $v_{s,v}^{-1}(\Gamma_s \cap \Gamma_e)$ of $\pi_1(\partial_s M_v)$ is contained in the outer subgroup $\operatorname{im}(\pi_1(\Xi_e) \to \pi_1(\partial_s M_v))$. If $\Gamma_s \cap \Gamma_e \neq \{1\}$, it follows from Proposition 4.2 that the outer subgroup $v_{s,v}^{-1}(\Gamma_s \cap \Gamma_e)$ of $\pi_1(\partial_s M_v)$ is contained in the outer subgroup $\operatorname{im}(\pi_1(\Phi_e) \to \pi_1(\partial_s M_v))$ for some component $\Phi_e \subset \partial_s M_v$ of Φ_v. In this case it follows from the definitions of Ξ and Ξ_0 that Φ_e is a component of Ξ_0 and hence of Ξ. If $\Gamma_s \cap \Gamma_e = \{1\}$, then since $\Xi \neq \emptyset$, it is trivially true that the (trivial) outer subgroup $v_{s,v}^{-1}(\Gamma_s \cap \Gamma_e)$ is contained in the outer subgroup $\operatorname{im}(\pi_1(\Xi_e) \to \pi_1(\partial_s M_v))$ for some component Ξ_e of Ξ. Thus condition (iii) of 7.4.2 is verified in all cases. This completes the proof of Proposition 8.2. □

9. Fibroids and platings, II

9.1. This section is devoted to the proof of the following result, which like Proposition 8.2 is purely topological and gives a sufficient condition for an incompressible surface to be a fibroid.

Theorem. *Let M be a simple, closed, orientable 3-manifold containing a connected incompressible bi-collared surface c. Let $\Gamma \cong \pi_1(M)$ denote the group of deck* *transformations of the universal covering space $\tilde{M}$ of M, and st $T=T(c)$. Suppose that there is a uniform Γ-*plating $(\mathcal{I}, \mathcal{X}, \mathcal{Z})$ *of the Γ-space $\tilde{M}$, where $\mathcal{Z}=(Z_i)_{i\in\mathcal{I}}$ and $\mathcal{X}=(X_i)_{i\in\mathcal{I}}$, satisfying the following conditions.*

(i) *For each $i\in\mathcal{I}$ the group X_i is a T-hyperbolic cyclic subgroup of Γ.*

(ii) *For any $i, i'\in\mathcal{I}$ such that $Z_i\cap Z_{i'}\neq\emptyset$, we have $\mathcal{E}(A_T(X_i))\cap \mathcal{E}(A_T(X_{i'}))\neq\emptyset$.*

Then the core of c is a fibroid.

9.2. Throughout the section, M will denote a manifold, and c a bi-collared surface, satisfying the hypotheses of Theorem 9.1. We shall fix a PL structure on M in which c is a PL bi-collared surface. As in the statement of the theorem we shall denote the universal covering space of M by $\tilde{M}$ and its group of deck transformations by Γ, and we shall set $T=T(c)$. We shall fix a uniform Γ-plating $\mathcal{Z}=(Z_i)_{i\in\mathcal{I}}$ of $\tilde{M}$ satisfying conditions (i) and (ii) of the theorem.

9.3. Corollary 6.4 gives a closed, locally finite neighborhood $(\mathcal{I}, \mathcal{X}, \mathcal{Z}')$ of $(\mathcal{I}, \mathcal{X}, \mathcal{Z})$ which is itself a uniform plating of $\tilde{M}$. Furthermore, if we write $\mathcal{Z}'=(Z'_i)_{i\in\mathcal{I}}$, then the Z'_i are all subcomplexes of some Γ-invariant triangulation of $\tilde{M}$; and for any two indices $i, j\in\mathcal{I}$ such that $Z_i\cap Z_j=\emptyset$ we have $Z'_i\cap Z'_{j'}=\emptyset$. It follows from this last property of $\mathcal{Z}'$ that the hypotheses of Theorem 9.1 continue to hold if $(\mathcal{I}, \mathcal{X}, \mathcal{Z})$ is replaced by $(\mathcal{I}, \mathcal{X}, \mathcal{Z}')$. But since $(\mathcal{I}, \mathcal{X}, \mathcal{Z}')$ is a neighborhood of $(\mathcal{I}, \mathcal{X}, \mathcal{Z}')$ we have $\tilde{M}=\bigcup_{i\in\mathcal{I}} \operatorname{int} Z'_i$. This means that in proving Theorem 9.1 we may assume, without loss of generality, that $\tilde{M}=\bigcup_{i\in\mathcal{I}} \operatorname{int} Z_i$, and that there is a Γ-invariant triangulation of $\tilde{M}$ in which the Z_i are all subcomplexes. We shall make these assumptions for the remainder of this section.

The proof of Theorem 9.1 proceeds by constructing from the Γ-plating $(\mathcal{I}, \mathcal{X}, \mathcal{Z})$ a new Γ-plating $(\mathcal{E}(T), \mathcal{G}, \mathcal{W})$ of $\tilde{M}$, indexed by the Γ-set $\mathcal{E}(T)$ (8.1), and satisfying the hypotheses of Proposition 8.2. The construction of the map f appearing in the hypothesis of Proposition 8.2 will precede the construction of the Γ-plating $(\mathcal{E}(T), \mathcal{G}, \mathcal{W})$.

9.4. **Lemma.** *There exists a continuous, Γ-equivariant map $f\colon \tilde{M}\to T$ such that for every $i\in\mathcal{I}$ we have $f(Z_i)\subset A_T(X_i)$.*

Proof. For each $P\in\tilde{M}$ let us denote by $\mathcal{I}_P\subset\mathcal{I}$ the set of all indices i such that $P\in Z_i$. Since the plating $\mathcal{Z}$ is by definition locally finite, $\mathcal{I}_P$ is a finite subset of $\mathcal{I}$ for each $P\in\tilde{M}$. By condition (ii) in the hypothesis of Theorem 9.1, for any two indices $i, i'\in\mathcal{I}$ we have $\mathcal{E}(A_T(X_i))\cap\mathcal{E}(A_T(X_{i'}))\neq\emptyset$, and in particular $A_T(X_i)\cap A_T(X_{i'}))\neq\emptyset$. Hence by Corollary 2.3.1, the set $\Psi_P=\bigcap_{i\in\mathcal{I}_P} A_T(X_i)$ is a segment in T, and is in particular non-empty. Since $Z_{\gamma\cdot i}=\gamma\cdot Z_i$ for all $i\in\mathcal{I}$, $\gamma\in\Gamma$, we have $\mathcal{I}_{\gamma\cdot P}=\gamma\cdot\mathcal{I}_P$ and hence $\Psi_{\gamma\cdot P}=\gamma\cdot\Psi_P$ for all $P\in\tilde{M}$, $\gamma\in\Gamma$.

Let us equip $\tilde{M}$ with the Γ-invariant triangulation given by 9.3, in which the Z_i are all subcomplexes. Let $\tilde{M}^{(d)}$ denote the d-skeleton of $\tilde{M}$ for $d=0, \ldots, 3$.

We claim that there is a Γ-equivariant map $f^0: \tilde{M}^{(0)} \to T$ such that $f^{(0)}(P) \in \Psi_P$ for every $P \in \tilde{M}^{(0)}$. To prove this we use a set $S^0 \subset \tilde{M}^{(0)}$ which is a complete set of orbit representatives for the action of Γ on $\tilde{M}^{(0)}$, in the sense that $M^{(0)} = \coprod_{P \in S^0} \Gamma \cdot P$. For each $P \in S^0$ we choose a vertex v_P in the segment Ψ_P. Since Γ acts freely on $\tilde{M}$, and in particular on $\mathscr{T}^0$, we may define f^0 unambiguously by setting $f^0(\gamma \cdot P_0) = \gamma \cdot v_{P_0}$ for all $\gamma \in \Gamma$, $P \in S^0$. The Γ-equivariance of f^0 is clear. Furthermore, for any $P \in \tilde{M}^{(0)}$, writing $P = \gamma \cdot P_0$ with $P_0 \in S^0$, we have $f^0(P) = \gamma \cdot v_{P_0} \in \gamma \cdot \Psi_{P_0} = \Psi_{\gamma \cdot P_0} = \Psi_P$. This proves the claim.

For any finite set Φ of vertices of T let us denote by hull(Φ) the union of all segments in T having their endpoints in Φ. If Φ is non-empty then hull(Φ) is a connected subcomplex of T and hence a subtree. If the non-empty finite set Φ is contained in a line $L \subset T$, then hull(Φ) is a segment in L.

By induction on $d \in \{0, \ldots, 3\}$, we shall show that f^0 can be extended to a map $f^d: \tilde{M}^{(d)} \to T$ such that for every closed simplex τ of $\tilde{M}$ we have $f(\tau) \subset \mathrm{hull}(f^0(\tau \cap \tilde{M}^{(0)}))$. For $d=0$ the assertion is trivial. Suppose that f^d has been constructed for a given $d \in \{0, 1, 2\}$. Let S^{d+1} be a complete set of orbit representatives for the action of Γ on the set of all closed $(d+1)$-simplices of $\tilde{M}$. For each $\tau \in S^{d+1}$, and each proper face τ' of τ, we have $f^d(\tau') \subset \mathrm{hull}(f^0(\tau' \cap \tilde{M}^{(0)})) \subset \mathrm{hull}(f^0(\tau \cap \tilde{M}^{(0)}))$. Thus f^d maps $\partial\tau$ into the tree $\mathrm{hull}(f^0(\tau \cap \mathscr{T}^0))$. Since a tree is contractible, $f^d | \partial\tau$ can be extended to a continuous map $f_\tau: \tau \to \mathrm{hull}(f^0(\tau \cap \tilde{M}^{(0)}))$. Now Γ acts freely on the set of $(d+1)$-simplices of $\mathscr{T}$, so any $(d+1)$-simplex τ can be written in a unique way as $\gamma \cdot \tau_0$ with $\gamma \in \Gamma$ and $\tau_0 \in S^{d+1}$. We define a continuous map $f_\tau: \tau \to T$ by setting $f(\gamma \cdot P) = \gamma \cdot f(P)$ for every $P \in \tau$. The Γ-equivariance of f^d implies that $f_\tau | \partial\tau = f^d | \partial\tau$. Hence we may extend f^d to a continuous map $f^{d+1}: \tilde{M}^{(d+1)} \to T$ by setting $f^{d+1} | \tau = f_\tau$ for every closed $(d+1)$-simplex τ. It is now clear that f is Γ-equivariant and that $f(\tau) \subset \mathrm{hull}(f^0(\tau \cap \tilde{M}^{(0)}))$ for every closed simplex τ of $\tilde{M}^{(d+1)}$. This completes the induction.

Set $f = f^3$. We shall complete the proof of the lemma by showing that $f(Z_i) \subset A_T(X_i)$ for every $i \in \mathscr{I}$. Since Z_i is a subcomplex of $\tilde{M}$ we need only show that if $\tau \subset Z_i$ is a closed simplex then $f(\tau) \subset A_T(X_i)$. For any vertex P of τ we have $P \in Z_i$ and hence $i \in \mathscr{I}_P$. Therefore $f^0(P) \in \Psi_P \subset A_T(X_i)$. Thus $f^0(\tau \cap \mathscr{T}^0) \subset A_T(X_i)$. It follows that $\mathrm{hull}(f^0(\tau \cap \mathscr{T}^0))$ is a segment in the line $A_T(X_i)$. Hence $f(\tau) \subset \mathrm{hull}(f^0(\tau \cap \tilde{M}^{(0)})) \subset A_T(X_i)$. □

9.5. In this subsection we construct a closed set $W_e \subset \tilde{M}$ for each $e \in \mathscr{E}(T)$. Most of the section will be devoted to proving that the sets W_e constitute a plating, and that they define a Γ-plating satisfying the hypotheses of Proposition 8.2.

For each edge e of T, we define a set $U_e \subset \tilde{M}$ as follows. A point $P \in \tilde{M}$ lies in U_e if and only if for every $i \in \mathscr{I}$ such that $P \in \mathrm{int}\, Z_i$, we have $e \in \mathscr{E}(A_T(X_i))$. Thus U_e is the complement of a union of sets of the form $\mathrm{int}\, Z_i$, where i ranges over a certain subset of $\mathscr{I}$. Since the sets $\mathrm{int}\, Z_i$ are open, it follows that U_e is closed.

For each triod Y in T we define a set $V_Y \subset \tilde{M}$ as follows. A point $P \in \tilde{M}$ belongs to V_Y if and only if for each length-2 segment $\sigma \subset Y$ there is an index $i \in \mathscr{I}$ such that $\sigma \subset A_T(X_i)$ and $P \in Z_i$. It is clear from this definition that V_Y is a union of sets of the form $Z_i \cap Z_j \cap Z_k$, where (i, j, k) ranges over some subset of $\mathscr{I} \times \mathscr{I} \times \mathscr{I}$. But the plating $\mathscr{Z}$ is by definition a locally finite family of closed sets. It follows that V_Y is a locally finite union of closed sets, and hence a closed subset of $\tilde{M}$.

Now for each edge e of T, we set

$$W_e = (U_e \cap f^{-1}(\bar{e})) \cup \bigcup_{y \in \mathscr{Y}_e} V_Y,$$

where $\mathscr{Y}_e$ denotes the set of all triods containing e. Since the U_e and the V_Y are closed, and since f is continuous, it is clear that W_e is closed in $\tilde{M}$.

9.6. **Lemma.** *For every edge e of T we have $f(W_e) \subset \bar{e}$.*

Proof. By definition W_e is the union of $U_e \cap f^{-1}(\tilde{e})$ with the sets V_Y, where Y ranges over all triods having e as an edge. We therefore need only prove that if e is an edge of a triod Y then $f(W_e) \subset \bar{e}$. Actually we will prove slightly more, namely that $f(W_e) = \{v\}$ where v is the center of the triod Y. Let $\sigma, \sigma', \sigma''$ be the length-2 segments contained in Y. Let P be any point of V_Y. By the definition of V_Y there are indices $i, i', i'' \in \mathscr{I}$ such that $P \in Z_i \cap Z_{i'} \cap Z_{i''}$ and such that $\sigma \subset A_T(X_i)$, $\sigma' \subset A_T(X_{i'})$ and $\sigma'' \subset A_T(X_{i''})$. By Lemma 9.4 we have $f(P) \in f(Z_i) \cap f(Z_{i'}) \cap f(Z_{i''}) \subset A_T(X_i) \cap A_T(X_{i'}) \cap A_T(X_{i''})$. But Lemma 2.4, applied to the set $\mathscr{L} = \{A_T(X_i), A_T(X_{i'}), A_T(X_{i''})\}$, shows that $A_T(X_i) \cap A_T(X_{i'}) \cap A_T(X_{i''}) = \{v\}$. Hence $f(P) = v$ as required. □

9.7. **Lemma.** *We have $\tilde{M} = \bigcup_{e \in \mathscr{E}(T)} W_e$.*

Proof. Given any point $P \in \tilde{M}$, we must show that $P \in W_e$ for some edge e of T. We let $\mathscr{I}_P$ denote the set of all indices $i \in \mathscr{I}$ such that $P \in Z_i$. We have $\mathscr{I}_P \neq \emptyset$ since $\tilde{M} = \bigcup_{i \in \mathscr{I}} \operatorname{int} Z_i$ by 9.3. But the plating $\mathscr{Z}$ is by definition a locally finite family; hence $\mathscr{I}_P$ is finite. Now let $\mathscr{L}$ denote the set of all lines of the form $A_T(X_i)$ for $i \in \mathscr{I}_P$. Then $\mathscr{L}$ is a finite, non-empty collection of lines in T. Since $P \in Z_i$ for every $i \in \mathscr{I}_P$, hypothesis (ii) of the theorem implies that $\mathscr{E}(L) \cap \mathscr{E}(L') \neq \emptyset$ for all $L, L' \in \mathscr{L}$. By Corollary 2.3.1, $\bigcap \mathscr{L}$ is a segment in T, possibly degenerate.

Consider first the case where $\bigcap \mathscr{L}$ is non-degenerate. Note that for every $i \in \mathscr{I}_P$ we have $P \in Z_i$ and hence $f(P) \in f(Z_i) \subset A_T(X_i)$, in view of the defining property of f. By the definition of $\mathscr{L}$ this means that $P \in \bigcap \mathscr{L}$. As $\bigcap \mathscr{L}$ is a non-degenerate segment, we have $f(P) \in \bar{e}$ for some edge $e \in \mathscr{E}(\bigcap \mathscr{L})$. But to say that $e \in \mathscr{E}(\bigcap \mathscr{L})$ means that $e \in \mathscr{E}(A_T(X_i))$ for every $i \in \mathscr{I}_P$; by the definition of U_e this says that $P \in U_e$. Hence in this case we have $P \in U_e \cap f^{-1}(\bar{e}) \subset W_e$.

Now consider the case where the segment $\bigcap \mathscr{L}$ is degenerate. In this case, according to Proposition 2.4, there is a triod $Y \subset T$ such that every length-2 segment contained in Y is contained in a line in $\mathscr{L}$. This means that for every length-2 segment σ in Y there is an index i such that $P\in\operatorname{int} Z_i$ and $\sigma\subset A_T(X_i)$. Since $P\in\operatorname{int} Z_i$ implies that in particular $P\in Z_i$, it follows from the definition of V_Y that in this case $P\in V_Y$, and so $P\in W_e$ for any edge e of Y. □

9.8. **Lemma.** *Every point $P\in\tilde{M}$ has a neighborhood N in $\tilde{M}$ such that there are at most two edges $e\in\mathscr{E}(T)$ for which $N\cap U_e\cap f^{-1}(\bar{e})\neq\emptyset$.*

Proof. By 9.3 there is an index $i\in\mathscr{I}$ such that $P\in\operatorname{int} Z_i$. Let v be a vertex of T such that $f(P)\in\operatorname{Star}(v)$, where $\operatorname{Star}(v)$ denotes the open star of v in T. Then $N=\operatorname{int} Z_i\cap f^{-1}(\operatorname{Star}(v))$ is a neighborhood of P in $\tilde{M}$. Let e be an edge such that $N\cap U_e\cap f^{-1}(\bar{e})\neq\emptyset$. Then in particular we have $\operatorname{int} Z_i\cap U_e\neq\emptyset$, and it follows from the definition of U_e that $e\in\mathscr{E}(A_T(X_i))$. But we also have $f^{-1}(\operatorname{Star}(v))\cap f^{-1}(\bar{e})\neq\emptyset$ and hence $(\operatorname{Star}(v))\cap\bar{e}\neq\emptyset$; thus the edge e must be incident to the vertex v. But there can be at most two closed edges that lie in the line $A_T(X_i)$ and are incident to the vertex v. □

9.9. **Lemma.** *For any $P\in\tilde{M}$ there is at most one triod $Y\subset T$ such that $P\in V_Y$. Furthermore, if $P\in V_Y$ for a given triod Y and $P\in U_e\cap f^{-1}(\bar{e})$ for a given edge e, then $e\in\mathscr{E}(Y)$.*

Proof. As in the proof of Lemma 9.7, we let $\mathscr{I}_P$ denote the set of all indices $i\in\mathscr{I}$ such that $P\in Z_i$, and we let $\mathscr{L}$ denote the set of all lines of the form $A_T(X_i)$ for $i\in\mathscr{I}_P$. As we observed in the proof of Lemma 9.7, we have $\mathscr{E}(L)\cap\mathscr{E}(L')\neq\emptyset$ for all $L, L'\in\mathscr{L}$. If Y is a triod such that $P\in V_Y$ then every length-2 segment in Y is contained in a line in $\mathscr{L}$. It follows from Proposition 2.4 that there is at most one triod with this property. This proves the first assertion.

Now suppose that $P\in V_Y$ for a triod Y and that $P\in U_e\cap f^{-1}(\bar{e})$ for an edge e. Let v denote the center of the triod Y. It follows from Proposition 2.4 that $\bigcap\mathscr{L}=\{v\}$. By the defining property of the map f we have

$$f(P)\in\bigcap_{i\in\mathscr{I}_P} f(Z_i)\subset\bigcap_{i\in\mathscr{I}_P} A_T(X_i)=\bigcap\mathscr{L}=\{v\},$$

i.e. $f(P)=v$. Hence the edge e is incident to v.

By 9.3 there is an index $k\in\mathscr{I}$ such that $P\in\operatorname{int} Z_k$. Since $P\in U_e$ it follows that $e\in\mathscr{E}(A_T(X_k))$. On the other hand, since $P\in\operatorname{int} Z_k$ we have in particular $P\in Z_k$ and hence $k\in\mathscr{I}_P$. Hence the line $A_T(X_k)$ belongs to $\mathscr{L}$. By Proposition 2.4, $A_T(X_k)$ meets the triod Y in a length-2 segment. Since e is an edge of $A_T(X_k)$ and is incident to v, it follows that e is an edge of Y. □

9.10. **Lemma.** *The indexed family $(\mathscr{V}_Y)_Y$ of subsets of $\tilde{M}$, where Y ranges over all triods in T, is locally finite.*

Proof: Since the plating $(Z_i)_{i\in\mathscr{I}}$ is by definition a locally finite family, the family $(Z_i\cap Z_j\cap Z_k)_{(i,j,k)\in\mathscr{I}}$ is also locally finite. Each set V_Y, where Y is a triod in T, is by definition a union of sets of the form $Z_i\cap Z_j\cap Z_k$ for $(i,j,k)\in\mathscr{I}$.

Furthermore, for $Y \neq Y'$ we have $V_Y \cap V_{Y'} = \emptyset$ by Lemma 9.9. Hence if P is a point of $\tilde{M}$ and N is a neighborhood of P that meets $Z_i \cap Z_j \cap Z_k$ for at most $n < \infty$ triples (i, j, k), then N meets V_Y for at most n triods Y. □

9.11. **Lemma.** *The family* $\mathscr{W} = (W_e)_{e \in \mathscr{E}(T)}$ *is a plating of* $\tilde{M}$.

Proof. We observed in 9.5 that each W_e is closed. By Lemma 9.7 we have $\tilde{M} = \bigcup_{e \in \mathscr{E}(T)} W_e$. It remains to verify that $\mathscr{W}$ is a locally finite family. Let P be any point of $\tilde{M}$. By Lemma 9.8, P has a neighborhood N in $\tilde{M}$ such that there are at most two edges $e \in \mathscr{E}(T)$ for which $N \cap U_e \cap f^{-1}(\bar{e}) \neq \emptyset$. By Lemma 9.10 there is a neighborhood N' of P such that $N \cap N' \neq \emptyset$ for only a finite number k of triods Y. Since each triod contains only three edges of T, it follows from the definition of the W_e that the neighborhood $N \cap N'$ of P meets W_e for at most $3k+2$ edges $e \in \mathscr{E}(T)$. This proves local finiteness. (A slightly more careful application of the lemmas above would give a bound of 3 in place of $3k+2$: compare Lemma 9.12 below.) □

9.12. **Lemma.** *The plating* $\mathscr{W} = (W_e)_{e \in \mathscr{E}(T)}$ *has order at most* 2.

Proof. Let P be any point of $\tilde{M}$. If there is no triod $Y \subset T$ such that $P \in V_Y$, it follows from Lemma 9.8 that there are at most two edges $e \in \mathscr{E}(T)$ such that $P \in W_e$. On the other hand, if $Y \subset T$ is a triod such that $P \in V_Y$, then it follows from Lemma 9.9 that the only edges of T for which P can lie in W_e are the three edges of Y. Thus in either case there are at most three edges e for which $P \in W_e$. □

9.13. Now let us set $\mathscr{G} = (\Gamma_e)_{e \in \mathscr{E}(T)}$ for every $e \in \mathscr{E}(T)$.

Lemma. *The triple* $(\mathscr{E}(T), \mathscr{G}, \mathscr{W})$ *is a* Γ*-plating of the* Γ*-space* $\tilde{M}$.

Proof. According to Lemma 9.11, $\mathscr{W}$ is a plating of $\tilde{M}$. Thus we need only establish conditions (i)–(iii) of Definition 5.2. Conditions (i) and (ii) are obvious since we have defined $\mathscr{G} = (\Gamma_e)_{e \in \mathscr{E}(T)}$. Condition (iii) asserts that $\gamma \cdot W_e = W_{\gamma \cdot e}$ for every $e \in \mathscr{E}(T)$ and every $\gamma \in \Gamma$. To prove this, we first show that $\gamma \cdot U_e = U_{\gamma \cdot e}$. From the definition of U_e we have

$$\tilde{M} - \gamma \cdot U_e = \gamma \cdot \bigcup_{\substack{i \in \mathscr{I} \\ e \notin A_T(X_i)}} Z_i = \bigcup_{\substack{i \in \mathscr{I} \\ e \notin A_T(X_i)}} Z_{\gamma \cdot i} = \bigcup_{\substack{i \in \mathscr{I} \\ e \notin A_T(X_{\gamma^{-1} \cdot i})}} Z_i.$$

But for all i and γ we have $A_T(X_{\gamma^{-1} \cdot i}) = A_T(\gamma^{-1} X_i \gamma) = \gamma^{-1} \cdot A_T(X_i)$. Hence

$$\tilde{M} - \gamma \cdot U_e = \bigcup_{\substack{i \in \mathscr{I} \\ e \notin \gamma^{-1} \cdot A_T(X_i)}} Z_i = \bigcup_{\substack{i \in \mathscr{I} \\ \gamma \cdot e \notin A_T(X_i)}} Z_i = \tilde{M} - U_{\gamma \cdot e},$$

so that $\gamma \cdot U_e = U_{\gamma \cdot e}$. A similar naturality argument shows that $\gamma \cdot V_Y = V_{\gamma \cdot Y}$ for every triod $Y \subset T$ and every $\gamma \in \Gamma$. The Γ-equivariance of f implies that

$\gamma \cdot f^{-1}(e) = f^{-1}(\gamma \cdot e)$ for all e and γ. Using the definition of the W_e one immediately concludes that $\gamma \cdot W_e = W_{\gamma \cdot e}$. □

9.14. The next three lemmas will allow us to prove that the Γ-plating $(\mathscr{E}(T), \mathscr{G}, \mathscr{W})$ is doubly uniform.

9.14.1. **Lemma.** *Let σ be any segment in $\tilde{M}$,* and let P be a point in $\bigcap_{e \in \mathscr{E}(\sigma)} W_e$. *Then there is an index $i \in \mathscr{I}$ such that $P \in Z_i$ and $\sigma \subset A_T(X_i)$.*

Proof. If e is any edge of σ, we have $P \in W_e$; thus according to the definition of W_e, either $P \in U_e \cap f^{-1}(e)$, or $P \in P_Y$ for some triod Y containing e. We first consider the case in which $P \in U_e \cap f^{-1}(e)$ for every $e \in \mathscr{E}(\sigma)$, so that in particular $P \in U_e \cap \mathrm{f}^{-1}(e)$ for every $e \in \mathscr{E}(\sigma)$. In this case, using 9.3, we choose an index i such that $P \in \operatorname{int} Z_i$. By the definition of the U_e we have $e \in \mathscr{E}(A_T(X_i))$ for every edge e of σ; this means that $\sigma \subset A_T(X_i)$, and the proof is complete in this se. Now consider the case in which there is an edge s of σ such that $P \notin U_s$, and therefore $P \in V_Y$ for some triod Y containing s. Let e be any edge of σ. If $P \in U_e \cap f^{-1}(e)$, then it follows from the second assertion of Lemma 9.9 that e is an edge of Y. On the other hand, if $P \in V_{Y'}$, for some triod Y' containing e, then the first assertion of Lemma 9.9 implies that $Y' = Y$, so that again e is an edge of Y. Thus every edge of σ is an edge of Y. This means that $\sigma \subset Y$, and hence that $\sigma \subset \sigma'$ for some length-2 segment $\sigma' \subset Y$. Since $P \in V_Y$, the definition of V_Y says that for some index i we have $A_T(X_i) \supset \sigma' \supset \sigma$ and $P \in Z_i$. Thus the proof is complete in this case as well. □

9.14.2. For any segment $\sigma \subset T$ we let $\mathscr{I}_\sigma$ denote the set of all indices $i \in \mathscr{I}$ such that $\sigma \subset A_T(X_i)$. It is clear that $\mathscr{I}_\sigma$ is invariant under Γ_σ.

Lemma. *For any segment $\sigma \subset T$, the Γ_σ-set $\mathscr{I}_\sigma$ contains only finitely many Γ_σ-orbits.*

Proof. By Proposition 5.4, there are only finitely many Γ-orbits in $\mathscr{I}$. Hence there is a finite set $S \subset \mathscr{I}_\sigma$ such that $\mathscr{I}_\sigma \subset \Gamma \cdot S$. Now let n denote the length of σ, and for any $s \in S$ let Σ_s denote the set of all segments of length n in $A_T(X_s)$. Then Σ_s is an X_s-set, and since the cyclic group X_s is T-hyperbolic, Σ_s is a finite union of X_s-orbits.

For each $s \in S$ set $\Sigma'_s = \Sigma_s \cap \Gamma \cdot \{\sigma\}$. Since Σ_s contains only finitely many X_s-orbits, there is a finite set $Q_s \subset \Sigma'_s$ such that $\Sigma'_s \subset X_s \cdot Q_s$. Now we have $Q_s \subset \Sigma'_s \subset \Gamma \cdot \sigma$. Hence for each $s \in S$ and each $\rho \in Q_s$, we may choose an element $\gamma_\rho \in \Gamma$ such that $\gamma_\rho \cdot \rho = \sigma$. We have $\rho \in \Sigma'_s \subset \Sigma_s$ and hence $\rho \subset A_T(X_s)$. Thus $\sigma = \gamma_\rho \cdot \rho \subset \gamma_\rho \cdot A_T(X_s) = A_T(\gamma_\rho X_s \gamma_\rho^{-1}) = A_T(X_{\gamma_\rho \cdot s})$. This shows that $\gamma_\rho \cdot s \in \mathscr{I}_\sigma$. Thus the finite set

$$\mathscr{J} = \bigcup_{s \in S} \{\gamma_\rho \cdot s : \rho \in Q_s\}$$

is a subset of $\mathscr{I}_\sigma$. We shall complete the proof by showing that $\Gamma_\sigma \cdot \mathscr{J} = \mathscr{I}_\sigma$.

Let $i \in \mathscr{I}_\sigma$ be given. We may write $i = \gamma \cdot s$ for some $s \in S$ and some $\gamma \in \Gamma$. We have $\sigma \subset A_T(X_i)$ and hence $\gamma^{-1} \cdot \sigma \subset A_T(X_s)$. Thus $\gamma^{-1} \cdot \sigma \in \Sigma'_s$. Thus we may write $\gamma^{-1} \cdot \sigma = x \cdot \rho$ for some $x \in X_s$ and some $\rho \in Q_s$. By definition we have $\gamma_\rho \cdot \rho = \sigma$. Hence $\gamma x \gamma_\rho^{-1} \in \Gamma_\sigma$. But since $x \in X_s$ we have $x \cdot s = s$. Therefore $i = \gamma \cdot s = (\gamma x \gamma_\rho^{-1})(\gamma_\rho \cdot s)$. This shows that i is in the Γ_σ-orbit of $\gamma_\rho \cdot s$, which by definition is an element of $\mathscr{J}$. This completes the proof. □

9.14.3. **Lemma.** *For every non-degenerate segment $\sigma \subset T$, the Γ_σ-set $W_\sigma = \bigcap_{e \in \mathscr{E}(\sigma)} W_e$ is uniform.*

Proof. By Lemma 9.14.1, we have $W_\sigma \subset \bigcup_{e \in \mathscr{I}_\sigma} Z_i$. By Lemma 9.14.2, there is a finite set $S \subset \mathscr{I}_\sigma$ such that $\Gamma_\sigma \cdot S = \mathscr{I}_\sigma$. Thus $W_\sigma \subset \Gamma_\sigma \cdot \bigcup_{i \in S} Z_i$. As $\bigcap_{e \in \mathscr{E}(\sigma)} W_e$ is Γ_σ-invariant, it follows that $W_\sigma \subset \Gamma_\sigma \cdot \bigcup_{i \in S} (Z_i \cap W_\sigma)$. Hence it suffices to show that $Z_i \cap W_\sigma$ is compact for every non-degenerate segment $\sigma \subset T$ and every $i \in \mathscr{I}$.

By the hypothesis of Theorem 9.1, $\mathscr{Z}$ is uniform. Thus there is a compact set $R_i \subset Z_i$ such that $X_i \cdot R_i = Z_i$. Let e_0 be any edge of σ. By Lemma 9.6 we have $W_\sigma \subset W_{e_0} \subset f^{-1}(\overline{e_0})$. Now let x_i be a generator of X_i. Since x_i is T-hyperbolic and $f(R) \subset T$ is compact, there is an integer $N > 0$ such that for every integer n with $|n| > N$ we have $x_i^n \cdot R \cap \overline{e_0} = \emptyset$. It follows that

$$Z_i \cap W_\sigma \subset Z_i \cap f^{-1}(\overline{e_0}) \subset \bigcup_{n=-N}^{N} x_i^n \cdot R.$$

Thus $Z_i \cap W_\sigma$ is indeed compact. □

9.14.4. **Lemma.** *The Γ-plating $(\mathscr{E}(T), \mathscr{G}, \mathscr{W})$ is doubly uniform.*

Proof. We must show that if e and e' are edges of T then the $(\Gamma_e \cap \Gamma_{e'})$-set $W_e \cap W_{e'}$ is uniform. If e and e' have no common vertex, then by Lemma 9.6 we have $f(W_e \cap W'_e) \subset \bar{e} \cap \bar{e}' = \emptyset$; thus in this case, $W_e \cap W'_e$ is empty and the assertion follows trivially. If e and e' have a common vertex, then $\sigma = \bar{e} \cup \bar{e}'$ is a segment of length 1 or 2. According to Lemma 9.14.3 the Γ_σ-set $W_e \cap W'_e$ is uniform. But $\Gamma_e \cap \Gamma_{e'}$ is the subgroup of Γ_σ consisting of elements that preserve the orientation of σ. Hence $\Gamma_e \cap \Gamma_{e'}$ has index at most 2 in Γ_σ. It follows by 1.4 that the $(\Gamma_e \cap \Gamma_{e'})$-set $W_e \cap W'_e$ is uniform. □

9.15. Proof of Theorem 9.1. By Lemmas 9.13 and 9.14.4, $(\mathscr{E}(T), \mathscr{G}, \mathscr{W})$ is a doubly uniform Γ-plating. By Lemma 9.12, $\mathscr{W}$ has order at most 2. By Lemma 9.4, $f: \tilde{M} \to T$ is Γ-equivariant. By Lemma 9.6, we have $f(W_e) \subset \bar{e}$ for every edge e of T. By Proposition 8.2 it follows that the core of c is a fibroid. □

10. Proof of Theorem A

10.1. Hyperbolic space. The proof of Theorem A will use some elementary properties of hyperbolic space $\mathbf{H}^n$.

Recall that an isometry x of $\mathbf{H}^n$ is *loxodromic* if x leaves some line $A \subset \mathbf{H}^n$ invariant and $x|A$ is a translation through some strictly positive distance l. The line A and the number l, which are uniquely determined by x, are called the *axis* and *translation length* of x; we shall write $A = A_{\mathbf{H}^n}(x)$ and $l = l_{\mathbf{H}^n}(x)$,

As in [2] and [3], we shall use the following notation. If x is an isometry of $\mathbf{H}^n$ and λ is a positive number, we shall denote by $Z_\lambda(x)$ the open subset of $\mathbf{H}^n$ consisting of all points P such that $\operatorname{dist}(P, x^m \cdot P) < \lambda$ for some $m \geqq 1$. The set $Z_\lambda(x)$ is empty if $\lambda \leqq l_{\mathbf{H}^n}(x)$. If $\lambda > l_{\mathbf{H}^n}(x)$ then $Z_\lambda(x)$ is a neighborhood of $A_{\mathbf{H}^n}(x)$.

We denote the closure of $Z_\lambda(x)$ by $\bar{Z}_\lambda(x)$. It is clear that for $0 < \lambda < \lambda'$ we have $\bar{Z}_\lambda(x) < \bar{Z}'_\lambda(x)$.

10.2. Hyperbolic manifolds. If M is a closed, hyperbolic 3-manifold, we shall identify the universal covering space of M with $\mathbf{H}^3$. The group $\Gamma \cong \pi_1(M)$ of deck transformations of the universal covering space is a discrete, torsion-free subgroup of the group $\mathrm{PSL}_2(\mathbf{C})$ of orientation-preserving isometries of $\mathbf{H}^3$. Recall that since M is closed, each non-trivial element x of Γ is loxodromic. We may identify M with $\mathbf{H}^3/\Gamma$.

Recall that for each non-trivial element $x \in \Gamma$, the centralizer of x is the unique maximal cyclic subgroup X of Γ containing x, and that $X - \{1\}$ consists of all non-trivial elements of Γ having the same axis in $\mathbf{H}^3$ as x.

Note that since every element of Γ has a cyclic centralizer, $\pi_1(M) \cong \Gamma$ has no free abelian subgroup of rank 2. Furthermore, the universal cover $\mathbf{H}^3$ of M is homeomorphic to $\mathbf{R}^3$ and hence M is irreducible. Thus M is a simple 3-manifold. If X is any maximal cyclic subgroup of Γ and λ is any positive number, then as in [2] we shall write $Z_\lambda(X) = Z_\lambda(x)$, where x is any generator of X; this is unambiguous because $Z_\lambda(x^{-1}) = Z_\lambda(x)$.

The discreteness of Γ implies that for any $P \in \mathbf{H}^3$ and any $\lambda > 0$ there are only finitely many elements $x \in \Gamma$ such that $\operatorname{dist}(P, x \cdot P) < \lambda$. Since each non-trivial element of Γ lies in a unique maximal cyclic subgroup, it follows that for any $P \in \mathbf{H}^3$ there are only finitely many maximal cyclic subgroups X of Γ such that $P \in Z_\lambda(X)$. Thus for any λ the sets $Z_\lambda(X)$, where X ranges over the maximal cyclic subgroups of Γ, form a locally finite family. Since for any λ and any X we have $\bar{Z}_\lambda(X) \subset Z_{\lambda+1}(X)$, it follows that for any λ the sets $\bar{Z}_\lambda(X)$, where X ranges over the maximal cyclic subgroups of Γ, again form a locally finite family.

10.3. Now suppose that we are given an incompressible surface $S \subset M$. Let c be a bi-collared surface with core S (3.1).

The following result, the proof of which will be based on the main theorem of [1], allows one to compare the action of Γ on the tree $T(c)$ with its action on $\mathbf{H}^3$.

Proposition. *Let $M=\mathbf{H}^3/\Gamma$ be a closed, orientable hyperbolic 3-manifold, and let x and y be elements of Γ such that $Z_{\log 3}(x)\cap Z_{\log 3}(y)\cap \neq\emptyset$. Suppose that M contains an incompressible bi-collared surface c; set $T=T(c)$, and suppose that x and y are T-hyperbolic. Then $\mathscr{E}(A_T(x))\cap\mathscr{E}(A_T(y))\neq\emptyset$.*

Proof. Let us fix a point $P\in Z_{\log 3}(x)\cap Z_{\log 3}(y)$. By the definition of $Z_{\log 3}(x)$ and $Z_{\log 3}(y)$, there are positive integers r and s such that $\mathrm{dist}(P, x^r\cdot P)<\log 3$ and $\mathrm{dist}(P, y^s\cdot P)<\log 3$.

We set $\xi=x^r$, $\eta=y^s$, $A=A_T(x)=A_T(\xi)$ and $B=A_T(y)=A_T(\eta)$. We are required to show that $\mathscr{E}(A)\cap\mathscr{E}(B)\neq\emptyset$. If ξ and η commute, their axes A and B coincide. Thus we may assume that ξ and η do not commute.

Suppose that $\mathscr{E}(A)\cap\mathscr{E}(B)=\emptyset$. Then by Proposition 2.6, the group $\Theta=\langle\xi,\eta\rangle$ is free on ξ and η, and the action of Θ on T is free. Hence by Proposition 3.6, the hyperbolic 3-manifold H^3/Θ is homeomorphic to the interior of a handlebody. In particular this means that the discrete group $\Theta\leqq \mathrm{PSL}_2(\mathbf{C})$ is non-co-compact, and it is *topologically tame* in the sense that H^3/Θ is homeomorphic to the interior of a compact 3-manifold. Note also that since M is closed and $\Theta\leqq\Gamma$, every non-trivial element of Θ is loxodromic.

Now we recall the main theorem of [1]. Suppose that ξ and η are non-commuting elements of $\mathrm{PSL}_2(\mathbf{C})$, and that the group generated by ξ and η is discrete, torsion-free, non-co-compact and topologically tame, and contains no parabolic elements. Then for any point $P\in H^3$ we have

$$\max(\mathrm{dist}(P,\xi\cdot P),\mathrm{dist}(P,\eta\cdot P))\geqq\log 3.$$

Applied to the elements η and ξ that we have defined, this gives a contradiction. □

10.4. Proof of Theorem A. We write $M=H^3/\Gamma$, where $\Gamma\leqq\mathrm{PSL}_2(\mathbf{C})$ is discrete and torsion-free. Since M is closed, all the non-trivial elements of Γ are loxodromic. Let us say that an element $x\in\Gamma-\{1\}$ is *short* if its translation length is less than λ. We choose a bi-collared surface c with core S, and we set $T=T(c)$.

Consider the case in which some short element $x\in\Gamma$ has a fixed point in T. Since x is short, its conjugacy class in Γ corresponds to a conjugacy class in $\pi_1(M)$ which is represented by a closed geodesic of length $<\lambda$. On the other hand, since x has a fixed point in T it follows from 3.1 that the conjugacy class in $\pi_1(M)$ defined by x is also represented by a closed curve in $M-|c|$. Thus conclusion (i) of Theorem A holds in this case.

Now suppose that no short element of Γ has a fixed point in T. By 2.5 this means that every short element is T-hyperbolic. In this case we shall show that conclusion (ii) of Theorem A holds.

Let us say that a maximal cyclic subgroup X of Γ is *short* if it has a short generator, and let us denote by $\mathscr{X}$ the set of all short maximal cyclic subgroups of Γ. We set $\mathscr{Z}=(\bar{Z}_\lambda(X))_{X\in\mathscr{X}}$.

Assume that conclusion (ii) of Theorem A does not hold, i.e. that M contains no hyperbolic ball of radius $\lambda/2$. Then by [2, Proposition 3.2] we have

$$\mathbf{H}^3=\bigcup_X Z_\lambda(X),$$

where X ranges over all maximal cyclic subgroups of Γ. Since $\bar{Z}_\lambda(X)=\emptyset$ when X is not short, this means that

$$\mathbf{H}^3=\bigcup_{X\in\mathscr{X}} Z_\lambda(X).$$

In particular we have $\mathbf{H}^3=\bigcup_{X\in\mathscr{X}} Z_\lambda(X)$. As we observed in 10.1, the family $\mathscr{Z}$ is a locally finite family, and by definition it consists of closed sets. Thus $\mathscr{Z}$ is a plating of $\mathbf{H}^3$.

Now as Γ acts on $\mathscr{X}$ by conjugation, we may regard $\mathscr{X}$ as a Γ-set. If we set $\mathscr{G}=(X)_{X\in\mathscr{X}}$, it is clear from Definition 5.2 that $(\mathscr{X},\mathscr{G},\mathscr{Z})$ is a Γ-plating of H^3. Since M is compact, it contains only finitely many conjugacy classes of short elements. Hence X is finite modulo Γ. Furthermore, each set $\bar{Z}_\lambda(X)$ is compact modulo X; this follows from [2, Proposition 1.6], which asserts that all the points of $Z_\lambda(X)$ lie within a bounded radius of the axis of X in H^3. This shows that the Γ-plating $(\mathscr{X},\mathscr{G},\mathscr{Z})$ is uniform.

We now apply Theorem 9.1 to the uniform Γ-plating $(\mathscr{X},\mathscr{G},\mathscr{Z})$. Hypothesis (i) of 9.1 is immediate from the construction since we are in the case where every short element of Γ is T-hyperbolic. To see that hypothesis (ii) holds, note that by the hypothesis of Theorem A we have $\lambda<\log 3$, and hence $Z_\lambda(X)\subset Z_{\log 3}(X)$ for every $X\in\mathscr{X}$. Thus if X, $X'\in\mathscr{X}$ satisfy $Z_\lambda(X)\cap Z_\lambda(X')\neq\emptyset$, then $Z_{\log 3}(X)\cap Z_{\log 3}(X')\neq\emptyset$, so that $\mathscr{E}(A_T(X))\cap\mathscr{E}(A_T(X'))\neq\emptyset$ by Proposition 10.3. It therefore follows from Theorem 9.1 that S is a fibroid. But this contradicts the hypothesis of Theorem A. □

11. Proof of the Main Theorem

11.1. The Main Theorem of the introduction will be proved by combining Theorem A with the results of [3]. We begin with the following result, which is of independent interest and from which the Main Theorem will be deduced by specializing to a suitable value of λ.

Proposition. *Let M be a closed, orientable hyperbolic 3-manifold. Let us write $M=H^3/\Gamma$, where Γ is a discrete, torsion-free group of isometries of H^3. Suppose that M contains a non-separating incompressible surface Σ which is not a fibroid.*

Let λ be a number such that $0<\lambda<\log 3$. Then either

(i) *there is an element $x\in\Gamma$ of length $<\lambda$ such that M contains an open set isometric to $Z_{\log 3}(x)/\langle x\rangle$ where x is some loxodromic isometry of H^3, or*

(ii) *M contains a hyperbolic ball of radius $\lambda/2$.*

Proof. It is enough to show that conclusion (i) of Theorem A implies conclusion (i) of the present proposition. Suppose that some closed curve $\alpha: S^1\to M-\Sigma$ is homotopic in M to a non-trivial closed geodesic of length $<\lambda$. Let a be an element of the conjugacy class in Γ corresponding to the free homotopy class of α in M. Since α is homotopic to a closed geodesic of length $<\lambda$ we have length $a<\lambda$. On the other hand, since α is carried by the complement of the non-separating surface Σ, the element a lies in the kernel of some homomorphism of Γ onto $\mathbf{Z}$. The centralizer X of a in Γ is infinite cyclic. Let x be a generator of X. Since a is a power of x, the translation length of x is less than λ, and x lies in the kernel of a homomorphism of Γ onto $\mathbf{Z}$. By [3, Proposition 2.2], any two conjugates of x generate a group which is topologically tame and has infinite index in Γ. By [3, Proposition 2.1], it follows that M contains an open set isometric to $Z_{\log 3}(x)/\langle x\rangle$. □

11.2. Proof of the Main Theorem. We apply Proposition 11.1, taking $\lambda=0.8$. If conclusion (i) holds then it follows from [3, Proposition 4.1] that vol $M>0.34$. If conclusion (ii) holds, i.e. if M contains a hyperbolic ball of radius 0.4, then using density estimates for sphere-packings as in [10], one computes that the volume of M is at least 0.35. □

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