Free Kleinian groups and volumes of hyperbolic 3-manifolds

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1 Introduction

The central result of this paper, Theorem 6.1, gives a constraint that must be satisfied by the generators of any free, topologically tame Kleinian group without parabolic elements. The following result is case (a) of Theorem 6.1.

Main Theorem *Let $k \geq 2$ be an integer and let Φ be a purely loxodromic, topologically tame discrete subgroup of* $\mathrm{Isom}_+(\mathbf{H}^3)$ *which is freely generated by elements $\xi_1, \ldots, \xi_k$. Let z be any point of $\mathbf{H}^3$ and set $d_i = \mathrm{dist}(z, \xi_i \cdot z)$ for $i = 1, \ldots, k$. Then we have*

$$\sum_{i=1}^{k} \frac{1}{1 + e^{d_i}} \leq \frac{1}{2}.$$

In particular there is some $i \in \{1, \ldots, k\}$ such that $d_i \geq \log(2k - 1)$.

The last sentence of the Main Theorem, in the case $k = 2$, is equivalent to the main theorem of [13]. While most of the work in proving this generalization involves the extension from rank 2 to higher ranks, the main conclusion above is strictly stronger than the main theorem of [13] even in the case $k = 2$.

Like the main result of [13], Theorem 6.1 has applications to the study of large classes of hyperbolic 3-manifolds. This is because many subgroups of the

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fundamental groups of such manifolds can be shown to be free by topological arguments. The constraints on these free subgroups impose quantitative geometric constraints on the shape of a hyperbolic manifold. As in [13] these can be applied to give volume estimates for hyperbolic 3-manifolds satisfying certain topological restrictions. The volume estimates obtained here, unlike those proved in [13], are strong enough to have qualitative consequences, as we shall explain below.

The following result is proved by combining the case $k = 3$ of the Main Theorem with the techniques of [14].

Corollary 9.2 *Let N be a closed orientable hyperbolic 3-manifold. Suppose that the first betti number $\beta_1(N)$ is at least 4, and that $\pi_1(N)$ has no subgroup isomorphic to the fundamental group of a surface of genus 2. Then N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$. Hence the volume of N is greater than 3.08.*

The volumes of hyperbolic 3-manifolds form a well-ordered set of ordinal type ω^ω. As there are infinitely many hyperbolic 3-manifolds of volume less than 3.08, the above result implies that each of the manifolds realizing the first ω volumes either has betti number at most 3 or has a fundamental group containing an isomorphic copy of a genus-2 surface group. (This conclusion is stated as Corollary 9.3.)

It was not possible to deduce qualitative consequences of this sort in [13] because the lower bound of 0.92, obtained there for the volume of a closed hyperbolic 3-manifold of first betti number at least 3, is smaller than the least known volume of any hyperbolic 3-manifold.

Our techniques also apply to non-compact manifolds. For example, Corollary 9.4 is similar to the above corollary but illustrates the applicability of our techniques to the geometric study of *infinite-volume* hyperbolic 3-manifolds. It asserts that a non-compact, topologically tame, orientable hyperbolic 3-manifold N without cusps always contains a hyperbolic ball of radius $\frac{1}{2}\log 5$ unless $\pi_1(N)$ either is a free group of rank 2 or contains an isomorphic copy of a genus-2 surface group.

An application of Theorem 6.1 to non-compact finite-volume manifolds is the following result, which uses only the case $k = 2$ of the Main Theorem, but does not follow from the weaker form of the conclusion which appeared in [13].

Theorem 11.1 *Let $N = \mathbf{H}^3/\Gamma$ be a non-compact hyperbolic 3-manifold. If N has betti number at least 4, then N has volume at least π.*

Theorem 11.1 is deduced via Dehn surgery techniques from Proposition 10.1 and its Corollary 10.3, which are of independent interest. These results imply that if a hyperbolic 3-manifold satisfies certain topological restrictions, for example if its first betti number is at least 3, then there is a good lower bound for the radius of a tube about a short geodesic, from which one can deduce a lower bound for the volume of the manifold in terms of the length of a short geodesic. The lower bound approaches π as the length of the shortest geodesic tends to 0. Corollary 10.3 will be used in [12] as one ingredient in a proof of a new lower bound for the volume of a hyperbolic 3-manifold of betti number 3. This lower bound is greater than the smallest known volume of a hyperbolic 3-manifold, and therefore has the qualitative consequence that any smallest-volume hyperbolic 3-manifold has betti number at most 2.

The proof of the Main Theorem follows the same basic strategy as the proof of the main theorem of [13]. The Main Theorem is deduced from Theorem 6.1(d), which gives the same conclusion under somewhat different hypotheses. In 6.1(d), rather than assuming that the free Kleinian group Φ is topologically tame and has no parabolics, we assume that the manifold $\mathbf{H}^3/\Phi$ admits no non-constant positive superharmonic functions. As in [13], the estimate is proved in this case by using a Banach-Tarski-style decomposition of the area measure based on a Patterson construction. The deduction of the Main Theorem from 6.1(d) is based on Theorem 5.2, which asserts that, in the variety of representations of a free group F_k, the boundary of the set $\mathcal{CC}(F_k)$ of convex-cocompact discrete faithful representations contains a dense G_δ consisting of representations whose images are "analytically tame" Kleinian groups without parabolics. This was proved in [13] in the case $k = 2$.

By definition, a rank-k free Kleinian group Γ without parabolics is analytically tame if the convex core of $\mathbf{H}^3/\Gamma$ can be exhausted by a sequence of geometrically well behaved compact submanifolds (a more exact definition is given in Section 5). The case $k = 2$ of Theorem 5.2 was established in [13] by combining a theorem of McMullen's [27] on the density of maximal cusps on the boundary of $\mathcal{CC}(F_k)$ with a special argument involving the canonical involution of a 2-generator Kleinian group. The arguments used in the proof of Theorem 5.2 make no use of the involution. This makes possible the generalization to arbitrary k, while also giving a new proof in the case $k = 2$. The ideas needed for the proof are developed in sections 2 through 5, and will be sketched here.

In Section 3 we prove a general fact, Proposition 3.2, about a sequence

(ρ_n) of discrete faithful representations of a finitely generated, torsion-free, non-abelian group G which converges to a maximal cusp ω. (For our purposes a maximal cusp is a discrete faithful representation ω of G into $\mathrm{PSL}_2(\mathbf{C})$ such that $\omega(G)$ is geometrically finite and every boundary component of the convex core of $\mathbf{H}^3/\omega(G)$ is a thrice-punctured sphere.) After passing to a subsequence one can assume that the Kleinian groups $\rho_n(G)$ converge geometrically to a Kleinian group $\widehat{\Gamma}$, which necessarily contains $\omega(G)$ as a subgroup. Proposition 3.2 then asserts that the convex core of $N = \mathbf{H}^3/\omega(G)$ embeds isometrically in $\mathbf{H}^3/\widehat{\Gamma}$. To prove this, we use Proposition 2.7, which combines an algebraic characterization of how conjugates of $\omega(G)$ can intersect in the geometric limit (Lemma 2.4), and a description of the intersection of the limit sets of two subgroups of a Kleinian group (Lemma 2.6).

In Section 4 we construct a large submanifold D of the convex core of N which is geometrically well-behaved in the sense that ∂D has bounded area and the radius-2 neighborhood of ∂D has bounded volume. We use Proposition 3.2 to show that if ρ is a discrete faithful representation near enough to ω, then $\mathbf{H}^3/\rho(G)$ contains a nearly isometric copy of D. This copy is itself geometrically well-behaved in the same sense.

In Section 5 we specialize to the case $G = F_k$. We show that if a discrete faithful representation ρ is well-approximated by infinitely many maximal cusps, then its associated quotient manifold contains infinitely many geometrically well-behaved submanifolds. In fact, we show that the resulting submanifolds exhaust the convex core of the quotient manifold and hence that the quotient manifold is analytically tame. We then apply McMullen's theorem to prove that there is a dense G_δ in the boundary of $\mathcal{CC}(F_k)$ consisting of representations which can be well approximated by maximal cusps.

In the argument given in [13], the involution of a 2-generator Kleinian group is used not only in the deformation argument, but also in the calculation based on the decomposition of the area measure in the case where $\mathbf{H}^3/\Phi$ supports no non-constant superharmonic functions. The absence of an involution in the k-generator case is compensated for by a new argument based on the elementary inequality established in Lemma 6.2. This leads to the stronger conclusion of the main theorem in the case $k = 2$.

Section 6 is devoted to the proof of Theorem 6.1.

We have mentioned that the application of Theorem 6.1 to the geometry of hyperbolic manifolds depends on a criterion for subgroups of fundamental groups of such manifolds to be free. The first such criterion in the case of a

2-generator subgroup was proved in [17] and independently in [35]. A partial generalization to k-generator subgroups, applying only when the given manifold is closed, was given in [3]. In Section 7 of this paper we give a criterion that includes the above results as special cases and is adapted to the applications in this paper.

In Section 8 we introduce some formalism which we have found useful for organizing the applications of the results of the earlier sections to the study of hyperbolic manifolds. The applications are presented in Sections 9, 10 and 11.

We close the introduction by describing several notational conventions which appear herein. Subgroups of a group are denoted by inequalities; that is, for a group G, we use $H \leq G$ to denote that H is a subgroup of G, and $H < G$ to denote that H is a proper subgroup of G. The translate of a set X by a group element γ is denoted $\gamma \cdot X$. Finally, we use $\mathrm{dist}(z, w)$ to denote the hyperbolic distance between points z and w in $\mathbf{H}^3$.

2 On the sphere at infinity

In this section we introduce the notion of the geometric limit of a sequence of Kleinian groups. We will consider a convergent sequence of discrete faithful representations into $\mathrm{PSL}_2(\mathbf{C})$ whose images converge geometrically. In general, the geometric limit of the images contains the image of the limit as a subgroup. The results in this section characterize the intersection of two conjugates of this subgroup, and the intersection of their limit sets.

The group $\mathrm{PSL}_2(\mathbf{C})$ will be considered to act either by isometries on $\mathbf{H}^3$ or, via extension to the sphere at infinity, by Möbius transformations on the Riemann sphere $\overline{\mathbf{C}}$. The action of a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{C})$ partitions $\overline{\mathbf{C}}$ into two pieces, the *domain of discontinuity* $\Omega(\Gamma)$ and the *limit set* $\Lambda(\Gamma)$. The domain of discontinuity is the largest open subset of $\overline{\mathbf{C}}$ on which Γ acts properly discontinuously. If $\Lambda(\Gamma)$ contains two or fewer points, we say Γ is *elementary.* If Γ has an invariant circle in $\overline{\mathbf{C}}$ and preserves an orientation of the circle then we say that Γ is *Fuchsian.*

By a *Kleinian group* we will mean a discrete non-elementary subgroup Γ of $\mathrm{PSL}_2(\mathbf{C})$. We will say that a Kleinian group Γ is *purely parabolic* if every non-trivial element is parabolic, or *purely loxodromic* if every non-trivial element is loxodromic.

Given a finitely generated group G, let $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ denote the variety of representations of G into $\mathrm{PSL}_2(\mathbf{C})$. A choice of k elements which generate G determines a bijection from $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ to an algebraic subset of $(\mathrm{PSL}_2(\mathbf{C}))^k$. We give $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ the topology that makes this bijection a homeomorphism onto the algebraic set with its complex topology. This topology on $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ is independent of the choice of generators of G.

For the rest of this section, and throughout sections 2 and 3, we will assume that G is a finitely generated, non-abelian, torsion-free group.

Let $\mathcal{D}(G)$ denote the subspace of $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ which consists of those representations which are injective and have discrete image. It is a fundamental result of Jørgensen's [18] that $\mathcal{D}(G)$ is a closed subset of $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$. The proof of Jørgensen's result is based on an inequality for discrete subgroups of $\mathrm{PSL}_2(\mathbf{C})$. A second consequence of this inequality is the following lemma.

Lemma 2.1 *(Lemma 3.6 in [19]) Let (ρ_n) be a convergent sequence in $\mathcal{D}(G)$. If (g_n) is a sequence of elements of G such that $\rho_n(g_n)$ converges to the identity, then there exists n_0 such that $g_n = 1$ for $n \geq n_0$.*

2.1

A sequence of discrete subgroups (Γ_n) is said to converge *geometrically* to a Kleinian group $\widehat{\Gamma}$ if and only if

(1) for every $\gamma \in \widehat{\Gamma}$, there exist elements $\gamma_n \in \Gamma_n$ such that the sequence (γ_n) converges to γ, and

(2) whenever (Γ_{n_j}) is a subsequence of (Γ_n) and $\gamma_{n_j} \in \Gamma_{n_j}$ are elements such that the sequence (γ_{n_j}) converges to a Möbius transformation γ, we have $\gamma \in \widehat{\Gamma}$.

We call $\widehat{\Gamma}$ the *geometric limit* of (Γ_n).

The following basic fact is proved in Jørgensen-Marden [19].

Proposition 2.2 *(Proposition 3.8 in [19]) Let (ρ_n) be a sequence of elements of $\mathcal{D}(G)$ converging to ρ. Then $(\rho_n(G))$ has a geometrically convergent subsequence. If $\widehat{\Gamma}$ is the geometric limit of any such subsequence, then $\rho(G) \leq \widehat{\Gamma}$.*

The following fact will also be used.

Lemma 2.3 *Let (ρ_n) be a convergent sequence in $\mathcal{D}(G)$ such that $\rho_n(G)$ converges geometrically to $\widehat{\Gamma}$. Then $\widehat{\Gamma}$ is torsion-free.*

Proof of 2.3: Suppose that $\gamma \in \widehat{\Gamma}$ has finite order d. Let (g_n) be a sequence of elements of G such that $\rho_n(g_n)$ converges to γ. Then $\rho_n(g_n^d)$ converges to the identity. Hence by Lemma 2.1 we have $g_n^d = 1$ for large n. Since G is torsion-free, we have $g_n = 1$ for large n and therefore $\gamma = 1$.

2.3

The following lemma characterizes the intersection of two conjugates of $\rho(G)$ in the geometric limit $\widehat{\Gamma}$.

Lemma 2.4 *Let (ρ_n) be a sequence of elements of $\mathcal{D}(G)$ converging to ρ. Suppose that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$. Then $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$ is a (possibly trivial) purely parabolic group for each $\gamma \in \widehat{\Gamma} - \rho(G)$.*

Proof of 2.4: Suppose that $\rho(a)$ is a nontrivial element of $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$ for some $\gamma \in \widehat{\Gamma} - \rho(G)$. We will show that $\rho(a)$ is parabolic.

Assume to the contrary that $\rho(a)$ is loxodromic. We may write $\rho(a) = \gamma\rho(b)\gamma^{-1}$ for some $b \in G$. Choose $g_n \in G$ such that $\rho_n(g_n)$ converges to γ. We then have that $\rho_n(g_n b g_n^{-1})$ converges to $\rho(a)$, and hence that $\rho_n(a^{-1} g_n b g_n^{-1})$ converges to 1. It follows from Lemma 2.1 that there exists an integer n_0 such that $a = g_n b g_n^{-1}$ for all $n \geq n_0$. Hence $g_{n_0}^{-1} g_n$ is contained in the centralizer of b for all $n \geq n_0$. Applying ρ_n and passing to the limit, we have that $\rho(g_{n_0}^{-1})\gamma$ commutes with $\rho(b)$.

Since the Kleinian group $\widehat{\Gamma}$ is torsion-free by Lemma 2.3 and the element $\rho(b) \in \widehat{\Gamma}$ is loxodromic, the centralizer of $\rho(b)$ in $\widehat{\Gamma}$ is cyclic. Thus there are integers j and k such that $(\rho(g_{n_0}^{-1})\gamma)^j = \rho(b)^k$. A second application of Lemma 2.1 shows that for some $n_1 \geq n_0$ we have $(g_{n_0}^{-1} g_n)^j = b^k$ for all $n \geq n_1$. But since G is isomorphic to a torsion-free Kleinian group, each element of G has at most one j^{th} root. Hence b^k has a unique j^{th} root c and we have $g_{n_0}^{-1} g_n = c$ for $n \geq M$. Thus $g_n = g_{n_0} c$ for large n, so the sequence (g_n) is eventually constant. Since $\rho_n(g_n)$ converges to γ, this implies that γ is an element of $\rho(G)$, which contradicts our hypothesis that $\gamma \in \widehat{\Gamma} - \rho(G)$.

2.4

Next we consider the intersection of the limit set of $\rho(G)$ with its image under an element of the geometric limit $\widehat{\Gamma}$.

The following definition will be useful. Let Γ_1 and Γ_2 be subgroups of the Kleinian group Γ. We will say that a point $p \in \Lambda(\Gamma_1) \cap \Lambda(\Gamma_2)$ is in $\mathrm{P}(\Gamma_1, \Gamma_2)$ if and only if $\mathrm{Stab}_{\Gamma_1}(p) \cong \mathbf{Z}$, $\mathrm{Stab}_{\Gamma_2}(p) \cong \mathbf{Z}$, and $\langle \mathrm{Stab}_{\Gamma_1}(p), \mathrm{Stab}_{\Gamma_2}(p) \rangle \cong \mathbf{Z} \oplus \mathbf{Z}$. In particular, it must be that p is a parabolic fixed point of both Γ_1 and Γ_2.

We will make use of the following result, due to Soma [32] and Anderson [1], which provides the link between the intersection of the limit sets of a pair of subgroups and the limit set of the intersection of the subgroups.

Recall that a Kleinian group Γ is *topologically tame* if $\mathbf{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold.

Theorem 2.5 *Let Γ_1 and Γ_2 be nonelementary, topologically tame subgroups of the Kleinian group Γ. Then*

$$\Lambda(\Gamma_1) \cap \Lambda(\Gamma_2) = \Lambda(\Gamma_1 \cap \Gamma_2) \cup \mathrm{P}(\Gamma_1, \Gamma_2).$$

The next lemma shows that the term $\mathrm{P}(\Gamma_1, \Gamma_2)$ may be ignored in the case where Γ_1 and Γ_2 are distinct conjugates of $\rho(G)$ by elements of the geometric limit $\widehat{\Gamma}$.

Lemma 2.6 *Let (ρ_n) be a sequence in $\mathcal{D}(G)$ converging to ρ. Suppose that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$. Then for each $\gamma \in \widehat{\Gamma} - \rho(G)$, the set $\mathrm{P}(\rho(G), \gamma\rho(G)\gamma^{-1})$ is empty.*

Proof of 2.6: The argument runs along much the same line as the proof of Proposition 2.4. Suppose that $p \in \mathrm{P}(\rho(G), \gamma\rho(G)\gamma^{-1})$, that $\mathrm{Stab}_{\rho(G)}(p) \cong \mathbf{Z}$ is generated by $\rho(a)$ and that $\mathrm{Stab}_{\gamma\rho(G)\gamma^{-1}}(p) \cong \mathbf{Z}$ is generated by $\gamma\rho(b)\gamma^{-1}$. Note that each of the elements a and b generates its own centralizer.

Choose $g_n \in G$ so that $\rho_n(g_n)$ converges to γ. Since $\rho(a)$ commutes with $\gamma\rho(b)\gamma^{-1}$, we conclude as in the proof of Lemma 2.4 that there exists an integer n_0 such that a commutes with $g_n b g_n^{-1}$ for all $n \geq n_0$. Since a generates its centralizer in G, each of the elements $g_n b g_n^{-1}$ for $n \geq n_0$ must be a power of a. But these are all conjugate elements, while distinct powers of a are not conjugate. Therefore we must have that $g_n b g_n^{-1} = g_{n_0} b g_{n_0}^{-1}$ for all $n \geq n_0$.

Thus $g_{n_0}^{-1} g_n$ commutes with b for all $n \geq n_0$. Since the centralizer of b is cyclic, we may now argue exactly as in the proof of 2.4 that the sequence (g_n)

must be constant for $n \geq n_1 \geq n_0$, obtaining a contradiction to our hypothesis that $\gamma \in \widehat{\Gamma} - \rho(G)$.

$\boxed{2.6}$

As an immediate consequence of Lemma 2.4, Theorem 2.5, and Lemma 2.6, we have the following proposition.

Proposition 2.7 *Let (ρ_n) be a sequence in $\mathcal{D}(G)$ converging to ρ. Suppose that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$ and that $\rho(G)$ is topologically tame. Then for any $\gamma \in \widehat{\Gamma} - \rho(G)$ the group $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$ is purely parabolic and*

$$\Lambda(\rho(G)) \cap \gamma \cdot \Lambda(\rho(G)) = \Lambda(\rho(G) \cap \gamma\rho(G)\gamma^{-1}).$$

Hence if $\Lambda(\rho(G)) \cap \gamma \cdot \Lambda(\rho(G))$ is non-empty then it must contain only the fixed point of $\rho(G) \cap \gamma\rho(G)\gamma^{-1}$.

3 In the convex core

We continue to assume that G is a finitely generated nonabelian torsion-free group. We consider a sequence (ρ_n) in $\mathcal{D}(G)$ which converges to a representation ω which is a maximal cusp (defined below). If we assume that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$ then the hyperbolic manifold $N = \mathbf{H}^3/\omega(G)$ is a covering space of $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$. The main result of this section says that in this situation the restriction of the covering projection gives an embedding of the convex core of N into $\widehat{N}$.

Given a Kleinian group Γ, define its *convex hull* $\mathrm{CH}(\Gamma)$ in $\mathbf{H}^3$ to be the smallest nonempty convex set in $\mathbf{H}^3$ which is invariant under the action of Γ. (Thus $\mathrm{CH}(\Gamma)$ is the intersection of all half-spaces in $\mathbf{H}^3$ whose closures in the compactification $\mathbf{H}^3 \cup \overline{\mathbf{C}}$ contain $\Lambda(\Gamma)$.)

The *convex core* of $N = \mathbf{H}^3/\Gamma$ is $\mathrm{C}(N) = \mathrm{CH}(\Gamma)/\Gamma$. We say that N, or equivalently Γ, is *geometrically finite* if Γ is finitely generated and $\mathrm{C}(N)$ has finite volume.

The *injectivity radius* $\mathrm{inj}_N(x)$ of N at the point x is half the length of the shortest homotopically nontrivial closed loop passing through x. Note that injectivity radius increases under lifting to a covering space. That is, if N is a cover of $\widehat{N}$ with covering map $\pi : N \to \widehat{N}$, and x is any point of N, then $\mathrm{inj}_N(x) \geq \mathrm{inj}_{\widehat{N}}(\pi(x))$.

Given a hyperbolic 3-manifold N, define the *ϵ-thick part* of N as

$$N_{\mathrm{thick}(\epsilon)} = \{x \in N | \operatorname{inj}_N(x) \geq \frac{\epsilon}{2}\}$$

and the *ϵ-thin part* of N as

$$N_{\mathrm{thin}(\epsilon)} = \{x \in N | \operatorname{inj}_N(x) \leq \frac{\epsilon}{2}\}.$$

We recall that $\mathrm{C}(N) \cap N_{\mathrm{thick}(\epsilon)}$ is compact for all $\epsilon > 0$ if and only if N is geometrically finite (see Bowditch [7]). Hence, for a geometrically finite hyperbolic 3-manifold N, the sets $\mathrm{C}(N) \cap N_{\mathrm{thick}(1/m)}$ for $m \geq 1$ form an exhaustion of $\mathrm{C}(N)$ by compact subsets.

We say that a representation ω in $\mathcal{D}(G)$ is a *maximal cusp* if $N = \mathbf{H}^3/\omega(G)$ is geometrically finite and every component of the boundary $\partial\,\mathrm{C}(N)$ of its convex core is a thrice-punctured sphere. We will further require that $\omega(G)$ not be a Fuchsian group. (This rules out only the case that $\omega(G)$ is in the (unique) conjugacy class of finite co-area Fuchsian groups uniformizing the thrice-punctured sphere.) Maximal cusps are discussed at length by Keen, Maskit and Series in [20], where the image groups are termed maximally parabolic.

A proof of the following lemma appears in [20]; since we will be using the lemma heavily, we include a sketch of the proof here.

Lemma 3.1 *Let $\omega \in \mathcal{D}(G)$ be a maximal cusp, and let $N = \mathbf{H}^3/\omega(G)$. Then each component of $\partial\,\mathrm{C}(N)$ is totally geodesic.*

Proof of 3.1: Since the universal cover of $\mathrm{C}(N)$ is $\mathrm{CH}(\omega(G))$, it suffices to show that each component of $\partial\,\mathrm{CH}(\omega(G))$ is a totally geodesic hyperplane or, equivalently, that each component of $\Omega(\omega(G))$ is a disk bounded by a circle on the sphere at infinity.

Recall, for example from Epstein-Marden [16], that $\partial\,\mathrm{C}(N)$ is homeomorphic to $\Omega(\omega(G))/\omega(G)$. Moreover, by the Ahlfor's Finiteness theorem $\Omega(\omega(G))/\omega(G)$ has finite area. Thus each component S of $\Omega(\omega(G))/\omega(G)$ must be a thrice-punctured sphere. Write $S = \Delta/\Gamma_\Delta$, where Δ is a component of $\Omega(\omega(G))$ and Γ_Δ is the subgroup of $\omega(G)$ stabilizing Δ.

Since Δ/Γ_Δ is a thrice-punctured sphere, the group Γ_Δ must be a Fuchsian group and Δ must be a disk bounded by a circle on the sphere at infinity. (See, for example, Chapter IX.C of Maskit's book [25]).

3.1

We are now ready to prove the main result of this section. A map between locally compact spaces will be called an *embedding* if it is proper and one-to-one.

Proposition 3.2 *Let (ρ_n) be a sequence of elements of $\mathcal{D}(G)$ converging to a maximal cusp ω. Suppose that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$. Let $N = \mathbf{H}^3/\omega(G)$, $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$, and let $\pi : N \to \widehat{N}$ be the covering map. Then $\pi|_{\mathrm{C}(N)}$ is an embedding.*

Proof of 3.2: We first note that if (x_i) is a sequence in $\mathrm{C}(N)$ leaving every compact set, then $\lim_{i\mapsto\infty} \mathrm{inj}_N(x_i) = 0$. Hence, $\lim_{i\mapsto\infty} \mathrm{inj}_{\widehat{N}}(\pi(x_i)) = 0$, which implies that $(\pi(x_i))$ leaves every compact subset of $\widehat{N}$. Thus, $\pi|_{\mathrm{C}(N)}$ is a proper mapping.

It remains to show that π is injective. The universal cover of $\mathrm{C}(N)$ is $\mathrm{CH}(\omega(G))$. Thus it suffices to show that $\mathrm{CH}(\omega(G)) \cap \gamma \cdot \mathrm{CH}(\omega(G))$ is empty for each $\gamma \in \widehat{\Gamma} - \omega(G)$. For notational convenience, set $X = \mathrm{CH}(\omega(G))$.

Since ω is assumed to be a maximal cusp, each component of $\partial\,\mathrm{C}(N)$ is totally geodesic. Hence each component of ∂X is a plane H in $\mathbf{H}^3$ whose boundary at infinity is a circle C which lies in $\Lambda(\omega(G))$.

If $X \cap \gamma \cdot X$ is not empty, there are two possibilities. Either there is a point in $\partial X \cap \gamma \cdot \partial X$, or a component of ∂X lies entirely within $\gamma \cdot X$ (or vice versa).

We begin with the case that there is a point x in $\partial X \cap \gamma \cdot \partial X$. There then exist a plane H in ∂X and a plane H' in $\gamma \cdot \partial X$ with $x \in H \cap H'$. If two planes in $\mathbf{H}^3$ meet, either their intersection is a line or they are equal. If we let C denote the boundary at infinity of H and C' the boundary at infinity of H', then either $C \cap C'$ contains exactly two points or $C = C'$. However, $C \cap C'$ is contained in $\Lambda(\omega(G)) \cap \gamma \cdot \Lambda(\omega(G))$, so that $\Lambda(\omega(G)) \cap \gamma \cdot \Lambda(\omega(G))$ contains at least two points, which contradicts Proposition 2.7.

The second possibility is that a component H of ∂X lies entirely within $\gamma \cdot X$. In this case, the boundary at infinity C of H lies in the boundary at infinity of $\gamma \cdot X$, which is exactly $\gamma \cdot \Lambda(\omega(G))$. However, C also lies in $\Lambda(\omega(G))$; hence $\Lambda(\omega(G)) \cap \gamma \cdot (\Lambda(\omega(G)))$ contains C. This also contradicts Proposition 2.7.

3.2

Remark 3.3 The conclusion of 3.2 holds, by the same argument, whenever N is geometrically finite and $\partial C(N)$ is totally geodesic. Anderson and Canary [2] use techniques similar to those developed in this section to undertake a more general study of the relationship between algebraic and geometric limits.

4 Near a maximal cusp

In this section, we will prove that if a representation in $\mathcal{D}(G)$ is near enough to a maximal cusp, then its associated hyperbolic 3-manifold contains a nearly isometric copy of an ϵ-truncated convex core of the maximal cusp. We first define this ϵ-truncated object and describe some of its useful attributes.

We recall that it follows from the Margulis lemma that there exists a constant λ_0, such that if $\epsilon < \lambda_0$ and N is a hyperbolic 3-manifold, then every component P of $N_{\mathrm{thin}(\epsilon)}$ is either a solid torus neighborhood of a closed geodesic, or the quotient of a horoball H by a group Θ of parabolic elements fixing H (see for example [4]). In the second case, Θ is isomorphic either to $\mathbf{Z}$ or to $\mathbf{Z} \oplus \mathbf{Z}$. Moreover, H is precisely invariant under $\Theta < \Gamma$, by which we mean that if $\gamma \in \Gamma$ and $\gamma \cdot H \cap H \neq \emptyset$, then $\gamma \in \Theta$ and $\gamma \cdot H = H$. If $\Theta \cong \mathbf{Z}$, we call P a *rank-one cusp*, and if $\Theta \cong \mathbf{Z} \oplus \mathbf{Z}$, we call P a *rank-two cusp.* Recall also that there exists $L(\epsilon) > 0$, such that any two components of $N_{\mathrm{thin}(\epsilon)}$ are separated by a distance of at least $L(\epsilon)$.

The next lemma gives the structure of $N_{\mathrm{thin}(\epsilon)}$ for sufficiently small ϵ. In fact the lemma holds for all geometrically finite hyperbolic 3-manifolds. We restrict to the setting of maximal cusps purely for ease of exposition. (The extra technical difficulties that arise in extending the lemma to the geometrically finite case may be handled using the techniques developed in Bowditch [7].)

Lemma 4.1 *Let $\omega \in \mathcal{D}(G)$ be a maximal cusp. Set $\Gamma = \omega(G)$ and $N = \mathbf{H}^3/\Gamma$. There exists $\delta(N) < \lambda_0$, such that if $\epsilon \leq \delta(N)$ and P is a component of $N_{\mathrm{thin}(\epsilon)}$, then*

- *(i) P is non-compact,*
- *(ii) ∂P meets $\mathrm{C}(N)$ orthogonally along each component of their intersection.*
- *(iii) $E = \partial P \cap \mathrm{C}(N)$ is a Euclidean surface with geodesic boundary, and* $\operatorname{diam} E \leq 1$*;*

(iv) if P is a rank-one cusp then E is an annulus, and if P is a rank-two cusp then E is a torus; and

(v) $\mathrm{C}(N) \cap P$ is homeomorphic to $E \times [0, \infty)$.

In particular, for any $\epsilon \leq \delta(N)$ the set $N_{\mathrm{thick}(\epsilon)} \cap \mathrm{C}(N)$ is a 3-manifold with piecewise smooth boundary.

Proof of 4.1: Consider any positive number $\epsilon < \lambda_0$. We notice that any compact component Q of $N_{\mathrm{thin}(\epsilon)}$ intersects $\mathrm{C}(N)$, since they both must contain the closed geodesic which is the core of Q. Moreover, $\mathrm{C}(N) \cap \partial Q$ is non-empty since otherwise we would have $\mathrm{C}(N) \subset Q$ (which would imply that N is elementary.) Since N is geometrically finite, $\mathrm{C}(N) \cap N_{\mathrm{thick}(\epsilon)}$ is compact. So we see immediately that there are only finitely many compact components of $N_{\mathrm{thin}(\epsilon)}$. Thus we may choose $\delta_0 < \lambda_0$ such that each component of $N_{\mathrm{thin}(\delta_0)}$ is non-compact.

Let $P_1, \ldots, P_k$ be the components of $N_{\mathrm{thin}(\delta_0)}$. Each P_i is non-compact and is the quotient of a horoball H_i in $\mathbf{H}^3$ by a free abelian group Θ_i of parabolic isometries of $\mathbf{H}^3$ under which H_i is precisely invariant. It suffices to show that for each i there exists $\delta_i \leq \delta_0$ such that for $0 < \epsilon < \delta_i$ the component of $N_{\mathrm{thin}(\epsilon)}$ contained in P_i has the properties listed in the statement. We may then take $\delta(N) = \inf\{\delta_1, \ldots, \delta_k\}$.

Let i be given and set $P = P_i$, $H = H_i$ and $\Theta = \Theta_i$. We will show how to construct $\delta = \delta_i$ as above. We will work in an upper half-space model of $\mathbf{H}^3$ chosen so that H is based at ∞. We may also assume that our model has been chosen so that $\theta_1 \colon z \mapsto z + 1$ is a primitive element of Θ. Let us write $H(T)$ for the horoball $\{(z, t) \in \mathbf{H}^3 | t > T\}$. For some choice of T_0 we have $H = H(T_0)$, and for any $T \geq T_0$ the horoball $H(T)$ is precisely invariant under $\Theta < \Gamma$.

First suppose that Θ has rank one. Then, since N is geometrically finite, there exist two components D_1 and D_2 of $\Omega(\Gamma)$ such that $\infty \in \partial D_1 \cap \partial D_2$. (See, for example, Corollary VI.C.3 in [25].) We know that each D_j is bounded by a circle or a line. Since $\Omega(\Gamma)$ is invariant by θ_1, the boundaries of the D_j must in fact be horizontal lines. Thus each D_j is a half-plane in $\mathbf{C}$ bounded by a line $y = r_j$. After re-indexing we may assume $r_1 > r_2$. If $T > T_0$ then the region

$$U(T) = \mathrm{CH}(\Gamma) \cap H(T) = \{(z, t) | r_1 \geq \mathrm{Im}\, z \geq r_2, t \geq T\}$$

is precisely invariant under $\Theta < \Gamma$. Note that $\partial H(T)$ meets exactly two components of $\partial\,\mathrm{C}(N)$, which are half-planes given by the equations $y = r_1$ and $y = r_2$ and are therefore orthogonal to $\partial H(T)$. Let $E(T)$ be the quotient of $U(T) \cap \partial H(T)$ by Θ and let ds_T^2 be its intrinsic metric. Notice that $E(T)$ is a Euclidean annulus of diameter less than $\frac{|r_1 - r_2| + 1}{T}$. The quotient $Q(T) = U(T)/\Theta$ is isometric to $E(T) \times [0, \infty)$ with the metric $ds^2 = e^{-2t} ds_T^2 + dt^2$. Moreover, for every $\delta \leq \delta_0$ we have $(P \cap N_{\mathrm{thin}(\delta)}) \cap \mathrm{C}(N) = Q(T)$ for some $T \geq T_0$. Note that T increases monotonically to ∞ as $\delta \to 0$. Thus, we may choose $\delta \leq \delta_0$, such that for any $\epsilon < \delta$, the component of $\mathrm{C}(N) \cap N_{\mathrm{thin}(\epsilon)}$ contained in P has the form $Q(T)$ for some $T \geq \max(T_0, |r_1 - r_2| + 1)$. The Euclidean surface $E(T)$ will then have diameter less than 1.

Now suppose that Θ has rank 2 and let $\theta_\omega\colon z \mapsto z + \omega$ be a complementary generator of Θ. Since $\Omega(\Gamma)$ is invariant under Θ, the components of $\Omega(\Gamma)$ are bounded by circles which do not contain ∞. Moreover, since $\theta_1 \in \Theta$ these circles cannot have diameter bigger than 1. Therefore, $H(T) \subset \mathrm{CH}(\Gamma)$ for all $T \geq 1$. So, if $T \geq \max\{1, T_0\}$, then the quotient $Q(T)$ of $\{(z,t) \in \mathbf{H}^3 | t \geq T\}$ is isometric to $E(T) \times [0, \infty)$ with the metric $ds^2 = e^{-2t} ds_T^2 + dt^2$, where $E(T)$ is the quotient of $\partial H(T)$ by Θ. Hence, $E(T)$ is a Euclidean torus with diameter less than $\frac{1+|\omega|}{T}$. As before, we may choose $\delta \leq \delta_0$ such that if $\epsilon < \delta$, then the component of $N_{\mathrm{thin}(\epsilon)}$ contained in P has the form $Q(T)$ for some $T > \max\{T_0, 1 + |\omega|\}$. The boundary $E(T)$ of $Q(T)$ then has diameter less than 1.

4.1

Let $\omega \in \mathcal{D}(G)$ be a maximal cusp and $N = \mathbf{H}^3/\omega(G)$. If $\epsilon > 0$, then we define $\eta(N, \epsilon)$ to be $\min\{\epsilon, \delta(N)\}$. The ϵ-truncated convex core of N is defined to be

$$D_\epsilon(N) = \mathrm{C}(N) \cap N_{\mathrm{thick}(\eta(N,\epsilon))}.$$

Recall that a compact submanifold of N is said to be a *compact core* for N if the inclusion map is a homotopy equivalence.

Lemma 4.2 *Suppose that $\omega \in \mathcal{D}(G)$ is a maximal cusp and that $\epsilon > 0$. Then $D_\epsilon(N)$ is a compact core for $N = \mathbf{H}^3/\omega(G)$.*

Proof of 4.2: First recall that the inclusion of $\mathrm{C}(N)$ into N is a homotopy equivalence. Moreover, each component of $\mathrm{C}(N) - D_\epsilon(N)$ is homeomorphic to

$E \times (0, \infty)$ for some Euclidean surface E. Thus, the inclusion of $D_\epsilon(N)$ into $\mathrm{C}(N)$ is a homotopy equivalence. The result follows.

4.2

For any subset X of a hyperbolic manifold N we will denote by $\mathcal{N}_r(X)$ the closed neighborhood of radius r of X. In the case $N = \mathbf{H}^3/\omega(G)$, where ω is a maximal cusp, Proposition 4.4 will provide bounds for both the area of $\partial D_\epsilon(N)$ and the volume of $\mathcal{N}_2(D_\epsilon(N))$. These bounds will depend only on the topological type of N and not on ϵ. We first recall the following special case of lemma 8.2 in [8] (see also Proposition 8.12.1 of Thurston [34]).

Lemma 4.3 *There is a constant $\kappa > 0$, such that for any maximal cusp $\omega \in \mathcal{D}(G)$ and any collection S of components of the boundary of the convex core of $N = \mathbf{H}^3/\omega(G)$, the neighborhood $\mathcal{N}_3(S)$ has volume less than $\kappa\,\mathrm{area}\,S$.*

4.3

If $N = \mathbf{H}^3/\omega(G)$, where $\omega(G)$ is a maximal cusp, we will denote by $\sigma(N)$ the number of components of $\partial\,\mathrm{C}(N)$, and by $\tau(N)$ the number of rank-two cusps of N. We set

$$\alpha(N) = \frac{7}{2}\pi\sigma(N) + 2\pi\tau(N)$$

and

$$\beta(N) = 2\pi\kappa\sigma(N) + \pi e^4\tau(N),$$

where κ is the constant given by Lemma 4.3.

Lemma 4.4 *Let ω be a maximal cusp and let $N = \mathbf{H}^3/\omega(G)$. Then*

(i) $\mathrm{area}\,\partial D_\epsilon(N) \leq \alpha(N)$, *and*

(ii) $\mathrm{vol}(\mathcal{N}_2(\partial D_\epsilon(N))) \leq \beta(N)$.

Proof of 4.4: Notice that $\partial D_\epsilon(N) = S \cup E$ where $S = \partial\,\mathrm{C}(N) \cap N_{\mathrm{thick}(\eta(N,\epsilon))}$ and $E = \mathrm{C}(N) \cap \partial N_{\mathrm{thick}(\eta(N,\epsilon))}$. Since $S \subset \partial\,\mathrm{C}(N)$ and since each component of $\partial\,\mathrm{C}(N)$ is a thrice-punctured sphere, we have $\mathrm{area}\,S \leq \mathrm{area}\,\partial\,\mathrm{C}(N) = 2\pi\sigma(N)$. By Lemma 4.1, each component of E is a Euclidean manifold of diameter at most 1, so each component of E has area at most π. Since each component of

$\partial\,\mathrm{C}(N)$ contains three components of ∂E there are $\frac{3}{2}\sigma(N)$ annular components of E. Moreover, there are $\tau(N)$ toroidal components of E. The first assertion follows.

Let $\widehat{S}$ denote the union of S with the annular components of E. Since each annular component of E has diameter less than 1,

$$\mathcal{N}_2(\widehat{S}) \subset \mathcal{N}_3(S) \subset \mathcal{N}_3(\partial\,\mathrm{C}(N)).$$

Thus Lemma 4.3 guarantees that

$$\mathrm{vol}\,\mathcal{N}_2(\widehat{S}) \leq \kappa\,\mathrm{area}(\partial\,\mathrm{C}(N)) = 2\pi\kappa\sigma(N).$$

Now, if T is a toroidal component of E, then $\mathcal{N}_2(T)$ is (the quotient of) a region isometric to $T \times (-2, 2)$ with the metric $ds^2 = e^{-2t}ds_T^2 + dt^2$, which has volume less than $2\pi e^4$. The second assertion now follows.

4.4

Our next result, Proposition 4.5, asserts (among other things) that the hyperbolic manifold associated to a representation which is near enough to a maximal cusp contains a biLipschitz copy of the ϵ-truncated convex core of the manifold associated to the maximal cusp.

We first outline the argument. Suppose that (ρ_n) is a sequence of representations in $\mathcal{D}(G)$ which converges to a maximal cusp ω and that the groups $\rho_n(G)$ converge geometrically to $\widehat{\Gamma}$. Let $N = \mathbf{H}^3/\omega(G)$ and $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$. If $\pi : N \to \widehat{N}$ is the covering map, Proposition 3.2 implies that $\pi|_{D_\epsilon(N)}$ is an embedding. Since $\rho_n(G)$ converges geometrically to $\widehat{\Gamma}$, larger and larger chunks of $\widehat{N}$ are nearly isometric to larger and larger chunks of $N_n = \mathbf{H}^3/\rho_n(G)$. In particular, for all large enough n there exists a 2-biLipschitz embedding $f_n : V_n \to N_n$ where $\pi(D_\epsilon(N)) \subset V_n \subset \widehat{N}$. The desired biLipschitz copy of $D_\epsilon(N)$ is $f_n(\pi(D_\epsilon(N)))$.

In order to carry out the program outlined above, it will be necessary to make consistent choices of base points in different hyperbolic 3-manifolds. We will use the following convention. If z is a point in $\mathbf{H}^3$ and Γ is a Kleinian group, we will let z_Γ denote the image of z in the hyperbolic manifold $\mathbf{H}^3/\Gamma$. In the case that $\Gamma = \rho(G)$ for some representation $\rho \in \mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C}))$ we will write $z_\rho = z_{\rho(G)}$.

If a codimension-0 submanifold X of a hyperbolic manifold N is connected and has piecewise smooth boundary, then it has two natural distance functions.

In the *extrinsic metric* the distance between two points of X is equal to their distance in N, while in the *intrinsic metric* the distance is the infimum of the lengths of rectifiable paths in X joining the two points. Observe that if X and Y are submanifolds of hyperbolic manifolds and if $f\colon X \to Y$ is a K-biLipschitz map with respect to the extrinsic metrics, then f is also K-biLipschitz with respect to the intrinsic metrics.

Proposition 4.5 *Suppose that $\omega \in \mathcal{D}(G)$ is a maximal cusp and set $N = \mathbf{H}^3/\omega(G)$. Let $\epsilon > 0$ be given and let z be a point of $\mathbf{H}^3$ such that z_ω lies in the interior of $D_\epsilon(N)$. Then there is a neighborhood $U(\epsilon, z, \omega)$ of ω in $\mathcal{D}(G)$ such that for each $\rho \in U(\epsilon, z, \omega)$, there exists a map $\phi : D_\epsilon(N) \to N' = \mathbf{H}^3/\rho(G)$, with the following properties:*

(1) ϕ maps $D_\epsilon(N)$ homeomorphically onto a manifold with piecewise smooth boundary, and is 2-biLipschitz with respect to the intrinsic metrics on $D_\epsilon(N)$ and $\phi(D_\epsilon(N))$,

(2) $\phi(z_\omega) = z_\rho$,

(3) $\operatorname{vol}\mathcal{N}_1(\partial(\phi(D_\epsilon(N)))) \leq 8\beta(N)$, and

(4) $\phi(N_{\text{thin}(\delta)} \cap D_\epsilon(N)) \subset N'_{\text{thin}(2\delta)}$ for any $\delta < \frac{\lambda_0}{2}$.

Proof of 4.5: Let (ρ_n) be a sequence in $\mathcal{D}(G)$ that converges to ω, and set $N_n = \mathbf{H}^3/\rho_n(G)$. It suffices to prove that (ρ_n) has a subsequence (ρ_{n_i}) such that there exist maps $\phi_i : D_\epsilon(N) \to N_{n_i}$ which have properties (1)—(4).

Given any sequence $\rho_n \in \mathcal{D}(G)$ converging to ω, Proposition 2.2 guarantees that there exists a subsequence $(\rho_{n_i}(G))$ of $(\rho_n(G))$ which converges geometrically to a group $\widehat{\Gamma}$ such that $\omega(G) \subset \widehat{\Gamma}$. Let $\widehat{N} = \mathbf{H}^3/\widehat{\Gamma}$ and let $\pi : N \to \widehat{N}$ be the associated covering map. Proposition 3.2 guarantees that $\pi|_{C(N)}$ is an embedding and hence that $\pi|_{D_\epsilon(N)}$ is an embedding. Let $D = \pi(D_\epsilon(N))$. Since π is a local isometry we find using Lemma 4.4 that

$$\operatorname{vol}\mathcal{N}_2(\partial D) \leq \operatorname{vol}\mathcal{N}_2(\partial D_\epsilon(N)) \leq \beta(N).$$

Since $(\rho_{n_i}(G))$ converges geometrically to $\widehat{\Gamma}$, it follows from Corollary 3.2.11 in [10] or Theorem E.1.13 in [4] that there exist smooth submanifolds $V_i \subset \widehat{N}$, numbers r_i and α_i, and maps $f_i : V_i \to N_{n_i}$ such that

(i) V_i contains $B(r_i, z_{\widehat{\Gamma}})$, the closed radius-$r_i$ neighborhood of $z_{\widehat{\Gamma}}$,

(ii) $f_i(z_{\widehat{\Gamma}}) = z_{\rho_{n_i}}$,

(iii) r_i converges to ∞, and α_i converges to 1,

(iv) f_i maps V_i diffeomorphically onto $f(V_i)$ and is α_i-biLipschitz with respect to the extrinsic metrics on V_i and $f(V_i)$.

Choose d so that $D \subset B(d, z_{\widehat{\Gamma}})$. Set $\mu = \max\{1, \lambda_0/2\}$. We may assume that the subsequence (ρ_{n_i}) has been chosen so that $\alpha_i < 2$ and $r_i > d + 2\mu$ for all i. This condition on r_i implies that $\mathcal{N}_{2\mu}(D)$ is contained in the interior of V_i.

We claim that $\mathcal{N}_\mu f_i(D) \subset f_i(V_i)$. To prove the claim, we consider the frontier X of $\mathcal{N}_{2\mu}(D)$ in $\widehat{N}$ and the frontier Y_i of $f_i(\mathcal{N}_{2\mu}(D))$ in N_{n_i}. Since f_i is a homeomorphism onto its image it follows from invariance of domain that $f_i(X) = Y_i$. Since f_i^{-1} is extrinsically α_i-Lipschitz with $\alpha_i < 2$, and since every point of X has distance 2μ from D, every point of $f_i(X) = Y_i$ must be a distance greater than μ from $f_i(D)$. Thus Y_i is disjoint from $\mathcal{N}_\mu(f_i(D))$. Since $f_i(\mathcal{N}_{2\mu}(D))$ contains $f_i(D)$ and is disjoint from the frontier Y_i of $\mathcal{N}_\mu(f_i(D))$ we have $f_i(\mathcal{N}_{2\mu}(D)) \supset \mathcal{N}_\mu(f_i(D))$, and the claim follows.

In particular, $\mathcal{N}_1(f_i(\partial D)) \subset f_i(V_i)$. Again using that f_i^{-1} is extrinsically 2-Lipschitz we conclude that $\mathcal{N}_1(f_i(\partial D)) \subset f_i(\mathcal{N}_2(\partial D))$. On the other hand, since f_i is extrinsically 2-Lipschitz, it is intrinsically 2-Lipschitz and can therefore increase volume by at most a factor of 8. Therefore

$$\operatorname{vol} \mathcal{N}_1(\partial f_i(D)) \le 8 \operatorname{vol} \mathcal{N}_2(\partial D) \le 8\beta(N). \tag{1}$$

We now define $\phi_i\colon D_\epsilon(N) \to N_{n_i}$ to be $f_i \circ \pi$. We will complete the proof of the proposition by showing that ϕ_i has properties (1)—(4).

Since π is a local isometry and $\pi|_{D_\epsilon(N)}$ is an embedding, the map $\pi|_{D_\epsilon(N)}$ is an isometry between $D_\epsilon(N)$ and $\pi(D_\epsilon(N))$ with respect to their intrinsic metrics. Since $f_i\colon D \to f_i(D)$ is an extrinsically (and hence intrinsically) 2-biLipschitz homeomorphism, it follows that ϕ_i has property (1).

We have $\phi_i(z_\omega) = f_i(\pi(z_\omega)) = f_i(z_{\widehat{\Gamma}}) = z_{\rho_{n_i}}$. This is property (2). Property (3) follows from equation (1) since $\phi_i(D_\epsilon(N)) = f_i(D)$.

It remains to check property (4). Suppose that $x \in N_{\mathrm{thin}(\delta)} \cap D_\epsilon(N)$ and $\delta < \frac{\lambda_0}{2}$. Then there exists a homotopically non-trivial loop C (in N) based at x and having length at most δ. Notice that $\phi_i(C)$ has length at most 2δ. Hence $\phi_i(x)$ must lie in the 2δ-thin part of N_{n_i} unless $\phi_i(C)$ is homotopically trivial. But since $\phi_i(C)$ has length at most 2δ, it is contained in the closed δ-neighborhood of $\phi_i(x)$ in N_{n_i}. Thus if $\phi_i(C)$ were homotopically trivial in N, it would lift to a loop in a ball of radius δ in $\mathbf{H}^3$ whose center projects to $\phi_i(x)$. Hence $\phi_i(C)$ would be null-homotopic in $B(\delta, \phi_i(x)) \subset \mathcal{N}_\delta(\phi_i(D)) \subset \mathcal{N}_\mu(\phi_i(D)) \subset f_i(V_i)$. This would imply that $f_i^{-1}(\phi_i(C)) = \pi(C)$ is homotopically trivial in $\widehat{N}$, in contradiction to the fact that π is a covering map. Thus, ϕ_i has property (4), and the proof is complete.

4.5

Remark 4.6 Since Lemma 4.1 goes through for general geometrically finite hyperbolic 3-manifolds, we could have defined $D_\epsilon(N)$ in this generality. Lemmas 4.3 and 4.4 have analogues for general geometrically finite hyperbolic 3-manifolds, but the constants would also depend on the minimal length of a compressible curve in $\partial\,\mathrm{C}(N)$. Proposition 4.5 remain true, by similar arguments, whenever N is geometrically finite and every component of $\partial\,\mathrm{C}(N)$ is totally geodesic. One may use arguments similar to those in [11] to prove that, in a sufficiently small neighborhood of ω, the set $\phi(D_\epsilon(N))$ is a compact core for N.

5 All over the boundary of Schottky space

We now restrict to the case where G is the free group F_k on k generators, where $k \geq 2$. We set $\mathcal{D}_k = \mathcal{D}(F_k)$. Recall that $\mathcal{D}_k$ is a closed subset of $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$. Let $\mathcal{CC}_k$ denote the subset of $\mathcal{D}_k$ consisting of representations which are *convex-cocompact*, i.e. are geometrically finite and have purely loxodromic image. Moreover, let $\mathcal{B}_k = \overline{\mathcal{CC}_k} - \mathcal{CC}_k \subset \mathcal{D}_k$. It is known (see Marden [22]) that $\mathcal{CC}_k$ is an open subset of $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$. (The quotient of $\mathcal{CC}_k$ under the action of $\mathrm{PSL}_2(\mathbf{C})$ is often called Schottky space.)

Let $\mathcal{M}_k$ denote the set of maximal cusps in $\mathcal{D}_k$. It is a theorem of Maskit's [26] that $\mathcal{M}_k \subset \mathcal{B}_k$. McMullen has further proved that $\mathcal{M}_k$ is a dense subset of $\mathcal{B}_k$. This result, though not written down, is in the spirit of McMullen's earlier result [27] that maximal cusps are dense in the boundary of any Bers slice of quasi-Fuchsian space.

The main result of this section, Theorem 5.2, asserts that there is a dense G_δ-set of purely loxodromic, analytically tame representations in $\mathcal{B}_k$. This theorem generalizes and provides an alternate proof of Theorem 8.2 in [13].

The proof of Theorem 5.2 makes use of Proposition 4.5 and McMullen's theorem. We use Proposition 4.5 to show that if $\rho \in \mathcal{B}_k$ is well-approximated by an infinite sequence of maximal cusps, then it can be exhausted by nearly isometric copies of the truncated convex cores of the maximal cusps; this implies that ρ is analytically tame. McMullen's theorem guarantees that the G_δ-set of points in $\mathcal{B}_k$ which are well-approximated by an infinite sequence of maximal cusps is dense.

We now recall the definition of an analytically tame hyperbolic 3-manifold.

Definition: *A hyperbolic 3-manifold N with finitely generated fundamental group is* analytically tame *if* $\mathrm{C}(N)$ *may be exhausted by a sequence of compact submanifolds* $\{M_i\}$ *with piecewise smooth boundary such that*

(1) $M_i \subset \mathring{M}_j$ *if* $i < j$, *where* $\mathring{M}_j$ *denotes the interior of* M_j *considered as a subset of* $\mathrm{C}(N)$,

(2) $\cup \mathring{M}_i = \mathrm{C}(N)$,

(3) there exists a number $K > 0$ *such that the boundary* ∂M_i *of* M_i *has area at most* K *for all* i, *and*

(4) there exists a number $L > 0$ *such that* $\mathcal{N}_1(\partial M_i)$ *has volume at most* L *for every* i.

While the definition of analytic tameness is geometric in nature, it does have important analytic consequences. In particular, for an analytically tame group Γ one can control the behavior of positive Γ-invariant superharmonic functions on $\mathbf{H}^3$. Specifically, we will make extensive use of the following result, which is contained in Corollary 9.2 of [8].

Proposition 5.1 *If* $N = \mathbf{H}^3/\Gamma$ *is analytically tame and* $\Lambda(\Gamma) = \overline{\mathbf{C}}$ *then all positive superharmonic functions on* N *are constant.*

$\boxed{5.1}$

We are now in a position to state Theorem 5.2.

Theorem 5.2 *For all* $k \geq 2$, *there exists a dense* G_δ*-set* $\mathcal{C}_k$ *in* $\mathcal{B}_k$, *which consists entirely of analytically tame Kleinian groups whose limit set is the entire sphere at infinity.*

The proof of 5.2 involves the following three lemmas. The first follows from the argument used in [27] to prove the corresponding statement for the boundary of a Bers slice.

Lemma 5.3 *The set* $\mathcal{U}_k$ *of purely loxodromic representations in* $\mathcal{B}_k$ *is a dense* G_δ*-set in* $\mathcal{B}_k$*.*

5.3

The second lemma is an adaptation of Bonahon's bounded diameter lemma [5].

Lemma 5.4 *For every* $\delta > 0$*, there is a number* $c_k(\delta)$ *with the following property. Let* $\epsilon > 0$ *be given, let* ω *be any maximal cusp in* $\mathcal{D}_k$*, and set* $N = \mathbf{H}^3/\omega(F_k)$*. Then any two points in* $\partial D_\epsilon(N)$ *may be joined by a path* β *in* $\partial D_\epsilon(N)$ *such that* $\beta \cap N_{\text{thick}(\delta)}$ *has length at most* $c_k(\delta)$*.*

Proof of 5.4: Let $\delta > 0$ be given. In order to define $c_k(\delta)$ we consider a hyperbolic 2-manifold P which is homeomorphic to a thrice-punctured sphere; there is only one such hyperbolic 2-manifold up to isometry. Since $P_{\text{thick}(\delta)}$ is a compact subset of the metric space P, it has a finite diameter $d(\delta)$. It is clear that any two points in P may be joined by a path β such that $\beta \cap P_{\text{thick}(\delta)}$ has length at most $d(\delta)$. We set $c_k(\delta) = (2k-2)d(\delta) + 3k - 3$.

Now let ω be any maximal cusp in $\mathcal{D}_k$, and set $N = \mathbf{H}^3/\omega(F_k)$. We consider an arbitrary component S of $\partial\, \mathrm{C}(N)$. Then S is a totally geodesic thrice-punctured sphere. Hence S, with its intrinsic metric, is isometric to P. Furthermore, the inclusion homomorphism $\pi_1(S) \to \pi_1(N)$ is injective, and hence $S_{\text{thin}(\delta)} \subset N_{\text{thin}(\delta)}$. It follows that any two points in S may be joined by a path β in S such that $\beta \cap N_{\text{thick}(\delta)}$ has length at most $d(\delta)$.

Now, since $D_\epsilon(N)$ is a compact core for N and $\pi_1(N)$ is a free group of rank k, we see that $D_\epsilon(N)$ is a handlebody of genus k. In particular, there are exactly $2k-2$ components of $\partial\, \mathrm{C}(N)$ and exactly $3k-3$ annular components of $\partial D_\epsilon(N) - \partial\, \mathrm{C}(N)$. Also recall, from Lemma 4.1 that each component of $\partial D_\epsilon(N) - \partial\, \mathrm{C}(N)$ has diameter at most 1. Thus, since $\partial D_\epsilon(N)$ is connected, we see that any two points in $\partial D_\epsilon(N)$ may be joined by a path β such that $\beta \cap N_{\text{thick}(\delta)}$ has length at most $(2k-2)d(\delta) + 3k - 3 = c_k(\delta)$.

5.4

Lemma 5.5 *For each point $z \in \mathbf{H}^3$, let $\mathcal{X}_k(z)$ denote the set of all representations $\rho \in \mathcal{B}_k$ such that z_ρ lies in the interior of* $\mathrm{C}(\mathbf{H}^3/\rho(F_k))$ *relative to* $\mathbf{H}^3/\rho(F_k)$. *Then each $\mathcal{X}_k(z)$ is an open subset of $\mathcal{B}_k$ and $\bigcup_{z\in\mathbf{H}^3}\mathcal{X}_k(z)$ is dense in $\mathcal{B}_k$.*

Proof of 5.5: Given a point $z \in \mathbf{H}^3$ and a representation $\rho \in \mathcal{X}_k(z)$, we have that z lies in the interior of $\mathrm{CH}(\rho(F_k))$. Hence z lies in the interior of some ideal tetrahedron T with vertices in $\Lambda(\rho(F_k))$. Since the fixed points of elements of $\rho(F_k)$ are dense in $\Lambda(\rho(F_k))$, we may assume that the vertices of T are attracting fixed points of elements $\rho(g_1), \ldots, \rho(g_4)$ of $\rho(F_k)$. It follows that for any $\rho' \in \mathcal{B}_k$ sufficiently close to ρ, the attracting fixed points of $\rho'(g_1), \ldots, \rho'(g_4)$ span a tetrahedron having z as an interior point. Hence z lies in the interior of $\mathrm{CH}(\rho'(F_k))$. This shows that $\mathcal{X}_k(z)$ is an open subset of $\mathcal{B}_k$ for every $z \in \mathbf{H}^3$.

To show that the $\bigcup_{x\in\mathbf{H}^3}\mathcal{X}_k(z)$ is dense, it suffices by Lemma 5.3 to show that it contains $\mathcal{U}_k$. Consider an arbitrary representation $\rho \in \mathcal{U}_k$ and set $N = \mathbf{H}^3/\rho(F_k)$. If $\mathrm{C}(N)$ has empty interior then $\Lambda(\rho(F_k))$ must be contained in a circle and hence $\rho(F_k)$ has a Fuchsian subgroup of index at most 2. Thus $\rho(F_k)$ is geometrically finite and, since ρ is purely loxodromic, we have $\rho \in \mathcal{CC}_k$. This contradiction shows that $\mathrm{C}(N)$ has non-empty interior. If z is any interior point of $\mathrm{C}(N)$, we have $\rho \in \mathcal{X}_k(z)$.

5.5

Proof of 5.2: In view of Lemma 5.5 it suffices to show that for every $z \in \mathbf{H}^3$ there is a dense G_δ-set $\mathcal{C}_k(z)$ in $\mathcal{X}_k(z)$ which consists entirely of analytically tame Kleinian groups whose limit set is the entire sphere at infinity.

We set $\mathcal{M}_k(z) = \mathcal{M}_k \cap \mathcal{X}_k(z)$. By McMullen's result, $\mathcal{M}_k(z)$ is a dense subset of $\mathcal{X}_k(z)$. For each $\omega \in \mathcal{M}_k(z)$ and each $\epsilon > 0$ we will define a neighborhood $V(\epsilon, z, \omega)$ of ω in $\mathcal{X}_k(z)$.

Set $N_\omega = \mathbf{H}^3/\omega(F_k)$. By the definition of $\mathcal{M}_k(z)$, we have $z_\omega \in \mathrm{C}(N_\omega)$. Hence either $z_\omega \in D_\epsilon(N_\omega)$ or z_ω lies in the interior of the ϵ-thin part of N_ω. If $z_\omega \in D_\epsilon(N_\omega)$ then we set $V(\epsilon, z, \omega) = U(\epsilon, z, \omega) \cap \mathcal{X}_k(z)$, where $U(\epsilon, z, \omega)$ is the open set given by Proposition 4.5. If z_ω lies in the ϵ-thin part of N_ω, we take $V = V(\epsilon, z, \omega)$ to be a neighborhood of ω in $\mathcal{X}_k(z)$ such that for every $\rho \in V$ the point z_ρ lies in the interior of the ϵ-thin part of $\mathbf{H}^3/\rho(F_k)$. (Such a neighborhood exists because there is an element $g \in F_k$ such that $\mathrm{dist}(z, \omega(g)\cdot z) < \epsilon$. For any ρ sufficiently close to ω we have $\mathrm{dist}(z, \rho(g)\cdot z) < \epsilon$.)

We now set $W_k(\epsilon, z) = \cup_{\omega \in \mathcal{M}_k(z)} V(\epsilon, z, \omega)$. Since $\mathcal{M}_k(z)$ is dense in $\mathcal{X}_k(z)$, the set $W_k(\epsilon, z)$ is an open dense subset of $\mathcal{X}_k(z)$ for every $\epsilon > 0$. Since $\mathcal{U}_k$ is a dense G_δ-set in $\mathcal{B}_k$ it follows that

$$\mathcal{C}_k(z) = \mathcal{U}_k \cap \bigcap_{m \in \mathbf{Z}_+} W_k(\frac{1}{m}, z)$$

is a dense G_δ-set in $\mathcal{X}_k(z)$. In order to complete the proof, we need only to show that each element of $\mathcal{C}_k(z)$ is analytically tame and has the entire sphere as its limit set.

Let $\rho : F_k \to \mathrm{PSL}_2(\mathbf{C})$ be a representation in $\mathcal{C}_k(z)$. Set $N = \mathbf{H}^3/\rho(F_k)$. We first observe that since $\rho(F_k)$ is free and purely loxodromic and is not a Schottky group, we have $\Lambda(\rho(F_k)) = \overline{\mathbf{C}}$ (see Maskit [24].) In particular, $\mathrm{C}(N) = N$.

By the definition of $\mathcal{C}_k(z)$, for every $m \in \mathbf{Z}_+$ there exists a maximal cusp $\omega_m : F_k \to \mathrm{PSL}_2(\mathbf{C})$ such that $\rho \in V(\frac{1}{m}, z, \omega_m)$. Let m_0 be a positive integer such that $\frac{1}{2m_0}$ is less than the injectivity radius of N at z_ρ. In what follows we consider an arbitrary integer $m \geq m_0$.

By the definition of m_0 the point z_ρ lies in the $\frac{1}{m}$-thick part of N. By the definition of the sets $V(\epsilon, z, \omega_m)$, it follows that $V(\epsilon, z, \omega_m) = U(\epsilon, z, \omega_m)$ and hence that $\rho \in U(\epsilon, z, \omega_m)$. Thus according to the properties of $U(\frac{1}{m}, z, \omega_m)$ stated in 4.5, there is a map $\phi_m : D_{\frac{1}{m}}(N_m) \to N$, such that

(1) ϕ_m maps $D_{\frac{1}{m}}(N_m)$ homeomorphically onto a manifold with piecewise smooth boundary, and is 2-biLipschitz with respect to the intrinsic metrics on $D_{\frac{1}{m}}(N_m)$ and $\phi_m(D_{\frac{1}{m}}(N_m))$,

(2) $\phi_m(z_{\omega_m}) = z_\rho$,

(3) $\mathrm{vol}\,\mathcal{N}_1(\partial(\phi_m(D_{\frac{1}{m}}(N_m)))) \leq 8\beta(N_m) = 32\pi\kappa(k-1)$, and

(4) $\phi_m(N_{\mathrm{thin}(\delta)} \cap D_{\frac{1}{m}}(N_m)) \subset N_{\mathrm{thin}(2\delta)}$ for any $\delta < \frac{\lambda_0}{2}$.

(Here κ is the constant given by Lemma 4.3. Notice that since $\pi_1(N_m)$ is a free group we have $\tau(N_m) = 0$ and hence $\beta(N_m) = 2\pi$
$kappa\sigma(N) = 4\pi\kappa(k-1)$.)

We set $\delta_0 = \lambda_0/3$ and $M_m = \phi_m(D_{\frac{1}{m}}(N_m))$.

Then by (4) we have

$$M_m \cap \phi_m((N_m)_{\mathrm{thin}(\delta_0)}) \subset N_{\mathrm{thin}(2\delta_0)}.$$

Hence, by Lemma 5.4, any two points in ∂M_m may be joined by a path β in ∂M_m such that $\beta \cap N_{\mathrm{thin}(2\delta_0)}$ has length at most $2c_k(\delta_0)$.

Let $r > 0$, and let $X(r)$ denote the set of points $x \in N$ for which there exists a path β beginning in $\overline{B(r, z_\Gamma)}$ and ending at x such that $\beta \cap N_{\mathrm{thick}(2\delta_0)}$ has length at most $2c_k(\delta_0)$. Since Γ is purely loxodromic, each component of $N_{\mathrm{thin}(2\delta_0)}$ is compact. Moreover, the components of $N_{\mathrm{thin}(2\delta_0)}$ are separated by a distance of at least $L(2\delta_0)$, so there exist only a finite number of components of $N_{\mathrm{thin}(2\delta_0)}$ contained in $X(r)$. Therefore $X(r)$ is compact. Set $\zeta(r) = \min_{x \in X(r)} \mathrm{inj}_N(x)$, and set $m_1 = \max(m_0, 2/\zeta(r))$.

If $m > m_1$ then $\partial M_m \cap B(r, z_\rho) = \emptyset$, since any point in ∂M_m may be joined by a path of length at most $2c_k(\delta_0)$ to a point of injectivity radius $\leq \frac{2}{m}$ (namely any point in $\phi_m(\partial D_{\frac{1}{m}}(N_m) - \partial\, \mathrm{C}(N_m))$). Since $z_\rho \in M_m$ by (2), and since $\partial M_m \cap B(r, z_\rho) = \emptyset$, we see that $B(r, z_\rho) \subset M_m$ for every $m > m_1$.

We therefore have $\cup_{m > m_1} \mathring{M}_m = N = \mathrm{C}(N)$. We may pass to a subsequence M_{m_j} such that $M_{m_j} \subset \mathring{M}_{m_{j+1}}$ for all j and $\cup_{j \in \mathbf{Z}_+} \mathring{M}_{m_j} = N = \mathrm{C}(N)$.

By (3) we have

$$\mathrm{vol}\, \mathcal{N}_1(\partial M_m) \leq 32\pi\kappa(k-1)$$

for all $m > m_1$. Moreover, by (1) and Lemma 4.4 we have

$$\mathrm{area}\, \partial M_m \leq 4\alpha(N_m) = 14\pi\sigma(N_m) = 28\pi(k-1).$$

Thus N is analytically tame.

5.2

6 Free groups and displacements

This section is devoted to the proof of the following theorem, which includes the Main Theorem stated in the introduction.

Theorem 6.1 *Let $k \geq 2$ be an integer and let Φ be a Kleinian group which is freely generated by elements $\xi_1, \ldots, \xi_k$. Suppose that either* *(a)* Φ *is purely loxodromic and topologically tame, or*

(b) Φ *is geometrically finite, or*

(c) Φ *is analytically tame and* $\Lambda(\Phi) = \overline{\mathbf{C}}$*, or*

(d) the hyperbolic 3-manifold $\mathbf{H}^3/\Phi$ *admits no non-constant positive superharmonic functions.*

Let z *be any point of* $\mathbf{H}^3$ *and set* $d_i = \mathrm{dist}(z, \xi_i \cdot z)$ *for* $i = 1, \ldots, k$*. Then we have*

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \le \frac{1}{2}.$$

In particular there is some $i \in \{1, \ldots, k\}$ *such that* $d_i \ge \log(2k-1)$*.*

Note that if we had $d_i < \log(2k-1)$ for $i = 1, \ldots, k$ it would follow that

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} > k \cdot \frac{1}{2k} = \frac{1}{2}.$$

Thus the last sentence of Theorem 6.1 does indeed follow from the preceding sentence.

Conditions (a)–(d) of Theorem 6.1 are by no means mutually exclusive. In particular, according to Proposition 5.1, condition (c) implies condition (d).

The following elementary inequality will be needed for the proof of Theorem 6.1.

Lemma 6.2 *Let* x *and* y *be non-negative real numbers with* $x + y \le 1$*. Set* $p = \frac{1}{2}(x+y)$*. Then we have*

$$\left(\frac{1-x}{x}\right)\left(\frac{1-y}{y}\right) \ge \left(\frac{1-p}{p}\right)^2.$$

Proof of 6.2: We can write $x = p + \alpha$ and $y = p - \alpha$ for some $\alpha \in \mathbf{R}$. We find by direct calculation that $p^2(1-x)(1-y) - (1-p)^2xy = (1-2p)\alpha^2$. But $p \le 1/2$ since $x + y \le 1$. Hence $p^2(1-x)(1-y) \ge (1-p)^2xy$, and the assertion follows.

6.2

Proof of Theorem 6.1: We first prove that condition (d) implies the conclusion of the theorem.

We use the terminology of [13]. We set $\Psi = \{\xi_1, \xi_1^{-1}, \dots, \xi_k, \xi_k^{-1}\} \subset \Phi$. According to Lemma 5.3 of [13], there exists a number $D \in [0, 2]$, a Φ-invariant D-conformal density $\mathfrak{M} = (\mu_z)$ for $\mathbf{H}^3$ and a family $(\nu_\psi)_{\psi\in\Psi}$ of Borel measures on S_∞ such that

(i) $\mu_{z_0}(S_\infty) = 1$;

(ii) $\mu_{z_0} = \sum_{\psi\in\Psi} \nu_\psi$; and

(iii) for each $\psi \in \Psi$ we have

$$\int (\lambda_{\psi,z_0})^D d\nu_{\psi^{-1}} = 1 - \int d\nu_\psi.$$

If condition (d) of the theorem holds, it follows from Proposition 3.9 of [13] that any Φ-invariant conformal density for $\mathbf{H}^3$ is a constant multiple of the area density $\mathfrak{A}$. In view of condition (i) above we must in fact have $\mathfrak{M} = \mathfrak{A}$. In particular $D = 2$.

For $i = 1, \dots, k$ we set $\nu_i = \nu_{\xi_i}$ and $\nu'_i = \nu_{\xi_i^{-1}}$. We denote by α_i and β_i the total masses of the measures ν_i and ν'_i respectively. After possibly interchanging the roles of ξ_i and ξ_i^{-1} we may assume that $\alpha_i \le \beta_i$. (Interchanging the roles of ξ_i and ξ_i^{-1} does not affect the truth of the conclusion of the lemma, since $d_i = \mathrm{dist}(z_0, \xi_i \cdot z_0) = \mathrm{dist}(z_0, \xi_i^{-1} \cdot z_0)$.)

By conditions (i) and (ii) above we have $\sum_{i=1}^k (\alpha_i + \beta_i) = 1$. In particular for each i we have $0 \le \beta_i \le 1$, and since $\alpha_i \le \beta_i$ we have $0 \le \alpha_i \le 1/2$. Since $\mathfrak{M} = \mathfrak{A}$, condition (ii) also implies that $\nu_i \le A_{z_0}$, where A_{z_0} denotes the area measure on S_∞ centered at z_0. By applying condition (iii) above to $\psi = \xi_i^{-1}$ we get that $\int_{S_\infty} \lambda^2_{\xi_i^{-1},z_0} d\nu_i = 1 - \beta_i$. And by definition we have $\nu_i(S_\infty) = \alpha_i$. Thus the hypotheses of Lemma 5.5 of [13] hold with $\nu = \nu_i$, $a = \alpha_i$ and $b = 1 - \beta_i$. Hence by Lemma 5.5 of [13] we have

$$d_i = \mathrm{dist}(z_0, \xi_i^{-1} \cdot z_0) \ge \frac{1}{2} \log \frac{b(1-a)}{a(1-b)}.$$

(This is a corrected version of the conclusion of Lemma 5.5 of [13]. In the published version of [13] the inequality appeared with the roles of a and b reversed.)

Thus

$$d_i \geq \frac{1}{2} \log \frac{(1-\beta_i)(1-\alpha_i)}{\alpha_i \beta_i}.$$

Since $\alpha_i + \beta_i \leq \sum_{i=1}^{k}(\alpha_i + \beta_i) = 1$, it follows from Lemma 6.2 that

$$\frac{(1-\alpha_i)(1-\beta_i)}{\alpha_i \beta_i} \geq (\frac{1-p_i}{p_i})^2,$$

where $p_i = (\alpha_i + \beta_i)/2$. Hence

$$d_i \geq \log \frac{1-p_i}{p_i}.$$

It follows that

$$\frac{1}{1+e^{d_i}} \leq p_i.$$

Hence

$$\sum_{i=1}^{k} \frac{1}{1+e^{d_i}} \leq \sum_{i=1}^{k} p_i = \frac{1}{2} \sum_{i=1}^{k} (\alpha_i + \beta_i) = \frac{1}{2}.$$

This completes the proof that condition (d) implies the conclusion of the theorem. In view of Proposition 5.1, it follows that condition (c) also implies the conclusion of the theorem.

Next we assume that condition (a) holds and deduce the conclusion of the theorem.

Since Φ is purely loxodromic and free of finite rank, either $\Lambda(\Phi) = \overline{\mathbf{C}}$ or Φ is a Schottky group (see Maskit [24].) In the case that $\Lambda(\Phi) = \overline{\mathbf{C}}$, we use Theorem 8.1 in [8], which states that a topologically tame hyperbolic 3-manifold is analytically tame.

Thus in this case condition (c) holds, and hence the conclusion of the theorem is true.

Now suppose that Φ is a Schottky group. Let F_k denote the abstract free group generated by $\{x_1, \ldots, x_k\}$. Let $\rho_0 : F_k \to \Phi$ denote the unique isomorphism that takes x_i to ξ_i for $i = 1, \ldots, k$. We may regard ρ_0 as a representation of F_k in $PSL_2(\mathbf{C})$. Since Φ is a Schottky group we have $\rho_0 \in \mathcal{CC}_k$.

According to the hypothesis of the theorem we are given a point $z \in \mathbf{H}^3$. We define a continuous, non-negative real-valued function f_z on the representation space $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$ by setting

$$f_z(\rho) = \sum_{i=1}^{k} \frac{1}{1 + \exp \mathrm{dist}(z, \rho(x_i) \cdot z)} \quad .$$

Now consider the dense G_δ-set $\mathcal{C}_k \subset \mathcal{B}_k$ given by Theorem 5.2. Recall that every representation in $\mathcal{C}_k$ maps F_k isomorphically onto an analytically tame Kleinian group whose limit set is the entire sphere at infinity. Thus for any $\rho \in \mathcal{C}_k$ the group $\rho(F_k)$ satisfies condition (c) of the present theorem, and we therefore have $f_z(\rho) \leq \frac{1}{2}$. Since $\mathcal{C}_k$ is dense in $\mathcal{B}_k$ and f_z is continuous, we have $f_z(\rho) \leq \frac{1}{2}$ for every $\rho \in \mathcal{B}_k$.

We may define a one-parameter family $(\xi_1^t)_{0 \leq t \leq 1}$ of elements of $\mathrm{PSL}_2(\mathbf{C})$ such that $\xi_1^t \cdot z = \xi_1 \cdot z$, $\xi_1^0 = \xi_1$, and ξ_1^1 has the same fixed points on $\overline{\mathbf{C}}$ as ξ_2. We then define a continuous one-parameter family $(\rho_t)_{0 \leq t \leq 1}$ of representations in $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$ by setting $\rho_t(x_1) = \xi_1^t$ and $\rho_t(x_i) = \xi_i$ for $i = 2, \ldots, k$. Note that for $t = 0$ this definition agrees with our previous definition of ρ_0, but that ρ_1 does not lie in $\mathcal{D}_k$. Note also that the function f_z is constant on the one-parameter family (ρ_t).

Let U denote the set of all points $t \in [0, 1]$ such that $\rho_t \in \mathcal{CC}_k$. Notice that U is open, since $\mathcal{CC}_k$ is an open subset of $\mathrm{Hom}(F_k, \mathrm{PSL}_2(\mathbf{C}))$. We have $0 \in U$ since $\rho_0 \in \mathcal{CC}_k$, and $\rho_1 \notin U$ since $\rho_1 \notin \mathcal{D}_k$. Thus we may set $u = \sup U$ and conclude that $\rho_u \in \mathcal{B}_k$, and hence that $f_z(\rho_u) \leq \frac{1}{2}$. Since f_z is constant on (ρ_t), it follows that $f_z(\rho_0) \leq \frac{1}{2}$. This is the conclusion of the theorem.

Finally, we prove the conclusion of the theorem under the assumption that condition (b) holds.

We continue to denote by F_k the abstract free group on k generators. We fix an isomorphism $\rho : F_k \to \Phi$, which we regard as a discrete, faithful representation of F_k in $\mathrm{Isom}_+(\mathbf{H}^3)$. In view of the geometric finiteness of $\Phi = \rho(F_k)$, a theorem of Maskit's [26] guarantees that there exists a sequence of discrete faithful representations $\{\rho_j : F \to \mathrm{Isom}_+(\mathbf{H}^3)\}$ such that (for all j) $\rho_j(F)$ is geometrically finite and purely loxodromic, and ρ_j converges (as a sequence of representations) to ρ. Given $z \in \mathbf{H}^3$, we set

$$m_j = \sum_{i=1}^{k} \frac{1}{1 + e^{\mathrm{dist}(\rho_j(x_i) \cdot z, z)}} \qquad \text{and} \qquad m = \sum_{i=1}^{k} \frac{1}{1 + e^{d_i}}.$$

Since each ρ_j satisfies (a), we have $m_j \leq \frac{1}{2}$. But clearly $\{m_j\}$ converges to m, so $m \leq \frac{1}{2}$. This is the conclusion of the theorem.

Theorem 6.1

7 Topology and free subgroups

The results of the last section can be used in studying the geometry of a hyperbolic manifold N. One writes N in the form $\mathbf{H}^3/\Gamma$ where Γ is a torsion-free Kleinian group, and applies the estimate given by Theorem 6.1 to suitable free subgroups Φ of Γ to deduce geometric information about N. In order to do this in a concrete situation, it is necessary to have sufficient conditions for a subgroup of $\Gamma \cong \pi_1(N)$ to be free. Using 3-manifold theory one can deduce from topological hypotheses that certain subgroups of $\pi_1(N)$ are free. Results of this sort appeared in [35] and in Section VI.4 of [17] for the case of a 2-generator subgroup, and in the appendix to [3] for the case in which N is a closed manifold. In this section we prove a result that includes the latter results as special cases and is suitable for the applications in this paper.

In this section we shall follow a couple of conventions that are widely used in low-dimensional topology. Unlabeled homomorphisms between fundamental groups are understood to be induced by inclusion maps. Base points will be suppressed whenever it is clear from the context how to choose consistent base points.

Recall that an orientable piecewise linear 3-manifold N is said to be *irreducible* if every PL 2-sphere in N bounds a PL ball. We shall say that N is *simple* if N is irreducible and if for every rank-2 free abelian subgroup A of $\pi_1(N)$, there is a closed PL subspace E of N, piecewise linearly homeomorphic to $T^2 \times [0, \infty)$, such that A is contained in a conjugate of $\mathrm{im}(\pi_1(E) \to \pi_1(N))$. (The subgroup $\mathrm{im}(\pi_1(E) \to \pi_1(N))$ of $\pi_1(N)$ is itself well-defined up to conjugacy.)

We shall say that an orientable PL 3-manifold N without boundary has *cusp-like ends* if it is PL homeomorphic to the interior of a compact manifold-with-boundary M such that (i) every component of ∂M is a torus and (ii) for every component B of ∂M, the inclusion homomorphism $\pi_1(B) \to \pi_1(M)$ is injective. In particular, note that if N is closed then it has cusp-like ends.

Recall that the *rank* of a group Γ is the minimal cardinality of a generating set for Γ.

A group Γ is termed *freely indecomposable* if it is non-trivial and is not a free product of two non-trivial subgroups.

For any non-negative integer g we denote by S_g the closed orientable surface of genus g.

Theorem 7.1 *Let N be a simple orientable PL 3-manifold without boundary. Suppose that $k = \operatorname{rank} \pi_1(N) < \infty$, that $\pi_1(N)$ is freely indecomposable, and that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then either $\pi_1(N)$ is a free abelian group, or N has cusp-like ends.*

Proof of 7.1: According to [30], there is a compact PL manifold-with-boundary $M \subset N$ such that $\pi_1(M) \to \pi_1(N)$ is an isomorphism. Among all compact PL manifolds-with-boundary with this property we may suppose M to have been chosen so as to minimize the number of components $r = r_M$ of ∂M. We may assume if we like that $r > 0$, for if $r = 0$ then $N = M$ is closed, and in particular it has cusp-like ends. Let $B_1, \ldots, B_r$ denote the components of ∂M. Let g_i denote the genus of B_i for $i = 1, \ldots, r$. Since $\pi_1(M) \cong \pi_1(N)$ has rank k, the first betti number of M is at most k.

It follows from Poincaré-Lefschetz duality and the exact homology sequence of $(M, \partial M)$ that the total genus $\sum g_i$ of ∂M is at most the first betti number of M. Thus $\sum g_i \leq k$.

We must have $g_i > 0$ for $i = 1, \ldots, r$. Indeed, if B_i is a 2-sphere for some i, then since N is irreducible, B_i bounds a PL ball $K \subset N$. We have either $K \supset M$ or $K \cap M = B_i$. If $K \supset M$ then since $\pi_1(M) \to \pi_1(N)$ is an isomorphism we have $\pi_1(N) = 1$, in contradiction to the free indecomposability of $\pi_1(N)$. If $K \cap M = B_i$ then $M' = M \cup K$ has fewer boundary components than M and $\pi_1(M) \to \pi_1(N)$ is an isomorphism; this contradicts our choice of M.

Let us consider the case in which $\pi_1(B_i) \to \pi_1(M)$ has a non-trivial kernel for some $i \leq r$. According to the Loop Theorem [33], M contains a properly embedded disk D such that ∂D is homotopically non-trivial in ∂M. If D separates M, both components of $M - D$ have boundary components of positive genus and are therefore non-simply connected. This contradicts the free indecomposability of $\pi_1(M)$. Hence D does not separate M, and M is a free product of an infinite cyclic group with a group isomorphic to

$\pi_1(M - D)$. The latter group must be trivial in view of the free indecomposability of $\pi_1(M) \cong \pi_1(N)$. Thus $\pi_1(N) \cong \pi_1(M)$ is infinite cyclic in this case, and in particular free abelian.

From this point on we assume that $\pi_1(B_i) \to \pi_1(M)$ is injective for $i = 1, \ldots, r$. Since by the hypothesis of the theorem, $\pi_1(M) \cong \pi_1(N)$ has no subgroup isomorphic to $\pi_1(S_{g_i})$ for $2 \leq g_i \leq k - 1$, each g_i is either ≤ 1 or $\geq k$. We have seen that the g_i are all strictly positive and that their sum is at most k. Hence we must have either (i) $r = 1$ and $g_1 = k$, or (ii) $g_i = 1$ for $i = 1, \ldots, r$.

Suppose that (i) holds. Then ∂M is a connected surface of genus k. Hence the Euler characteristic $\chi(\partial M)$ is equal to $2-2k$. We have $\chi(M) = \frac{1}{2}\chi(\partial M) = 1-k$. Now as a compact PL 3-manifold with non-empty boundary, M admits a simplicial collapse to a 2-complex L. In particular M is homotopy-equivalent to L, and hence to the CW-complex L' obtained from L by identifying a maximal tree in the 1-skeleton of L to a point. If c_i denotes the number of i-cells in L', we have $c_0 = 1$ and $1 - c_1 + c_2 = \chi(L') = \chi(M) = 1 - k$, so that $c_1 - c_2 = k$. But $\pi_1(L') \cong \pi_1(M) \cong \pi_1(N)$ has a presentation with c_1 generators and c_2 relations, and $k = c_1 - c_2$ is by definition the deficiency of the presentation. On the other hand, k is by hypothesis the rank of $\pi_1(N)$. It is a theorem due to Magnus [21] that if a group Γ has rank k and admits a presentation of deficiency k, then Γ is free of rank k. Since $\pi_1(N)$ is freely indecomposable, we must have $k = 1$, and $\pi_1(N)$ must be infinite cyclic in this case as well.

Now suppose that (ii) holds. Then $B_1, \ldots, B_r$ are tori. Since the inclusion homomorphisms $\pi_1(B_i) \to \pi_1(M)$ are injective, the groups $A_i = \mathrm{im}(\pi_1(B_i) \to \pi_1(M))$ are free abelian groups of rank 2. Since N is simple, there are closed PL subspaces $E_1, \ldots, E_r$ of N, each piecewise linearly homeomorphic to $T^2 \times [0, \infty)$, such that A_i is contained in a conjugate of $\mathrm{im}(\pi_1(E_i) \to \pi_1(N))$ for $i = 1, \ldots, r$. It follows from Proposition 5.4 of [36] that B_i is isotopic to ∂E_i for $i = 1, \ldots, r$. Hence we may suppose the E_i to have been chosen so that $\partial E_i = B_i$. For each $i \leq r$ we have either $M \subset E_i$ or $M \cap E_i = B_i$.

If $M \cap E_i = B_i$ for $i = 1, \ldots, r$, we have $N = M \cup E_1 \ldots \cup E_r$. It follows that in this case N is PL homeomorphic to the interior of M, and hence that N has cusp-like ends.

There remains the case in which $M \subset E_i$ for some i. In the sequence of inclusion homomorphisms

$$\pi_1(B_i) \to \pi_1(M) \to \pi_1(E_i) \to \pi_1(N)$$

the composition of the first two arrows (from the left) is the isomorphism $\pi_1(B_i) \to \pi_1(E_i)$, and the composition of the last two arrows is the isomorphism $\pi_1(M) \to \pi_1(N)$. It follows that the entire sequence consists of isomorphisms, and hence that $\pi_1(N)$ is a rank-2 free abelian group in this case.

7.1

We shall say that a group is *semifree* if it is a free product of abelian groups.

Corollary 7.2 *Let N be an orientable hyperbolic 3-manifold of infinite volume. Suppose that $k = \operatorname{rank} \pi_1(N) < \infty$, and that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then $\pi_1(N)$ is semifree.*

Proof of 7.2: Let us write $N = \mathbf{H}^3/\Gamma$, where $\Gamma \cong \pi_1(N)$ is a discrete torsion-free subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. Since Γ is finitely generated it can be written as a free product $\Gamma_1 * \ldots * \Gamma_n$ of freely indecomposable subgroups. Since N has infinite covolume, so does the manifold H^3/Γ_i for $i = 1, \ldots, n$. The rank k_i of $\Gamma_i \cong \pi_1(N_i)$ is at most k. Hence the hypothesis of the corollary implies that Γ_i has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Applying Theorem 7.1 with N_i in place of N, we conclude that for each $i \leq n$, either Γ_i is free abelian or N_i has cusp-like ends. But the latter alternative is impossible because a hyperbolic manifold with cusp-like ends has finite volume (see [4], D.3.18). Thus all the Γ_i are free abelian and hence Γ is semifree.

7.2

Recall that a group Γ is termed *k-free,* where k is a cardinal number, if every subgroup of Γ whose rank is at most k is free. We shall say that Γ is *k-semifree* if every subgroup of Γ whose rank is at most k is semifree.

Corollary 7.3 *Let N be an orientable hyperbolic 3-manifold and let k be a non-negative integer. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. In addition suppose that either* *(i) N has infinite volume, or*

(ii) every subgroup of $\pi_1(N)$ whose rank is at most k is of infinite index in $\pi_1(N)$.

Then $\pi_1(N)$ is k-semifree.

Proof of 7.3: Let Δ be any subgroup of $\pi_1(N)$ whose rank is at most k. Let $\tilde{N}$ denote the covering space of N associated to H. If either (i) or (ii) holds, $\tilde{N}$ has infinite volume. Hence by Corollary 7.2, Δ is semifree.

7.3

Hypothesis (ii) of 7.3 clearly holds if the first betti number of N is at least $k+1$. According to Proposition 1.1 of [31], it also holds if $H_1(N, \mathbf{Z}/p)$ has rank at least $k+2$ for some prime p. Thus we have:

Corollary 7.4 *Let N be an orientable hyperbolic 3-manifold and let k be a non-negative integer. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \le g \le k-1$. In addition suppose that either*

(i) the first betti number of N is at least $k+1$, or

(ii) $H_1(N, \mathbf{Z}/p)$ has rank at least $k+2$ for some prime p.

Then $\pi_1(N)$ is k-semifree.

Remark 7.5 If the orientable hyperbolic 3-manifold N has no cusps, then every abelian subgroup of $\pi_1(N)$ is infinite cyclic; thus $\pi_1(N)$ is k-semifree for a given k if and only if it is k-free. Thus if N has no cusps we may replace "semifree" by "free" in the conclusions of Corollaries 7.3 and 7.4.

8 Strong Margulis numbers and k-Margulis numbers

In order to unify the different applications of the results of the last two sections it is useful to introduce a little formalism. Let Γ be a discrete torsion-free subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. Recall from [31] and [13] that a positive number λ is termed a *Margulis number* for the group Γ, or for the orientable hyperbolic 3-manifold $N = \mathbf{H}^3/\Gamma$, if whenever ξ and η are non-commuting elements of

Γ, and $z \in \mathbf{H}^3$, we have $\max\{\mathrm{dist}(\xi \cdot z, z), \mathrm{dist}(\eta \cdot z, z)\} \geq \lambda$. We shall say that λ is a *strong Margulis number* for Γ, or for N, if whenever ξ and η are non-commuting elements of Γ, we have

$$\frac{1}{1+e^{\mathrm{dist}(\xi\cdot z,z)}} + \frac{1}{1+e^{\mathrm{dist}(\eta\cdot z,z)}} \leq \frac{2}{1+e^{\lambda}} \ .$$

Notice that if λ is a strong Margulis number for Γ, then λ is also a Margulis number for Γ.

More generally, let $k \geq 2$ be an integer, and let λ be a positive real number. We shall say that λ is a *k-Margulis number* for the discrete torsion-free group $\Gamma \leq \mathrm{Isom}_+(\mathbf{H}^3)$, or for $N = \mathbf{H}^3/\Gamma$, if for any k elements $\xi_1, \ldots, \xi_k \in \Gamma$, and for any $z \in \mathbf{H}^3$, we have that either

(i) $\max_{i=1}^k \mathrm{dist}(\xi_i \cdot z, z) \geq \lambda$, or

(ii) the group $\langle \xi_1, \ldots, \xi_k \rangle$ is generated by at most $k-1$ abelian subgroups.

We say that λ is a *strong k-Margulis number* for Γ, or for N, if for any k elements $\xi_1, \ldots, \xi_k \in \Gamma$, and for any $z \in \mathbf{H}^3$, we have that either

(i)
$$\sum \frac{1}{1+e^{\mathrm{dist}(\xi_i\cdot z,z)}} \leq \frac{k}{1+e^{\lambda}} \ ,$$
or

(ii) the group $\langle \xi_1, \ldots, \xi_k \rangle$ is generated by at most $k-1$ abelian subgroups.

Note that λ is a (strong) 2-Margulis number for Γ if and only if it is a (strong) Margulis number for Γ. Note also that if λ is a strong k-Margulis number for Γ, then λ is also a k-Margulis number for Γ.

In this section we will use Theorem 6.1 and the corollaries of Theorem 7.1 to prove that under various conditions a hyperbolic 3-manifold has $\log(2k-1)$ as a strong k-Margulis number. In the following three sections these results will be used to obtain lower bounds for the volume of various classes of hyperbolic 3-manifolds.

Our first result is an easy consequence of Theorem 6.1(a). We shall say that a Kleinian group Γ is *k-tame*, where k is a positive integer, if every subgroup of Γ having rank at most k is topologically tame.

Proposition 8.1 *Let $k \geq 2$ be an integer and let Γ be a discrete subgroup of* $\mathrm{Isom}_+(\mathbf{H}^3)$*. Suppose that Γ is k-free, k-tame and purely loxodromic. Then* $\log(2k-1)$ *is a strong k-Margulis number for Γ.*

Proof of 8.1: If $\xi_1, \ldots, \xi_k \in \Gamma$ are elements of Γ, the group $\langle \xi_1, \ldots, \xi_k \rangle$ is topologically tame, purely loxodromic and free of some rank $\leq k$. If its rank is k then it is freely generated by $\xi_1, \ldots, \xi_k$; hence for any $z \in \mathbf{H}^3$, we have

$$\sum \frac{1}{1 + e^{\mathrm{dist}(\xi_i \cdot z, z)}} \leq \frac{1}{2} = \frac{k}{1 + e^{\log(2k-1)}}$$

by Theorem 6.1(a). If $\langle \xi_1, \ldots, \xi_k \rangle$ has rank $\leq k-1$, then in particular it is generated by at most $k-1$ abelian subgroups.

8.1

Corollary 8.2 *Let $k \geq 2$ be an integer and let N be a non-compact, topologically tame orientable hyperbolic 3-manifold without cusps. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then* $\log(2k-1)$ *is a strong k-Margulis number for N.*

Proof of 8.2: Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a discrete, non-cocompact, purely loxodromic subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. According to Proposition 3.2 in [8], every finitely generated subgroup of Γ is topologically tame. In particular Γ is k-tame. On the other hand, since N has infinite volume and $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$, Corollary 7.3 and Remark 7.5 guarantee that $\Gamma \cong \pi_1(N)$ is k-free. The desired conclusion therefore follows from Proposition 8.1.

8.2

It is worth pointing out that the following corollary can be deduced from Proposition 8.1, although a more general result, Corollary 8.7, will be proved below by a slightly different argument.

Corollary 8.3 *Let $k \geq 2$ be an integer and let N be a closed orientable hyperbolic 3-manifold. Suppose that the first betti number of N is at least $k+1$ and that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then* $\log(2k-1)$ *is a strong k-Margulis number for N.*

Proof of 8.3: Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a discrete, cocompact, purely loxodromic subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$. By Corollary 7.4 and Remark 7.5, $\Gamma \cong \pi_1(N)$ is k-free. On the other hand, since N has betti number at least $k+1$, any subgroup Γ' of Γ having rank at most k is contained in the kernel of a surjective homomorphism $\beta : \Gamma \to \mathbf{Z}$. According to Proposition 8.4 of [9], it follows that Γ' is topologically tame. Thus Γ is k-tame and the desired conclusion follows from Proposition 8.1.

$\boxed{8.3}$

The following result gives information not contained in Proposition 8.1 because the group Γ is allowed to have parabolic elements.

Proposition 8.4 *Let $k \geq 2$ be an integer and let Γ be a discrete subgroup of* $\mathrm{Isom}_+(\mathbf{H}^3)$*. Suppose that Γ is k-semifree. Suppose in addition that for every subgroup Γ' of Γ having rank at most k, either*

(i) Γ' *is geometrically finite, or*

(ii) $N' = \mathbf{H}^3/\Gamma'$ *admits no non-constant positive superharmonic functions.*

Then $\log(2k-1)$ *is a strong k-Margulis number for Γ.*

Proof of 8.4: If $\xi_1, \ldots, \xi_k \in \Gamma$ are elements of Γ, the group $\langle \xi_1, \ldots, \xi_k \rangle$ is semifree. Thus we may write it as a free product $A_1 * \ldots * A_r$, where r is an integer $\leq k$ and $A_1, \ldots, A_r$ are free abelian groups. The sum of the ranks of the A_i is at most k. If $r < k$, or if some A_i has rank > 1, then $\langle \xi_1, \ldots, \xi_k \rangle$ is generated by at most $k-1$ abelian subgroups. Now suppose that $r = k$ and that the A_i are all cyclic. Then $\langle \xi_1, \ldots, \xi_k \rangle$ is free of rank k and is therefore freely generated by $\xi_1, \ldots, \xi_k$. If condition (i) of the hypothesis of the proposition holds, it follows from Theorem 6.1(b) that for any $z \in \mathbf{H}^3$ we have

$$\sum \frac{1}{1 + e^{\mathrm{dist}(\xi_i \cdot z, z)}} \leq \frac{1}{2} = \frac{k}{1 + e^{\log(2k-1)}}.$$

If condition (ii) holds, the same conclusion follows from Theorem 6.1(d).

$\boxed{8.4}$

If a torsion-free Kleinian group Γ is geometrically finite and has infinite covolume, then a theorem of Thurston's (see Proposition 7.1 in Morgan [29]) guarantees that every finitely generated subgroup of Γ is geometrically finite. This yields the following corollary to Proposition 8.4.

Corollary 8.5 *Let $k \geq 2$ be an integer and let Γ be a discrete subgroup of $\mathrm{Isom}_+(\mathbf{H}^3)$ which is geometrically finite and k-semifree and has infinite covolume. Then $\log(2k-1)$ is a strong k-Margulis number for Γ.*

8.5

This result can also be combined with the results from Section 7 as in the following corollary.

Corollary 8.6 *Let $k \geq 2$ be an integer and let N be a geometrically finite orientable hyperbolic 3-manifold of infinite volume. Suppose that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then $\log(2k-1)$ is a strong k-Margulis number for N.*

Proof of 8.6: We write $N = \mathbf{H}^3/\Gamma$, where Γ is a geometrically finite Kleinian group. Since N has infinite volume and $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$, Corollary 7.3 guarantees that $\Gamma \cong \pi_1(N)$ is k-semifree. The assertion now follows from Corollary 8.5.

8.6

The next corollary generalizes Corollary 8.3.

Corollary 8.7 *Let $k \geq 2$ be an integer and let N be an orientable hyperbolic 3-manifold of finite volume. Suppose that the first betti number of N is at least $k+1$ and that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \leq g \leq k-1$. Then $\log(2k-1)$ is a strong k-Margulis number for N.*

Proof of 8.7: We write $N = \mathbf{H}^3/\Gamma$, where Γ is a Kleinian group of finite covolume. It follows from Corollary 7.4 that $\Gamma \cong \pi_1(N)$ is k-semifree. To complete the proof it suffices to show that for every subgroup Γ' of Γ whose rank is at most k, one of the hypotheses (i) or (ii) of Proposition 8.4 holds.

Since N has betti number at least $k+1$, the subgroup Γ' is contained in the kernel of a surjective homomorphism $\beta : \pi_1(N) \to \mathbf{Z}$. Therefore, by Corollary E in [9], Γ' is either geometrically finite, or N has a finite cover $\widehat{N}$ which fibers over the circle and Γ' is topologically tame and contains the fiber subgroup Γ'' of $\widehat{N}$. In the latter case we have $\Lambda(\Gamma') = \overline{\mathbf{C}}$, since $\Lambda(\Gamma'') = \overline{\mathbf{C}}$. Corollary 9.2 in [8] then guarantees that $N' = \mathbf{H}^3/\Gamma'$ admits no non-constant positive superharmonic functions.

8.7

Finally, by specializing some of the results stated above to the case $k = 2$ we obtain some sufficient conditions for $\log 3$ to be a Margulis number for a hyperbolic 3-manifold.

Corollary 8.8 *Let $N = \mathbf{H}^3/\Gamma$ be an orientable hyperbolic 3-manifold, such that either*

(i) N is geometrically finite and has infinite volume,

(ii) N is topologically tame, purely loxodromic, and has infinite volume, or

(iii) N has finite volume and its first betti number is at least 3.

Then $\log 3$ is a strong Margulis number for Γ.

Proof of 8.8: As we observed at the beginning of this section, a Margulis number is the same thing as a 2-Margulis number. Under the hypothesis (i), (ii) or (iii), the assertion follows respectively from Corollary 8.6, Corollary 8.2, or Corollary 8.7. The general version of each of these corollaries included the assumption that $\pi_1(N)$ has no subgroup isomorphic to any of the groups $\pi_1(S_g)$ for $2 \le g \le k-1$. For $k = 2$ this condition is vacuously true.

8.8

Remark 8.9 Given corollary 8.8 it seems reasonable to conjecture that log 3 is a strong Margulis number for any infinite volume hyperbolic 3-manifold. We notice that our conjecture would follow from the conjecture that every free 2-generator Kleinain group is a limit of Schottky groups. There appear to exist closed hyperbolic 3-manifolds for which $\log 3$ is not even a Margulis number: computations by Hodgson and Weeks give strong evidence that the Weeks manifold does not contain a ball of radius $(\log 3)/2$.

9 Geometric estimates for closed manifolds

In this section we will prove the results promised in the introduction concerning balls of radius $\frac{1}{2}\log 5$ and volume estimates for closed manifolds of betti number at least 4. This will be done by combining the results of the last section with the following result, which illustrates the use of the notion of a k-Margulis number for $k > 2$.

Theorem 9.1 *Let N be an orientable hyperbolic 3-manifold without cusps. Suppose that $\pi_1(N)$ is 3-free. Let λ be a 3-Margulis number for N. Then either N contains a hyperbolic ball of radius $\lambda/2$, or $\pi_1(N)$ is a free group of rank 2.*

Before giving the proof of Theorem 9.1 we shall point out how to use it to prove the corollaries stated in the introduction.

Corollary 9.2 *Let N be a closed orientable hyperbolic 3-manifold. Suppose that the first betti number $\beta_1(N)$ is at least 4, and that $\pi_1(N)$ has no subgroup isomorphic to $\pi_1(S_2)$. Then N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$. Hence the volume of N is greater than 3.08.*

Proof of 9.2: According to Corollary 7.4 and Remark 7.5, the group $\pi_1(N)$ is 3-free. According to Corollary 8.3, $\log 5$ is a strong 3-Margulis number, and *a fortiori* a 3-Margulis number, for N. It therefore follows from Theorem 9.1 that either N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$ or $\pi_1(N)$ is a free group of rank 2. The latter alternative is impossible, because Γ, as the fundamental group of a closed hyperbolic 3-manifold, must have cohomological dimension 3, whereas a free group has cohomological dimension 1. Thus N must contains a hyperbolic ball of radius $\frac{1}{2}\log 5$.

The lower bound on the volume now follows by applying Böröczky's density estimate for hyperbolic sphere-packings as in [13].

9.2

Let $\mathcal{W}$ denote the set of all finite volumes of orientable hyperbolic 3-manifolds. Then $\mathcal{W}$ is a set of positive real numbers, and by restricting the usual ordering of the real numbers we can regard $\mathcal{W}$ as an ordered set. It is a theorem of Thurston's, based on work due to Jorgensen and Gromov, that $\mathcal{W}$ is a well-ordered set having ordinal type ω^ω and that there are at most a finite number of isometry types of hyperbolic 3-manifolds with a given volume. (See [4], E.1) Thus there is a unique order-preserving bijection between $\mathcal{W}$ and the set of ordinal numbers less than ω^ω. Let us denote by v_c the element of $\mathcal{W}$ corresponding to the ordinal number c.

Corollary 9.3 *Let c be any finite ordinal number and let N be any orientable hyperbolic 3-manifold with $\operatorname{vol} N = v_c$. Then either the first betti number of N is at most 3, or $\pi_1(N)$ contains an isomorphic copy of $\pi_1(S_2)$.*

Proof of 9.3: Assume that N has first betti number at least 4 and contains no isomorphic copy of $\pi_1(S_2)$. Then by Corollary 9.2 we have $v_c = \operatorname{vol} N > 3.08$. On the other hand, the complement of the figure-eight knot in S^3 is diffeomorphic to a hyperbolic 3-manifold E of volume $w = 2.02988..$ (The volume w is twice the volume of a regular ideal hyperbolic tetrahedron.) In particular we have $w < v_c$. Moreover, there is a sequence of hyperbolic manifolds obtained by Dehn surgeries on the figure-eight knot whose volumes are strictly less than w but converge to w (see Theorem E.7.2 in [4].). Thus there are infinitely many elements of the set $\mathcal{W}$ that are less than w and are therefore less than v_c. This contradicts the hypothesis that c is a finite ordinal.

9.3

Corollary 9.4 L*et N be a non-compact, topologically tame, orientable hyperbolic 3-manifold without cusps. Suppose (i) that $\pi_1(N)$ is not a free group of rank 2, and (ii) that $\pi_1(N)$ has no subgroup isomorphic to $\pi_1(S_2)$. Then N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$.*

Proof of 9.4: According to Corollary 8.2 and Remark 7.5, the group $\pi_1(N)$ is 3-free. According to Corollary 8.3, $\log 5$ is a strong 3-Margulis number, and *a fortiori* a 3-Margulis number, for N. It therefore follows from Theorem 9.1 (and hypothesis (ii)) that N contains a hyperbolic ball of radius $\frac{1}{2}\log 5$.

9.4

The rest of this section is devoted to the proof of Theorem 9.1. The essential ideas of the proof appear in the proof of Theorem B in [14]. We begin by reviewing and extending a few notions from [14].

As in [14], we shall say that elements $z_1, \ldots, z_r$ of a group Γ are *independent* if they freely generate a (free, rank-r) subgroup of Γ. Recall that the *rank* of a finitely generated group G to be the minimal cardinality of a generating set for G.

As in [14], a Γ-*labeled complex*, where Γ is a group, is defined to be an ordered pair $(K, (X_v)_v)$, where K is a simplicial complex and $(X_v)_v$ is a family of cyclic subgroups of Γ indexed by the vertices of K. If $(K, (X_v)_v)$ is a Γ-labeled complex then for any subcomplex L of K we denote by $\Theta(L)$ the subgroup of Γ generated by all the groups X_v, where v ranges over the vertices of L.

In this paper we shall use one notion which appeared only implicitly in [14]. Let Γ be a group and let $(K, (X_v)_v)$ be a Γ-labeled complex. By a *natural action of* Γ *on* $(K, (X_v)_v)$ we shall mean a simplicial action of Γ on K such that for each vertex v of K we have $X_{\gamma \cdot v} = \gamma X_v \gamma^{-1}$. The following result could have been stated and used in [14].

Proposition 9.5 L*et* Γ *be a finitely generated 3-free group in which every non-trivial element has a cyclic centralizer. Let* $(K, (X_v)_v)$ *be a* Γ*-labeled complex which admits a natural* Γ*-action. Suppose that* X_v *is a maximal cyclic subgroup of* Γ *for every vertex* v *of* K*. Suppose that* K *is connected and has more than one vertex, and that the link of every vertex of* K *is connected. Suppose that for every 1-simplex* e *of* K *the group* $\Theta(|e|)$ *is non-abelian, and that there is no 2-simplex* σ *of* K *such that* $\Theta(|\sigma|)$ *is free of rank 3. Then* $\Theta(K)$ *is a free group of rank 2.*

Proof of 9.5: The hypotheses of the above proposition include those of Proposition 4.3 of [14]. According to the latter result, $\Theta(K)$ has *local rank* 2: according to the definitions given in [14], this means that every finitely generated subgroup of $\Theta(K)$ is contained in a subgroup of rank ≤ 2, but that not every finitely generated subgroup of $\Theta(K)$ is contained in a subgroup of rank ≤ 1. On the other hand, the existence of a natural action of Γ on $(K, (X_v)_v)$ clearly implies that $\Theta(K)$ is a normal subgroup of Γ.

Now choose any vertex v_0 of K and let x_0 denote a generator of $X_0 = X_{v_0}$. Since x_0 has a cyclic centralizer and X_0 is a maximal cyclic subgroup of Γ, the element x_0 generates its own centralizer in Γ. Now it is a special case of Proposition 4.4 of [14] that if Θ is a normal subgroup of a finitely generated 3-free group Γ, if Γ is 3-free over some finitely generated subgroup of Θ, and if Θ has local rank 2 and contains an element x_0 which generates its own centralizer in Γ, then Γ is a free group of rank 2.

This completes the proof.

9.5

Proof of Theorem 9.1: As in [14], for any infinite cyclic group X of isometries of $\mathbf{H}^3$, generated by a loxodromic isometry, and for any $\lambda > 0$, we denote by $Z_\lambda(X)$ denote the set of points $z \in \mathbf{H}^3$ such that $\operatorname{dist}(z, \xi \cdot z) < \lambda$ for some non-trivial element ξ of X.

Suppose that N satisfies the hypotheses of Theorem 9.1 but contains no ball of radius $\lambda/2$. We shall prove the theorem by showing that $\pi_1(N)$ is a free group of rank 2. Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a purely loxodromic Kleinian group. Then according to the discussion in subsection 3.4 of [14], the indexed family $(Z_\lambda(X))_{X \in \mathcal{X}}$, where $\mathcal{X} = \mathcal{X}_\lambda(\mathcal{N})$ denotes the set of all maximal cyclic subgroups X of Γ such that $Z_\lambda(X) \neq \emptyset$, is an open covering of $\mathbf{H}^3$, and the nerve $K = K_\lambda(N)$ of this covering is a simplicial complex. By definition the vertices of K are in natural one-one correspondence with the maximal cyclic subgroups in the set $\mathcal{X}$. If we denote by $X_v \in \mathcal{X}$ the maximal cyclic subgroup corresponding to a vertex v, then $(K, (X_v)_v)$ is a Γ-labeled complex.

We shall show that the group Γ and the Γ-labeled complex $(K, (X_v)_v)$ satisfy the hypotheses of Proposition 9.5. By the hypothesis of the theorem, Γ is 3-free. Since Γ is a purely loxodromic Kleinian group, it has the property that each of its non-trivial elements has a cyclic centralizer.

In order to construct a natural action of Γ on $(K, (X_v)_v)$, we first define an action of Γ on the set of vertices of K by $X_{\gamma \cdot v} = \gamma X_v \gamma^{-1}$. If $v_0, \ldots, v_m$ are the vertices of an m-simplex of K we have

$$\bigcap_{0 \le i \le m} Z_\lambda(\gamma X \gamma^{-1}) = \bigcap_{0 \le i \le m} \gamma \cdot Z_\lambda(X) = \gamma \cdot \bigcap_{0 \le i \le m} Z_\lambda(X) \neq \emptyset,$$

so that $\gamma \cdot v_0, \ldots, \gamma \cdot v_m$ are the vertices of an m-simplex of K. Thus the action of Γ on the vertex set extends to a simplicial action on K. It is immediate from the definitions that this is a natural action on $(K, (X_v)_v)$.

By Proposition 3.4 of [14], K is a connected simplicial complex with more than one vertex, and the link of every vertex of K is connected. Now let e be any 1-simplex of K, and let v and w denote its vertices. Let x_v and x_w be generators of X_v and X_w. We have $v \neq w$ and hence $X_v \neq X_w$; that is, the elements x_v and x_w generate distinct maximal cyclic subgroups of Γ. Since the abelian subgroups of Γ are cyclic, it follows that $\Theta(|e|) = \langle x_v, x_w \rangle$ is non-abelian.

Finally, we claim that if σ is a 2-simplex of K, the group $\Theta(|\sigma|)$ cannot be free of rank 3. To prove this, let u, v and w denote the vertices of σ, and let ξ_u, ξ_v and ξ_w be generators of X_u, X_v and X_w. By the definition of the nerve K we have $Z_\lambda(X_u) \cap Z_\lambda(X_v) \cap Z_\lambda(X_w) \neq \emptyset$. Let z be any point of $Z_\lambda(X_u) \cap Z_\lambda(X_v) \cap Z_\lambda(X_w)$. By definition there are non-trivial elements of X_u, X_v and X_w, say $\eta_u = \xi_u^{n_u}$, $\eta_v = \xi_v^{n_v}$ and $\eta_w = \xi_w^{n_w}$, such that $\mathrm{dist}(z, \eta_u \cdot z)$, $\mathrm{dist}(z, \eta_v \cdot z)$ and $\mathrm{dist}(z, \eta_w \cdot z)$ are less than λ. Since λ is a 3-Margulis number for Γ, it follows that $\langle \eta_u, \eta_v, \eta_w \rangle$ is generated by at most two abelian subgroups. Now if $\Theta(|\sigma|)$ were free of rank 3 then ξ_u, ξ_v and ξ_w would be independent, and so η_u, η_v and η_w would also be independent. This would mean that $\langle \eta_u, \eta_v, \eta_w \rangle$ would be free of rank 3, and thus could not be generated by two abelian subgroups. This proves the claim.

Thus Γ and $(K, (X_v)_v)$ satisfy all the hypotheses of Proposition 9.5. Hence Γ is a free group of rank 2, as required.

Theorem 9.1

Remark 9.6 It is possible to drop the hypothesis that N has no cusps in Theorem 9.1. Because $\pi_1(N)$ is 3-free, N could only have rank 1 cusps. The construction of the Γ-labelled complex in the proof of 9.1 can still be carried out, although the arguments in [14] must be extended to account for the fact that some of the sets $Z_\lambda(X)$ will be horoballs instead of cylinders.

10 Volumes and short geodesics

Let C be a non-trivial closed geodesic in a closed hyperbolic 3-manifold N. Let us write $N = \mathbf{H}^3/\Gamma$, where Γ is a cocompact, torsion-free, discrete group of isometries of $\mathbf{H}^3$. Then C is the image in N of the axis A_γ of some non-trivial (and hence loxodromic) element $\gamma \in \Gamma$ which is uniquely determined up to conjugacy. Let us set

$$R = \frac{1}{2} \min_{\delta} \operatorname{dist}(A_\gamma, \delta \cdot A_\gamma),$$

where δ ranges over all elements of Γ which do not commute with γ. If we denote by Z the set of all points in $\mathbf{H}^3$ whose distance from A_γ is less than R, it follows from the definition of R that $Z \cap \delta \cdot Z = \emptyset$ for every $\delta \in \Gamma$ not commuting with γ; hence the quotient $Z/\langle\gamma\rangle$ embeds in N. The resulting isometric copy of $Z/\langle\gamma\rangle$ in N is called the *maximal embedded tube* about the geodesic C, and the number R is called the *radius* of the tube. If the geodesic C has length l then the volume of the maximal embedded tube about C is given by the formula

$$\pi l \sinh^2 R, \tag{2}$$

which is therefore a lower bound for the volume of N.

In this section we prove the following result.

Proposition 10.1 *Let N be an orientable hyperbolic 3-manifold having* $\log 3$ *as a strong Margulis number. Let C be a closed geodesic in N, and let l denote its length. If R denotes the radius of the maximal embedded tube about C, we have*

$$\cosh 2R \geq \frac{e^{2l} + 2e^l + 5}{(\cosh \frac{l}{2})(e^l - 1)(e^l + 3)}.$$

Combining this with Corollary 8.8 we immediately obtain:

Corollary 10.2 *Let N be an orientable hyperbolic 3-manifold of finite volume whose first betti number is at least* 3. *Let C be a closed geodesic in N, and let l denote its length. If R denotes the radius of the maximal embedded tube about C, we have*

$$\cosh 2R \geq \frac{e^{2l} + 2e^l + 5}{(\cosh \frac{l}{2})(e^l - 1)(e^l + 3)}.$$

The above results will also be used to give volume estimates for hyperbolic 3-manifolds containing short geodesics (see 10.3, 10.5 and 10.6 below).

Proof of 10.1: By the definition of R there is an element δ of Γ, not commuting with γ, such that the distance from A_γ to $\delta \cdot A_\gamma = A_{\delta\gamma\delta^{-1}}$ is $2R$. Let B denote the common perpendicular to the lines A_γ and $\delta \cdot A_\gamma$, and let z and w denote the points of intersection of B with A_γ and $\delta \cdot A_\gamma$ respectively. Then $\text{dist}(z, w) = 2R$. Let us write $w = \delta \cdot u$ where u is a point of A_γ. Since γ acts on A_γ as a translation of length l, there is an integer m such that $\text{dist}(u, \gamma^m \cdot z) \leq l/2$. Hence $\text{dist}(w, \delta\gamma^m \cdot z) \leq l/2$. The triangle with vertices z, w and $\delta\gamma^m \cdot z$ has a right angle at w. Writing $\alpha = \text{dist}(z, \delta\gamma^m \cdot z)$ for the hypotenuse of this right triangle and applying the Hyperbolic Pythagorean Theorem, we obtain

$$\cosh \alpha = \cosh 2R \cosh \text{dist}(w, \delta\gamma^m \cdot z) \leq \cosh \frac{l}{2} \cosh 2R. \tag{3}$$

Since γ and δ do not commute, the elements γ and $\delta\gamma^m$ of Γ also do not commute. Applying the definition of a strong Margulis number with $\xi = \gamma$ and $\eta = \delta\gamma^m$, and using that $\alpha = \text{dist}(z, \delta\gamma^m \cdot z)$ and that $l = \text{dist}(z, \gamma \cdot z)$, we obtain

$$\frac{1}{1 + e^\alpha} + \frac{1}{1 + e^l} \leq \frac{1}{2},$$

which we rewrite in the form

$$e^\alpha \geq \frac{e^l + 3}{e^l - 1}. \tag{4}$$

On the other hand, using (2) we find that

$$\begin{aligned} e^\alpha &= \cosh \alpha + \sinh \alpha \\ &= \cosh \alpha + \sqrt{\cosh^2 \alpha - 1} \\ &\leq \cosh 2R \cosh \frac{l}{2} + \sqrt{\cosh^2 2R \cosh^2 \frac{l}{2} - 1}. \end{aligned}$$

Combining this with (3) we get

$$\cosh 2R \cosh \frac{l}{2} + \sqrt{\cosh^2 2R \cosh^2 \frac{l}{2} - 1} \geq \frac{e^l + 3}{e^l - 1}. \tag{5}$$

The equation

$$x \cosh \frac{l}{2} + \sqrt{x^2 \cosh^2 \frac{l}{2} - 1} = \frac{e^l + 3}{e^l - 1}$$

has the solution

$$x_0 = \frac{e^{2l} + 2e^l + 5}{(\cosh \frac{l}{2})(e^l - 1)(e^l + 3)}.$$

Since the function $x \cosh \frac{l}{2} + \sqrt{x^2 \cosh^2 \frac{l}{2} - 1}$ is monotone increasing for $x \geq 1$, it follows from (4) that

$$\cosh 2R \geq x_0.$$

This is the conclusion of Proposition 10.1.

10.1

Let us define a function $V(x)$ for $x > 0$ by

$$V(x) = \frac{\pi x}{e^x - 1}\left(\frac{e^{2x} + 2e^x + 5}{2(\cosh \frac{x}{2})(e^x + 3)}\right) - \frac{\pi x}{2}.$$

Note that

$$\lim_{x \to 0} V(x) = \pi.$$

Since $\sinh^2 R = \frac{1}{2}(\cosh 2R - 1)$, Proposition 10.1 and the above formula (1) for the volume of a maximal tube now imply:

Lemma 10.3 *Let N be an orientable hyperbolic 3-manifold having* $\log 3$ *as a strong Margulis number. Let C be a closed geodesic in N, and let l denote its length. Then the maximal embedded tube about C has volume at least $V(l)$.*

The following result will permit us to put the information given by the above lemma in a more useful form.

Proposition 10.4 *The function $V(x)$ is monotonically decreasing for $x > 0$.*

Proof of 10.4: For $x \geq 0$ we set

$$f(x) = \frac{V(2x)}{\pi}.$$

We have

$$
\begin{aligned}
&(\cosh^2 x)(e^{4x} + 2e^{2x} - 3)^2 f'(x) = \\
&(\cosh x - x \sinh x)(e^{4x} + 2e^{2x} - 3)(e^{4x} + 2e^{2x} + 5) \\
&-(\cosh^2 x)(e^{4x} + 2e^{2x} - 3)^2 \\
&-32x(\cosh x)(e^{4x} + e^{2x}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\cosh x)(e^{4x} + 2e^{2x} - 3)^2 f'(x) &\leq (e^{4x} + 2e^{2x} - 3)(e^{4x} + 2e^{2x} + 5) \\
&\quad -(e^{4x} + 2e^{2x} - 3)^2 - 32x(e^{4x} + e^{2x}) \\
&= 8(e^{4x}(1 - 4x) + 2e^{2x}(1 - 2x) - 3).
\end{aligned}
$$

But the function $e^{4x}(1 - 4x) + 2e^{2x}(1 - 2x) - 3$ is negative-valued for $x > 0$, because it vanishes at 0 and its derivative $-xe^{4x} - 4xe^{2x}$ is negative for $x > 0$. Thus $f'(x) < 0$ for $x > 0$.

$\boxed{10.4}$

Combining Lemma 10.3 with Proposition 10.4, we immediately obtain the following result.

Corollary 10.5 *Let N be an orientable hyperbolic 3-manifold having* $\log 3$ *as a strong Margulis number. Let λ be a positive number, and suppose that N contains a closed geodesic of length at most λ. Then the maximal embedded tube about C has volume at least $V(\lambda)$. In particular the volume of N is at least $V(\lambda)$.*

Corollary 10.6 *Let N be an orientable hyperbolic 3-manifold which has first betti number at least* 3. *Let λ be a positive number, and suppose that N contains a closed geodesic of length at most λ. Then the volume of N is at least $V(\lambda)$.*

Proof of 10.6: We may assume that N has finite volume, as otherwise the assertion is trivial. It then follows from Corollary 8.8 that $\log 3$ is a Margulis number for N. The assertion now follows from Corollary 10.5.

$\boxed{10.6}$

We observed above that $\lim_{x\to 0} V(x) = \pi$. Thus Corollary 10.6 implies that if an orientable hyperbolic 3-manifold N has betti number at least 3 and contains a very short geodesic, the volume of N cannot be much less than π. Explicitly, we can say for example that if N contains a geodesic of length at most 0.1, then the volume of N is at least $V(0.1) = 2.906\ldots$. We already get non-trivial information from 10.3 and 10.4 if N contains a closed geodesic of length at most 1: in this case the results imply that N has volume at least $V(1) = 0.956\ldots$. This is greater than the smallest *known* volume $0.943\ldots$ of a closed orientable hyperbolic 3-manifold, which is in turn greater than the lower bound 0.92 established in [13] for the volume an arbitrary closed orientable hyperbolic 3-manifold of betti number at least 3.

In [12], Corollary 10.6 will be used as one ingredient in a proof that any orientable hyperbolic 3-manifold with betti number at least 3 has a volume exceeding that of the smallest known example, and hence that any smallest-volume orientable hyperbolic 3-manifold has betti number at most 2.

11 A volume bound for non-compact manifolds

Theorem 11.1 *Let $N = \mathbf{H}^3/\Gamma$ be a non-compact hyperbolic 3-manifold. If N has betti number at least 4, then N has volume at least π.*

Proof of 11.1: We may assume that N has finite volume. In this case N is homeomorphic to the interior of a compact 3-manifold M with non-empty boundary ∂M which consists of a finite collection of tori. Let T_1 be a torus in ∂M and let M_n be the result of the $(1, n)$ Dehn filling of M along T_1, in terms of some fixed system of coordinates on T_1 . Notice that M_n has betti number at least 3, since N had betti number at least 4.

Thurston's Hyperbolic Dehn Surgery Theorem (see [34]) guarantees that the interior of M_n admits a hyperbolic structure for all large enough n (see also Theorem E.5.1 in [4].) Let $N_n = \mathbf{H}^3/\Gamma_n$ be a hyperbolic manifold homeomorphic to the interior of M_n. Then we have $\operatorname{vol} N_n < \operatorname{vol} N$ for all n and $\operatorname{vol} N_n$ converges to $\operatorname{vol} N$ (see Theorem E.7.2 in [4].) Moreover, we may assume that Γ_n converges geometrically to Γ (see Theorem E.6.2 in [4].)

Let γ_n denote an element of Γ_n representing the shortest closed geodesic in N_n. Then, since Γ_n converges geometrically to Γ, N has k cusps and N_n has

$k-1$ cusps (for every n), we see that $l_n = l(\gamma_n)$ converges to 0 (see Theorem E.2.4 in [4]). By Corollary 10.3 we have

$$\operatorname{vol} N_n \geq V(l_n) = \frac{\pi l}{e^{l_n}-1}\left(\frac{e^{2l_n}+2e^{l_n}+5}{2(\cosh\frac{l_n}{2})(e^{l_n}+3)}\right) - \frac{\pi l_n}{2}.$$

Recall that $V(l_n)$ converges to π, since l_n converges to 0. We therefore have $\operatorname{vol} N \geq \pi$.

11.1

References

[1] J. W. Anderson, "Intersections of topologically tame subgroups of Kleinian groups," preprint.

[2] J. W. Anderson and R. D. Canary, "Cores of hyperbolic 3-manifolds and limits of Kleinian groups," in preparation.

[3] G. Baumslag and P. B. Shalen, "Groups whose three-generator subgroups are free," *Bull. Austral. Math. Soc.* **40**(1989), 163–174.

[4] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag Universitext, 1992.

[5] F. Bonahon, "Bouts des varietes hyperboliques de dimension 3," *Ann. of Math.* **124**(1986), pp. 71-158.

[6] K. Böröczky, "Packing of spheres in spaces of constant curvature," *Acta Math. Acad. Sci. Hungar.* **32**(1978), 243–261.

[7] B. H. Bowditch, "Geometrical finiteness for hyperbolic groups," *J. Funct. Anal.* **113**(1993), 245–317.

[8] R. D. Canary, "Ends of hyperbolic 3-manifolds," *Journal of the A.M.S*, **6**(1993), 1–35.

[9] R. D. Canary, "A covering theorem for hyperbolic 3-manifolds and its applications," to appear in *Topology.*

[10] R. D. Canary, D. B. A. Epstein and P. Green, "Notes on notes of Thurston", in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, pp. 3-92.

[11] R. D. Canary and Y. N. Minsky, "On limits of tame hyperbolic 3-manifolds," preprint.

[12] M. Culler, S. Hersonsky and P. B. Shalen, "On the betti number of the smallest hyperbolic 3-manifold," in preparation.

[13] M. Culler and P. B. Shalen, "Paradoxical decompositions,2-generator Kleinian groups, and volumes of hyperbolic 3-manifolds," *Journal of the A.M.S.* **5**(1992), 231–288.

[14] M. Culler and P. B. Shalen, "Hyperbolic volume and mod p homology," *Comment. Math. Helvitici* **68** (1993), 494-509.

[15] M. Culler and P. B. Shalen, "The volume of a hyperbolic 3-manifold with Betti number 2," *Proc. Amer. Math. Soc.* **120** (1994), 1281-1288.

[16] D. B. A. Epstein and A. Marden, "Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces" in *Analytical and Geometrical Aspects of Hyperbolic Spaces*, Cambridge University Press, 1987, pp. 113-253.

[17] W. Jaco and P. B. Shalen, *Seifert fibered spaces in 3-manifolds.* Mem. Amer. Math. Soc. **21**, no. 220, 1979.

[18] T. Jorgensen, "On discrete groups of Möbius transformations," Amer. J. of Math. **98**(1976), pp. 739-49.

[19] T. Jorgensen and A. Marden, "Algebraic and geometric convergence of Kleinian groups," *Math. Scand.* **66**(1990), 47–72.

[20] L. Keen, B. Maskit and C. Series, "Geometric finiteness and uniqueness of groups with circle packing limit sets," *J. Reine u. Angew. Math.* **436**(1993), 209–219.

[21] W. Magnus, "Über freie Faktorgruppen und freie Untergruppen gegebener Gruppen," *Monatsh. Math. u Phys.* **47** (1939), 307–313.

[22] A. Marden, "The geometry of finitely generated Kleinian groups," *Ann. of Math.* **99**(1974), pp. 383-462.

[23] A. Marden, "Schottky groups and circles," in *Contributions to Analysis*, Academic Press, New York, 1974.

[24] B. Maskit, "A characterization of Schottky groups," *J. d'Analyse Math.* **19**(1967), 227–230.

[25] B. Maskit, *Kleinian Groups,* Springer-Verlag, 1988.

[26] B. Maskit, "On free Kleinian groups," *Duke Math. J.* **48**(1981), 755–765.

[27] C. T. McMullen, "Cusps are dense," *Ann. of Math.* **133**(1991), 217–247.

[28] R. Meyerhoff, "Sphere-packing and volume in hyperbolic 3-space," *Comm. Math. Helv.* **61**(1986), 271–278.

[29] J. W. Morgan, "On Thurston's Uniformization Theorem for three-dimensional manifolds," in *The Smith Conjecture*, ed. by J. Morgan and H. Bass, Academic Press, 1984, 37–125.

[30] G. P. Scott, "Compact submanifolds of 3-manifolds," *J. London Math. Soc.* **7** (1973), 246–250.

[31] P. B. Shalen and P. Wagreich, "Growth rates, $\mathbf{Z}_p$-homology, and volumes of hyperbolic 3-manifolds," *Trans. A.M.S.* **331**(1992), 895–917.

[32] T. Soma, "Function groups in Kleinian groups," *Math. Annalen* **292**(1992), 181–190.

[33] J. R. Stallings, "On the Loop Theorem," *Ann. of Math.* (2) **72** (1960), 12–19.

[34] W. P. Thurston, *The Geometry and Topology of 3-manifolds*, lecture notes.

[35] T. Tucker, "On Kleinian groups and 3-manifolds of Euler characteristic zero." Unpublished.

[36] F. Waldhausen, "On irreducible 3-manifolds which are sufficiently large," *Annals of Math.* (2) **87** (1968), 56-88.