

4-free groups and hyperbolic geometry

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ABSTRACT

We give new information about the geometry of closed, orientable hyperbolic 3-manifolds with 4-free fundamental group. As an application, we show that such a manifold has volume greater than 3.44. This is in turn used to show that if M is a closed orientable hyperbolic 3-manifold such that $\text{vol } M \leq 3.44$, then $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \leq 7$.

1. Introduction

The theme of this paper is the interaction between the geometric structure of hyperbolic 3-manifolds and their topologically defined properties. While the basic techniques of the paper are suggested by those of Culler and Shalen [12] and [5, Section 9], the present paper involves deeper topological results and much more subtle combinatorial ideas than those used in [12] or [5].

The following definitions give a context for some of our main results.

DEFINITION 1.1. The *rank* of a finitely generated group is defined to be the minimal cardinality of a generating set of the group. A group Γ is said to be *k-free*, where k is a given positive integer, if every finitely generated subgroup of Γ having rank at most k is free.

The property of having k -free fundamental group can often be deduced from natural conditions on the topology of a closed, orientable hyperbolic 3-manifold M . For example, according to a theorem proved by Jaco and Shalen [17, Theorem VI.4.1], $\pi_1(M)$ is always 2-free unless M has a finite cover \tilde{M} such that $\pi_1(\tilde{M})$ has rank 2. As a second example, it follows from [14, Proposition 7.1] that if $k \geq 3$, and if $H_1(M; \mathbb{Z}_2)$ has \mathbb{Z}_2 -dimension at least $\max(3k - 4, 6)$, then either $\pi_1(M)$ is k -free, or M contains a closed incompressible surface of genus at most $k - 1$.

On the other hand, via the ‘ $\log(2k - 1)$ -theorem’ proved in [5, 11], together with the Marden conjecture proved in [1, 9], the property of being k -free interfaces directly with the geometry of the manifold. For example, according to [3, Corollary 4.2], which is deduced from results in [1, 9, 11], if $\pi_1(M)$ is 2-free, then $\log 3$ is a ‘Margulis number’ for M in a sense that will be reviewed below in Subsection 1.5. In particular, this implies that there exists a point of M where the injectivity radius is at least $(\log 3)/2$. Likewise, it follows from [2, Corollary 9.3] that a closed orientable hyperbolic 3-manifold with 3-free fundamental group contains a point where the injectivity radius is $(\log 5)/2$.

The methods used to prove these results in the 2-free and 3-free cases depend not only on the $\log(2k - 1)$ theorem but also on subtle topological and combinatorial arguments involving

certain coverings of \mathbb{H}^3 by cylinders. These arguments do not extend directly to give stronger information in the 4-free case.

In this paper, we introduce much more subtle topological and combinatorial methods that lead to a fundamental new fact about the geometry of a 3-manifold with 4-free fundamental group, which is stated as Theorem 1.4. This result will in turn be applied to prove a volume estimate for such manifolds, which is stated as Theorem 1.6. Furthermore, combining this with the results of Culler and Shalen [14], we obtain a new connection between volume and homology, stated in Theorem 1.7.

The statement of Theorem 1.4 requires a definition.

DEFINITION 1.2. We shall say that a point P of a hyperbolic 3-manifold M is λ -thin, where λ is a given positive number, if there is a homotopically non-trivial loop of length less than λ based at P . We say that a point is λ -thick if it is not λ -thin.

Equivalently, a point of M is λ -thick if and only if it is the center of a hyperbolic ball of radius $\lambda/2$ in M . Thus, [2, Corollary 9.3] is equivalent to the assertion that if M is closed and orientable and $\pi_1(M)$ is 3-free, then M contains a $(\log 5)$ -thick point.

DEFINITION 1.3. We shall say that a point $P \in M$ is λ -doubly thin if there are two loops of length less than λ based at P that represent non-commuting elements of $\pi_1(M, P)$. A point $P \in M$ will be said to be λ -semithick if it is not λ -doubly thin.

Section 3 contains a general discussion of properties of doubly thin and semithick points. We can now state the main result of this paper, which is proved in Section 5.

THEOREM 1.4. *Suppose that M is a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 4-free. Then M has a $\log 7$ -semithick point.*

The proof begins from the same basic point of view as that of Agol, Culler and Shalen [2, Corollary 9.3], but requires much deeper arguments. In the following sketch, we use terms that are well-known and are defined precisely in the body of the paper.

1.5. Consider a closed, orientable hyperbolic 3-manifold written as $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete and cocompact. Fix a positive number λ . For each maximal cyclic subgroup C of Γ generated by a (loxodromic) element of translation length less than λ , we consider the hyperbolic cylinder consisting of points of \mathbb{H}^3 that are displaced through a distance less than λ by some non-trivial element of C . This gives a family $\mathcal{Z} = \{Z_C\}$ of cylinders indexed by certain maximal cyclic subgroups. This family of cylinders covers \mathbb{H}^3 if and only if the manifold M contains no λ -thick point.

If Γ is 2-free, then any pair of distinct maximal cyclic subgroups generates a free group of rank 2. Thus, if we take $\lambda = \log 3$, then it follows from the ‘log 3-theorem’, the case $k = 2$ of the $\log(2k - 1)$ -theorem, that $Z_C \cap Z_{C'} = \emptyset$ whenever $C \neq C'$. This is expressed by saying that $\log 3$ is a Margulis number for M ; see Subsection 3.6.

As hyperbolic space cannot be covered by pairwise disjoint open cylinders, this shows that M has a $\log 3$ -thick point.

When Γ is k -free, for $k > 2$, this argument can be refined by assuming that \mathbb{H}^3 is covered by the cylinders in the family \mathcal{Z} and considering the nerve of this covering. The nerve is an

abstract simplicial complex K whose associated space is denoted by $|K|$. By definition each vertex of K corresponds to an index for the family \mathcal{Z} , that is, a certain maximal cyclic subgroup of Γ . To each open simplex σ of K we assign the subgroup $\Theta(\sigma) \leq \Gamma$, which is generated by the cyclic subgroups corresponding to the vertices of σ . If we take $\lambda = \log(2k - 1)$, then the $\log(2k - 1)$ -theorem, together with the k -free assumption and the assumption that \mathcal{Z} covers, would give a contradiction if there existed a simplex σ of dimension $k - 1$ with $\Theta(\sigma) = k$. Thus, for every simplex σ of dimension $k - 1$, $\Theta(\sigma)$ must be a free group of rank at most $k - 1$. For $k = 3$ the group-theoretic arguments given in [5] use this structure to derive a contradiction from the assumption that \mathcal{Z} covers \mathbb{H}^3 , implying that M has a $\log 5$ -thick point.

In the case being considered in this paper, where Γ is 4-free, we argue by contradiction and assume that M has no $\log 7$ -semithick point. This implies that the family \mathcal{Z} ‘doubly covers’ \mathbb{H}^3 in the sense that every point lies in at least two different cylinders in the family. The ‘double covering’ property, together with the contractibility of \mathbb{H}^3 , implies that the complement of the 0-skeleton $|K^0|$ relative to the 3-skeleton $|K^3|$ is connected and simply connected.

We know that $\Theta(\sigma)$ must be a free group of rank 2 or 3 for any open simplex σ contained in $|K^3| - |K^0|$. Set-theoretically we may therefore regard $|K^3| - |K^0|$ as the disjoint union of sets X_2 and X_3 , where X_k is the union of all open simplices σ of K_3 for which $\Theta(\sigma)$ has rank k .

As in the corresponding argument in [5], we have a natural simplicial action of Γ on K . For $k = 2, 3$, this induces an action on the set of connected components of X_k . A key step is showing that the stabilizer of any component of X_k under this action is a free group. This is approached by the same basic ideas involving the lattice of free subgroups in a k -free group that was used in [5], but it is much more difficult. In particular, it depends in a crucial way on the result recently proved by Kent [18] and independently by Louder and McReynolds [20] that if two rank-2 subgroups of a free group have a rank-2 intersection, then they have a rank-2 join. The relevant group theory, incorporating our application of the Kent–Louder–McReynolds result, is done in Section 4.

The actions of Γ on the sets of components of the X_k give rise to an action of Γ on an abstract bipartite graph T . The vertices of T are the components of X_2 and of X_3 . Two vertices of T are joined by an edge if they correspond to subsets of $|K^3| - |K^0|$, one of which is a component of X_2 and one a component of X_3 , and if some simplex contained in one of these sets is a face of a simplex contained in one of the other sets.

The 1-connectedness of $|K^3| - |K^0|$ implies that T is a tree. The group Γ acts on T without inversions. The vertex stabilizers under the action of Γ on T are stabilizers of components of the X_k under the action of Γ , and are therefore free. As Γ is isomorphic to the fundamental group of a closed hyperbolic 3-manifold, basic facts about actions of 3-manifold groups on trees, which we quote from [10], then lead to a contradiction, and Theorem 1.4 is proved.

Having established that interesting geometric properties of M follow from the assumption that $\pi_1(M)$ is 4-free, it is natural to ask whether these geometric conditions in fact imply stronger volume estimates than the ones that follow from the 2-free and 3-free hypotheses. Sections 6–13 address this question.

It is pointed out in [2, Corollary 9.3] that the existence of a $(\log 5)$ -thick point in a hyperbolic 3-manifold M with 3-free fundamental group leads to a lower bound of 3.08 for the volume of M . As we explain below, in the case where $\pi_1(M)$ is 4-free, much more intricate methods are needed in order to pass from the existence of a $(\log 7)$ -semithick point to an estimate for the volume of M . The volume estimate that we eventually obtain with these methods is summarized by the following theorem, which we prove in Section 13.

THEOREM 1.6. *Let M be a closed, orientable hyperbolic 3-manifold. If $\pi_1(M)$ is 4-free, then $\text{vol } M > 3.44$.*

To compare the relative strengths of the bounds 3.08 and 3.44, we note that it is a consequence of Thurston [23, Corollary 6.6.3] that the set \mathcal{V} of all volumes of closed, orientable hyperbolic 3-manifolds is a well-ordered subset of \mathbb{R} . There are 34 known volumes of cusped orientable hyperbolic 3-manifolds between 3.08 and 3.44; this can be shown to imply that the ordinal type of the set $\mathcal{V} \cap (3.08, 3.44)$ is at least 34ω . By contrast, there are only eight known volumes of cusped orientable hyperbolic 3-manifolds less than 3.08, and the best available lower bound for the ordinal type of the set $\mathcal{V} \cap (0, 3.08)$ is $8\omega + 6$. Thus, from the point of view of ordinal numbers, the lower bound of 3.44 may be regarded as being more than four times stronger than the lower bound of 3.08.

Section 13 also contains a proof of the following result, which establishes a new connection between volume and homology.

THEOREM 1.7. *Let M be a closed orientable hyperbolic 3-manifold such that $\text{vol } M \leq 3.44$. Then $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \leq 7$.*

This result is analogous to [14, Theorem 1.2], which asserts that if $\text{vol } M \leq 3.08$, then $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \leq 5$.

1.8. We now sketch the proof of Theorem 1.6. In the case where M contains a $\log 7$ -thick point, the sphere-packing results established in [7] give a lower bound of more than 5.7 for $\text{vol } M$. If M contains a $\log 7$ -semithick point but contains no $\log 7$ -thick point, a continuity argument produces a point $P \in M$ that is $\log 7$ -semithick, but is not λ -semithick for any $\lambda > \log 7$. All loops based at the point P that have length less than $\log 7$ represent elements of a single maximal cyclic subgroup C of $\pi_1(M, P)$. (The relevant properties of the point P are summarized in Corollary 5.14, in somewhat different and more technically convenient language.)

If P is such a point, let $N \subset M$ denote the metric neighborhood of P with radius $(\log 7)/2$, and let D denote the minimal length of a loop based at P , which represents a generator of the maximal cyclic group C . Lemma 7.6 gives a lower bound for $\text{vol } N$ in terms of D , subject to a suitable lower bound δ on the length of the shortest geodesic in M .

To establish this lower bound, one begins by interpreting $\text{vol } N$ as the volume of a set X obtained by removing finitely many caps from a ball $B \subset \mathbb{H}^3$ of radius $(\log 7)/2$ (see Proposition 6.2). Each of these caps is associated with a non-trivial element γ of C . If we write $M = \mathbb{H}^3/\Gamma$ as above, take B to be centered at a point \tilde{P} in the pre-image of P under the quotient map, and identify C with a subgroup of the deck transformation group, then the cap corresponding to γ is one ‘half’ of the intersection of B with $\gamma \cdot B$. Thus, an element γ of C corresponds to a cap in the construction of X only if $B \cap \gamma \cdot B \neq \emptyset$. If P happens to lie close enough to the closed geodesic corresponding to the cyclic group C , then the set X is obtained by removing two disjoint caps from the ball; in general, however, there may be more than two caps, and some of them may overlap.

Imposing a lower bound on the shortest geodesic in M gives a lower bound on the translation length of a generator of C . When this translation length is sufficiently large, only relatively small powers of the generator of C can correspond to caps in the construction of X . Furthermore, the lower bound for the translation length of the generator gives lower bounds for the displacement of \tilde{P} under these powers of the generator, and this controls the volumes of the caps and of their intersections. The details involve quite a bit of hyperbolic trigonometry.

In order to apply Lemma 7.6 for numerical estimates, we need to be able to calculate volumes of caps and of intersections of caps. This is the subject of the Appendix.

The lower bound for $\text{vol } N$ provided by Lemma 7.6 is in particular a lower bound for $\text{vol } M$, but this lower bound decreases as D decreases, and by itself it turns out to be insufficient

to give the conclusion of Theorem 1.6 except for rather large values of D . To compensate for this, we obtain a lower bound for $\text{vol}(M - N)$, which increases as D decreases, and use $\text{vol} N + \text{vol}(M - N)$ as a lower bound for $\text{vol} M$. To obtain a lower bound for $\text{vol}(M - N)$, we exploit the fact that there are non-commuting elements of $\pi_1(M, P)$ represented by loops of length $\log 7$ and D , and again use the hypothesis that $\pi_1(M)$ is 4-free. These pieces of information are used, via results proved in [15] and adapted to the context of this paper in Section 8, to show that there is a point $Y \in M$ whose distance from P is ρ , where the quantity ρ is explicitly defined as a monotonically decreasing function of D . The results from [15] also guarantee that we may take Y to lie in the μ -thick part of M if μ is any Margulis number for M . (One could, for example, take μ to be $\log 3$, which by, Agol, Culler and Shalen [3, Corollary 4.2], is a Margulis number for any closed orientable hyperbolic 3-manifold with 2-free fundamental group.)

When ρ is sufficiently large, it is easy to use the existence of the point Y to give a non-trivial lower bound for $\text{vol}(M - N)$. For example, if $\rho > (\log 7)/2$, it follows from the triangle inequality that the $(\rho - (\log 7)/2)$ -neighborhood of Y is contained in $M - N$. If $(\rho - (\log 7)/2) < \mu/2$, this neighborhood is a hyperbolic ball and its volume, which can be calculated explicitly in terms of ρ , is a lower bound for $\text{vol}(M - N)$. If $(\rho - (\log 7)/2) > \mu/2$, then the volume of $M - N$ is bounded below by the volume of a hyperbolic ball of radius $\mu/2$.

In order to obtain the lower bound asserted in Theorem 1.6, this straightforward method for bounding $\text{vol}(M - N)$ from below must be refined in several ways. One of these involves the choice of the Margulis constant μ . We mentioned above that $\log 3$ is a Margulis number for M provided that $\pi_1(M)$ is 2-free. In Section 10, we use the methods of Culler and Shalen [15] to give a stronger result when $\pi_1(M)$ is k -free for a given $k > 2$ and the diameter Δ of M is known. Corollary 10.3 asserts that a certain quantity defined as a function of k and Δ , which is monotonically increasing in k and monotonically decreasing in Δ , is a Margulis number. By combining this with [2, Corollary 9.3], we show that if $\pi_1(M)$ were 4-free and if M had volume at most 3.44, then $\mu = 1.119$ would be a Margulis number for M . We may therefore use this improved choice of μ in the proof of Theorem 1.6.

A second refinement of the straightforward lower bound for $\text{vol}(M - N)$ is based on the sphere-packing arguments given in [7]. If $(\rho - (\log 7)/2) > h$, where h denotes the distance from the barycenter to a vertex of a regular hyperbolic tetrahedron with sides of length μ , then $M - N$ contains the metric neighborhood N' of radius h about Y , and arguments in [7] give an explicit lower bound for $\text{vol} N'$, which is significantly greater than the volume of a ball of radius $\mu/2$. While this lower bound applies to any $\mu/2$ -thick point in a hyperbolic manifold, we have more information in the present situation: not only is Y a $\mu/2$ -thick point, but M contains the point P whose distance from Y is ρ , a number that is often considerably larger than h . In Section 9, we show that the lower bounds given in [7] can be improved using the existence of such a 'distant point' P .

A third refinement of the lower bound for $\text{vol}(M - N)$ is based on an observation that was already used in [15]. When $\rho - (\log 7)/2 > h$, there is a point Y' of M whose minimum distances from both the $(\log 7)/2$ -neighborhood of P and the h -neighborhood of Y are at least $\rho - ((\log 7)/2 + h)/2$. It is often possible to take such a point Y' to be μ -thick, and thus to obtain an additional contribution to $\text{vol}(M - N)$ from a suitable metric neighborhood of Y' in the same way as from a metric neighborhood of Y .

We mentioned that our method for bounding $\text{vol} N$ from below requires a lower bound δ for the length of the shortest geodesic in M . It turns out that if we take $\delta = 0.58$, then the methods sketched above give the lower bound 3.44 for $\text{vol} M$. We have lower bounds for both $\text{vol} N$ and $\text{vol}(M - N)$ in terms of the parameter D , and their sum is a function of D that is a lower bound for $\text{vol} M$, provided that no geodesic in M has length less than 0.58. In Section 12, we prove by a rigorous sampling argument that this function is bounded below by 3.44 on the relevant range.

In the case where M does contain a short geodesic c of length $l < 0.58$, the argument does not use Theorem 1.4, but it does use many of the other ingredients described above. We fix a point P on the closed geodesic c and define $N \subset M$ to be the $\lambda/2$ -neighborhood of P , where λ is the length of the shortest loop based at P that does not represent an element of the cyclic subgroup of $\pi_1(M, P)$ determined by c . As P lies on c , $\text{vol} N$ is equal to the volume of a set obtained by removing two disjoint caps from a ball of radius $\lambda/2$ in \mathbb{H}^3 ; the volumes of the caps are determined by λ and the length l of c . Lemma 7.6 again applies to give a lower bound for $\text{vol}(M - N)$; the quantity λ plays the role that $\log 7$ played in the earlier application, and l plays the role of D . We now have a lower bound for $\text{vol} M$ involving the two parameters λ and l . In Section 13, we prove by a rigorous sampling argument that this function is bounded below by 3.44 on the relevant range of values of l and λ , except possibly when $l < 0.003$. In the latter case a different method, based on estimates for tube radius established in [5] and tube-packing estimates proved by Przeworski [21], gives the desired lower bound of 3.44 for $\text{vol} M$.

2. General conventions

2.1. Let S be a subset of a group Γ . We shall denote by $\langle S \rangle$ the subgroup of Γ generated by S . We shall say that S is *independent* (or that the elements of S are independent) if $\langle S \rangle$ is free on the generating set S .

It is a basic fact in the theory of free groups [19, vol. 2, p. 59] that a finite set $S \subset \Gamma$ is independent if and only if $\langle S \rangle$ is free of rank $|S|$. This fact will often be used without explicit mention.

2.2. For any positive real number R , let us define the *cylinder of radius R* about a line L in \mathbb{H}^3 to be the set of all points whose distance from L is less than r .

2.3. A real-valued function $f(x_1, \dots, x_n)$ of n variables will be said to be *monotone increasing or decreasing in the variable x_i* if $f(x_1, \dots, x_i, \dots, x_n) \leq f(x_1, \dots, x'_i, \dots, x_n)$ whenever $x_i < x'_i$ or $x_i > x'_i$ and both $(x_1, \dots, x_i, \dots, x_n)$ and $(x_1, \dots, x'_i, \dots, x_n)$ lie in the domain of f .

2.4. Let γ be a loxodromic isometry of \mathbb{H}^3 . Let l denote the translation length of γ and let θ denote its twist angle. If $z \in \mathbb{H}^3$ is a point, and if we set $D = \text{dist}(z, \gamma \cdot z)$, then the discussion in [13, Subsection 1.3] shows that the distance from z to the axis of γ is equal to $\omega(l, \theta, D)$, where ω is the function defined for $0 \leq l \leq D$ and $\theta \in \mathbb{R}$ by

$$\omega(l, \theta, D) \doteq \text{arcsinh} \left(\left(\frac{\cosh D - \cosh l}{\cosh l - \cos \theta} \right)^{1/2} \right). \quad (2.1)$$

The formula (2.1) shows that, for any θ and any $l > 0$, the function $\omega(l, \theta, \cdot)$ is a continuous, monotonically increasing function on (l, ∞) .

If λ is any positive real number, we shall denote by $Z_\lambda(\gamma)$ the set of points $z \in \mathbb{H}^3$ such that $\text{dist}(z, \gamma \cdot z) < \lambda$. Then $Z_\lambda(\gamma)$ is empty if the translation length l of γ is at least λ . If $l < \lambda$, then, in view of the monotonicity of ω in the third variable, $Z_\lambda(\gamma)$ is a cylinder of radius $\omega(l, \theta, \lambda)$ about the axis of γ . Furthermore, the continuity of ω in the third variable implies that if $l < \lambda$, then

$$\overline{Z_\lambda(\gamma)} = \{z \in \mathbb{H}^3 : \text{dist}(z, \gamma \cdot z) \leq \lambda\}. \quad (2.2)$$

2.5. Let C be a cyclic subgroup of $\text{Isom}_+(\mathbb{H}^3)$ generated by a loxodromic element γ_0 . The non-trivial elements of C have a common axis, which we denote by A_C . Now if λ is any positive real number, then it follows from the discussion in Subsection 2.4 that the set $Z_\lambda(C) \doteq \bigcup_{1 \neq \gamma \in C} Z_\lambda(\gamma)$ is empty if γ_0 has translation length at least λ , and is a cylinder about A_C if γ_0 has translation length less than λ . If γ_0 has length l and twist angle θ , then the radius of the cylinder $Z_\lambda(C)$ is

$$\Omega_C(\lambda) \doteq \max_{1 \leq n \leq \lfloor \lambda/l \rfloor} \omega(nl, n\theta, \lambda). \quad (2.3)$$

2.6. If Γ is a discrete subgroup of $\text{Isom}_+(\mathbb{H}^3)$, then we shall denote the quotient projection $\mathbb{H}^3 \rightarrow \mathbb{H}^3/\Gamma$ by q_Γ . If P is a point of $M = \mathbb{H}^3/\Gamma$, then each point \tilde{P} of $q_\Gamma^{-1}(P)$ determines an isomorphic identification of $\pi_1(M, P)$ with Γ .

2.7. If (X, d) is a metric space (for example, a hyperbolic manifold with the usual distance function) and A is a bounded subset of X , we define the *extrinsic diameter* of A in X to be the quantity $\sup_{x, y \in A} d(x, y)$.

If r is a non-negative real number and P is a point in X , then we shall let

$$N_X(P, r) \doteq \{x \in X \mid d(x, P) < r\}$$

denote the metric ball with radius r and center P . We abbreviate this as $N(P, r)$ when it is clear which metric space (X, d) is meant. Note that we are using the term ‘ball’ to mean an open ball.

2.8. Let M be a closed, orientable hyperbolic 3-manifold, and λ be a positive number. The notion of a λ -thin point of M was defined in Definition 1.2. We denote the set of λ -thin points of M by $M_{\text{thin}}(\lambda)$, and we set $M_{\text{thick}}(\lambda) = M - M_{\text{thin}}(\lambda)$.

Suppose we write $M = \mathbb{H}^3/\Gamma$, where Γ is a discrete subgroup of $\text{Isom}_+(\mathbb{H}^3)$. Let P be a point of M and let \tilde{P} be a point of $q_\Gamma^{-1}(P)$. Then P is a λ -thin point if and only if we have $\text{dist}_{\mathbb{H}^3}(\gamma \cdot \tilde{P}, \tilde{P}) < \lambda$ for some $\gamma \in \Gamma - \{1\}$. Equivalently, we have $P \in M_{\text{thin}}(\lambda)$ if and only if P lies in $Z_\lambda(C)$ for some maximal cyclic subgroup C of Γ .

3. Doubly thin and semithick points

The notion of a λ -doubly thin or λ -semithick point of a hyperbolic manifold, where λ is a given positive number, was defined in Definition 1.3.

PROPOSITION 3.1. *Let M be a closed, orientable hyperbolic 3-manifold. For every point $P \in M$ there is a unique number $\mathfrak{N} > 0$ such that P is λ -doubly thin for every $\lambda > \mathfrak{N}$ and is λ -semithick for every λ with $0 < \lambda \leq \mathfrak{N}$.*

Proof. Let $\mathcal{S} \subset \pi_1(M, P) \times \pi_1(M, P)$ denote the set of all non-commuting pairs of elements of $\pi_1(M, P)$. As M is closed, $\pi_1(M, P)$ is non-abelian and hence $\mathcal{S} \neq \emptyset$. For each $g \in \pi_1(M, P)$, there is a unique loop α_g of minimal length among all representatives of the based homotopy class g . The set $\mathcal{L} = \{\text{length } \alpha_g : g \in \pi_1(M, P)\}$ is discrete, and hence the set $\mathcal{L}' = \{\max(\text{length } \alpha_g, \text{length } \alpha_h) : (g, h) \in \mathcal{S}\} \subset \mathcal{L}$ has a least element \mathfrak{N} . We have $\mathfrak{N} > 0$ because non-commuting elements of $\pi_1(M, P)$ must be non-trivial. It is immediate from the definitions that P is λ -doubly thin for every $\lambda > \mathfrak{N}$ and is λ -semithick for every λ with $0 < \lambda \leq \mathfrak{N}$. Uniqueness is obvious. \square

NOTATION 3.2. If M is a closed, orientable hyperbolic 3-manifold, then, for every point $P \in M$, we denote by $\mathfrak{N}_M(P)$ the number \mathfrak{N} given by Proposition 3.1. Thus, \mathfrak{N}_M is a positive-valued function defined on M .

PROPOSITION 3.3. *If M is a closed, orientable hyperbolic 3-manifold, the function \mathfrak{N}_M is 2-Lipschitz; that is,*

$$|\mathfrak{N}_M(P) - \mathfrak{N}_M(Q)| \leq 2 \operatorname{dist}_M(P, Q),$$

for all $P, Q \in M$. In particular, \mathfrak{N}_M is continuous.

Proof. By symmetry it is enough to show that $\mathfrak{N}_M(Q) - \mathfrak{N}_M(P) \leq 2 \operatorname{dist}_M(P, Q)$ for all points $P, Q \in M$. Set $d = \operatorname{dist}_M(P, Q)$ and $\lambda = \mathfrak{N}_M(P)$. Then P is λ -doubly thin, so that there are loops α and β based at P , both of length less than λ , and representing non-commuting elements of $\pi_1(M, P)$. Let ζ be a path of length d from Q to P . Then $\zeta \star \alpha \star \bar{\zeta}$ and $\zeta \star \beta \star \bar{\zeta}$ have length less than $\lambda + 2d$ and representing non-commuting elements of $\pi_1(M, Q)$. It follows that Q is $(\lambda + 2d)$ -doubly thin, that is, that $\mathfrak{N}_M(Q) \leq \lambda + 2d = \mathfrak{N}_M(P) + 2 \operatorname{dist}_M(P, Q)$. \square

3.4. If M is a closed, orientable hyperbolic 3-manifold, then we may write $M = \mathbb{H}^3/\Gamma$ where $\Gamma \leq \operatorname{Isom}_+(\mathbb{H}^3)$ is discrete and torsion-free. As M is closed, Γ is purely loxodromic. Hence, each non-trivial element γ of Γ lies in a unique maximal cyclic subgroup, which is the centralizer of γ . In particular, non-trivial elements that lie in distinct maximal cyclic subgroups do not commute.

We denote by $\mathcal{C}(\Gamma) = \mathcal{C}(M)$ the set of all maximal cyclic subgroups of Γ . If λ is a positive number, then we denote by $\mathcal{C}_\lambda(\Gamma) = \mathcal{C}_\lambda(M)$ the subset consisting of all $C \in \mathcal{C}(\Gamma)$ such that a generator of C has translation length less than λ . It follows from Subsection 2.5 that $Z_\lambda(C)$ is a cylinder of radius $\Omega_C(\lambda)$ if $C \in \mathcal{C}_\lambda(\Gamma)$ and is empty if $C \in \mathcal{C}(\Gamma) - \mathcal{C}_\lambda(\Gamma)$.

The discreteness of the group Γ implies that the family $(Z_\lambda(\gamma))_{1 \neq \gamma \in \Gamma}$ is locally finite. As for each $C \in \mathcal{C}_\lambda(\Gamma)$, we have $Z_\lambda(C) = Z_\lambda(\gamma)$ for some $\gamma \in C - \{1\}$, the family $(Z_\lambda(C))_{C \in \mathcal{C}_\lambda(\Gamma)}$ is also locally finite.

PROPOSITION 3.5. *Let $M = \mathbb{H}^3/\Gamma$ be a closed, orientable hyperbolic 3-manifold, let P be a point of M , let \tilde{P} be a point of $q_\Gamma^{-1}(P)$, and let λ be a positive number. Then P is a λ -doubly thin point of M if and only if \tilde{P} belongs to the set*

$$\mathcal{D} \doteq \bigcup_{\substack{C, C' \in \mathcal{C}_\lambda(\Gamma) \\ C \neq C'}} Z_\lambda(C) \cap Z_\lambda(C').$$

Proof. We use the point \tilde{P} to identify $\pi_1(M, P)$ with Γ as in Subsection 2.6. If $\tilde{P} \in \mathcal{D}$, then there exist $C, C' \in \mathcal{C}_\lambda(\Gamma)$, with $C \neq C'$, such that \tilde{P} lies in both $Z_\lambda(C)$ and $Z_\lambda(C')$. Hence, for some $x \in C - \{1\}$ and $x' \in C' - \{1\}$, we have $\operatorname{dist}(x \cdot \tilde{P}, \tilde{P}) < \lambda$ and $\operatorname{dist}(x' \cdot \tilde{P}, \tilde{P}) < \lambda$. Thus, $x, x' \in \pi_1(M, P)$ are represented by loops of length less than λ . As C and C' are distinct maximal cyclic subgroups of Γ , the elements x and x' do not commute. By definition it follows that P is λ -doubly thin.

Conversely, suppose that P is λ -doubly thin, so that there are non-commuting elements $x, x' \in \pi_1(M, P)$ represented by loops of length less than λ . If we regard x and x' as elements of Γ , then they lie in distinct maximal cyclic subgroups $C, C' \in \mathcal{C}_\lambda(\Gamma)$. We have $\operatorname{dist}(x \cdot \tilde{P}, \tilde{P}) < \lambda$ and $\operatorname{dist}(x' \cdot \tilde{P}, \tilde{P}) < \lambda$, so that \tilde{P} lies in both $Z_\lambda(C)$ and $Z_\lambda(C')$ and hence $\tilde{P} \in \mathcal{D}$. \square

3.6. As in [15], we define a *Margulis number* for a closed, orientable hyperbolic 3-manifold M to be a positive number μ such that, for any two distinct subgroups $C, C' \in \mathcal{C}_\mu(\Gamma)$, we have $Z_\mu(C) \cap Z_\mu(C') = \emptyset$. If μ is a Margulis number for M , then the components of $M_{\text{thin}}(\mu)$ are tubes. In particular, $M_{\text{thick}}(\mu)$ is connected and non-empty.

PROPOSITION 3.7. *Let M be a closed, orientable hyperbolic 3-manifold, and let μ be a positive number. Then the following conditions are equivalent:*

- (1) $\mu > 0$ is a Margulis number for M ;
- (2) M has no μ -doubly thin points;
- (3) μ is a lower bound for the function \mathfrak{N}_M .

Proof. The equivalence of (1) and (2) follows from Proposition 3.5. The equivalence of (2) and (3) follows from Proposition 3.1 and the definition of \mathfrak{N}_M . \square

3.8. Let M be a closed, orientable hyperbolic 3-manifold. For every point P of M , we denote by ℓ_P the smallest length of any homotopically non-trivial loop in M based at P . The subgroup of $\pi_1(M, P)$ generated by all homotopy classes that contain loops of length ℓ_P will be denoted by \mathcal{L}_P .

We shall denote by \mathfrak{G}_M the set of all points $P \in M$ such that \mathcal{L}_P is cyclic. If $P \in \mathfrak{G}_M$, then there is a unique maximal cyclic subgroup containing \mathcal{L}_P ; we shall denote this subgroup by C_P .

We shall define real-valued functions D_M and \mathfrak{s}_M with domain \mathfrak{G}_M as follows. For any $P \in \mathfrak{G}_M$ we define $D_M(P)$ to be the minimal length of a loop based at P that represents a generator of C_P . (Note that $D_M(P)$ is not necessarily equal to ℓ_P , as the shortest homotopically non-trivial loop based at P may represent a proper power of a generator of C_P .)

For any $P \in \mathfrak{G}_M$ we define $\mathfrak{s}_M(P)$ to be the smallest length of any loop in M based at P that does *not* represent an element of the cyclic group C_P . In particular, we have $\mathfrak{s}_M(P) > \ell_P$.

PROPOSITION 3.9. *Let P be a point of a closed hyperbolic 3-manifold M , and set $\mathfrak{N} = \mathfrak{N}_M(P)$. Then either*

- (i) P is an \mathfrak{N} -thick point of M or
- (ii) $P \in \mathfrak{G}_M$ and $\mathfrak{s}_M(P) = \mathfrak{N}$.

Proof. Suppose that P is not an \mathfrak{N} -thick point of M . Then, by definition, there is a homotopically non-trivial loop of length less than \mathfrak{N} based at P . In particular, if α is the shortest homotopically non-trivial loop based at P , then we have $\ell_P = \text{length } \alpha < \mathfrak{N}$. If P did not lie in \mathfrak{G}_M , then there would be two loops based at P having length ℓ_P and representing non-commuting elements of $\pi_1(M, P)$. Hence, P would be a λ -doubly thin point of M for every $\lambda > \ell_P$. This is impossible because $\ell_P = \text{length } \alpha < \mathfrak{N} = \mathfrak{N}_M(P)$. Hence, $P \in \mathfrak{G}_M$.

Now, by the definition of $\mathfrak{s}_M(P)$, there is a loop β of length $\mathfrak{s}_M(P)$ based at P such that $[\beta] \notin C_P$. It follows that $[\alpha]$ and $[\beta]$ do not commute. As $\text{length } \alpha = \ell_P < \mathfrak{s}_M(P) = \text{length } \beta$, the point P is λ -doubly thin for every $\lambda > \mathfrak{s}_M(P)$. Hence, $\mathfrak{s}_M(P) \geq \mathfrak{N}$.

Now assume that $\mathfrak{s}_M(P) > \mathfrak{N}$. Then, as $\mathfrak{N} = \mathfrak{N}_M(P)$, the point P is $\mathfrak{s}_M(P)$ -doubly thin. Hence, there are loops β and β' based at P having length less than $\mathfrak{s}_M(P)$ and representing non-commuting elements of $\pi_1(M, P)$. In particular, $[\beta]$ and $[\beta']$ cannot both lie in C_P . After re-labeling if necessary, we may assume that $[\beta] \notin C_P$, and hence that $\text{length } \beta \geq \mathfrak{s}_M(P)$. This is a contradiction, which shows that $\mathfrak{N} = \mathfrak{s}_M(P)$. \square

4. Structure of 4-free groups

The results of this section are analogous to those of Culler and Shalen [12, Section 4], but are deeper because they require the recent group-theoretical results established in [18, 20].

4.1. We will follow the conventions of Spanier [22] regarding simplicial complexes and their associated topological spaces. By a *simplicial complex*, we shall mean a set V , whose elements are called *vertices*, together with a collection S of non-empty finite subsets of V , called *simplices*, such that every singleton subset of V is a simplex and every non-empty subset of a simplex is a simplex. For $q \geq 0$, a simplex consisting of $q + 1$ vertices will be called a q -simplex and will be said to have dimension q . The *dimension* of a simplicial complex is defined to be the supremum of the dimensions of its simplices.

We emphasize that we do not assume simplicial complexes to be locally finite, and indeed in the main application, in Section 5, the complexes that arise are locally infinite.

The q -dimensional skeleton of a simplicial complex K is the subcomplex K^q consisting of all p -simplices of K for $p \leq q$. (In particular, K^{-1} is the empty complex.)

The space $|K|$ of a simplicial complex K is given the coherent (that is, weak) topology (see [22, 3.1.14]). If L is a subcomplex of K , then $|L|$ is naturally identified with a subspace of $|K|$. For each q -simplex s of K , the space $|s|$, as a subspace of $|K|$, is called a *closed simplex* and is homeomorphic to the standard simplex in \mathbb{R}^{q+1} . In the coherent topology, a subset of $|K|$ is closed (or open) if and only if its intersection with $|s|$ is closed (or open) for every simplex s of K .

If s is a simplex in a simplicial complex K , then the set of all proper faces of s is a subcomplex denoted by \dot{s} . For each simplex s of K , the subspace $|s| - |\dot{s}|$ of $|K|$ will be called an *open simplex* in $|K|$. If t is a face of s , then the open simplex $|t| - |\dot{t}|$ will be said to be a face of the open simplex $|s| - |\dot{s}|$.

We shall say that a subset X of $|K|$ is *saturated* if X is a union of open simplices. A saturated subset will always be understood to be endowed with the subspace topology as a subset of $|K|$. Note that if X is saturated in $|K|$, then the connected components of X are also saturated.

If v is a vertex of a simplicial complex K , then we use the notation $\text{st}_K(v)$ to denote the open star of v in $|K|$ (see [22, 3.1.22]). The link of v is a subcomplex L_v of K , and we use the notation $\text{lk}_K(v)$ to denote the closed set $|L_v| \subset |K|$.

The first barycentric subdivision of a simplicial complex K is denoted by K' , and we shall identify the spaces $|K|$ and $|K'|$.

We define a *graph* to be a simplicial complex of dimension at most 1. (Thus, a graph has no loops or multiple edges.)

4.2. We recall some definitions from [12].

A group Γ will be said to have *local rank at most k* , where k is a positive integer, if every finitely generated subgroup of Γ is contained in a subgroup of rank at most k . The local rank is the smallest integer k with this property, and is defined to be ∞ if no such integer exists. Note that, for a finitely generated group, the local rank is equal to the rank.

Let Γ be a group. By a Γ -*labeled complex*, we mean an ordered pair $(K, (C_v)_v)$, where K is a simplicial complex and $(C_v)_v$ is a family of infinite cyclic subgroups of Γ indexed by the vertices of K . If $(K, (C_v)_v)$ is a Γ -labeled complex, then, for any saturated subset W of $|K|$, we shall denote by $\Theta(W)$ the subgroup of Γ generated by all the groups C_v , where v ranges over the vertices of open simplices contained in W .

The following simple group-theoretic lemma is needed for the next two results. The notion of a k -free group was defined in Definition 1.1.

LEMMA 4.3. *Suppose that Γ is a k -free group for a given integer $k > 0$, that $R \leq \Gamma$ is a subgroup of rank less than k , that C is a cyclic subgroup of Γ , and that the rank of $\langle R \cup C \rangle$ is strictly greater than that of R . Then C is infinite cyclic, and $\langle R \cup C \rangle$ is the free product of the subgroups R and C .*

Proof. Let $r < k$ denote the rank of R . As Γ is k -free, R is a free group and has a basis $\{x_1, \dots, x_r\}$. If t denotes a generator of C , then $\langle R \cup C \rangle$ is generated by x_1, \dots, x_r, t . In particular, its rank is at most $r + 1$. As by hypothesis $\langle R \cup C \rangle$ has rank greater than r , its rank is exactly $r + 1$. As $r + 1 \leq k$, the group $\langle R \cup C \rangle$ is free, and hence the set $\{x_1, \dots, x_r, t\}$ is a basis (see Subsection 2.1). The conclusions follow. \square

PROPOSITION 4.4. *Let k and r be integers with $k > r \geq 2$ and $k \geq 4$, let Γ be a k -free group, and let $(K, (C_v)_v)$ be a Γ -labeled complex. Let W be a connected, saturated subset of $|K|$ such that $\Theta(\sigma)$ has rank exactly r for every open simplex $\sigma \subset W$. Suppose in addition that either*

- (i) *there is an integer $n \geq 2$ such that every open simplex contained in W has dimension n or $n - 1$ or*
- (ii) *$r = 2$.*

Then $\Theta(W)$ has local rank at most r .

Proof. We must show that if E is a finitely generated subgroup of $\Theta(W)$, then E is contained in some finitely generated subgroup of $\Theta(W)$ whose rank is at most r . It follows from the definition of $\Theta(W)$ that $E \leq \Theta(W_0)$ for some saturated subset $W_0 \subset W$, which is a union of finitely many open simplices. As W is connected, there is a connected (non-empty) saturated set $V \supset W_0$ that is also a union of finitely many open simplices. We shall prove the proposition by showing that under either of the hypotheses (i) or (ii), the finitely generated group $\Theta(V) \geq E$ has rank at most r .

By connectedness we may list the open simplices contained in V as $\sigma_0, \dots, \sigma_m$, where $m \geq 0$, and, for each index j with $0 < j \leq m$, there is an index l with $0 \leq l < j$ such that either σ_l is a proper face of σ_j , or σ_j is a proper face of σ_l . We set $V_j = \sigma_0 \cup \dots \cup \sigma_j$ for each j with $0 \leq j \leq m$. We shall show by induction on j , for $j = 0, \dots, m$, that $\Theta(V_j)$ has rank at most r .

By hypothesis, $\Theta(\sigma_j)$ has rank r for $0 \leq j \leq m$. In particular, $\Theta(V_0) = \Theta(\sigma_0)$ has rank r ; this is the base case of the induction. Now suppose that $0 < j \leq m$ and that $\Theta(V_{j-1})$ has rank at most r . According to the rule for ordering the open simplices contained in V , we may fix an index l with $0 \leq l < j$ such that either σ_l is a proper face of σ_j , or σ_j is a proper face of σ_l . If σ_j is a face of σ_l , then $\Theta(V_j) = \Theta(V_{j-1})$, and the induction step is trivial.

Now suppose that σ_l is a proper face of σ_j . Set $P = \Theta(V_{j-1})$, $Q = \Theta(\sigma_j)$, and $R = \Theta(\sigma_l)$. Then $R \leq P \cap Q$ and $\Theta(V_j) = \langle P \cup Q \rangle$. Then P has rank at most r , whereas Q and R have rank exactly r . We are required to show that $\langle P \cup Q \rangle$ has rank at most r .

First consider the case in which hypothesis (i) holds. Then as σ_l is a proper face of σ_j , we must have $\dim \sigma_j = n$ and $\dim \sigma_l = n - 1$. Let v denote the vertex of σ_j that is not a vertex of σ_l , and set $C = C_v$. Then C is an infinite cyclic group, and $Q = \langle R \cup C \rangle$. Hence, $\langle P \cup Q \rangle = \langle P \cup C \rangle$.

We need to show that $\langle P \cup C \rangle$ has rank at most r . Assume to the contrary that its rank is greater than r . As Γ is k -free, with $k > r$, and P has rank at most r , it then follows from Lemma 4.3 that $\langle P \cup C \rangle$ is the free product of the subgroups P and C . In particular, as $R \leq P$, it follows that $Q = \langle R \cup C \rangle$ is the free product of the subgroups R and C . But this is impossible as Q and R have rank exactly r (and are free as $r < k$), whereas C is infinite cyclic. This completes the induction step in this case.

We now turn to the case in which hypothesis (ii) holds. In this case, Q and R have rank 2, whereas P has rank at most 2 and contains R . As a group of rank at most 1 cannot have a subgroup of rank 2, the rank of P must be equal to 2 as well. As P and Q are of rank 2, the group $\langle P \cup Q \rangle$ certainly has rank at most 4, and as $4 \leq k$, it follows that $\langle P \cup Q \rangle$ is free. As P and Q are rank-2 subgroups of a free group, it follows from the main theorem of Burns [8] that $P \cap Q$ has rank at most 2. But $P \cap Q$ contains the rank-2 subgroup R , and hence cannot have rank at most 1. Thus, $P \cap Q$ has rank exactly 2.

We now appeal to the main result of Kent [18] and Louder and McReynolds [20], which asserts that if P and Q are rank-2 subgroups of a free group and $P \cap Q$ has rank 2, then $\langle P \cup Q \rangle$ also has rank 2. This completes the induction in this case. \square

PROPOSITION 4.5. *Let $k > r \geq 1$ be integers. Let Δ be a k -free group, and suppose that Δ has a normal subgroup of local rank r . Then Δ has local rank at most r .*

Proof. We must show that if E is a finitely generated subgroup of Δ , then E is contained in some finitely generated subgroup of Δ whose rank is at most r . Let $\{x_1, \dots, x_m\}$ be a finite generating set for E . Let N be a normal subgroup of Δ whose local rank is r . As $r \geq 1$, we may select a non-trivial element t of N . The finitely generated subgroup $\langle t, x_1 t x_1^{-1}, \dots, x_m t x_m^{-1} \rangle$ of N is contained in some subgroup H_0 of N whose rank is at most r . For $j = 1, \dots, m$, we denote by H_j the subgroup $\langle H_0 \cup \{x_1, \dots, x_j\} \rangle$ of Δ . We shall show, by induction on $j = 0, \dots, m$ that H_j has rank at most r . As $E \leq H_m$, this implies the conclusion.

The base case is clear because H_0 was chosen to have rank at most r . Assume, for a given j , with $0 \leq j < m$, that H_j has rank at most r . If the rank of $H_{j+1} = \langle H_j \cup \{x_{j+1}\} \rangle$ is $> r$, then it follows from Lemma 4.3 that H_{j+1} is the free product of the subgroups H_j and $\langle x_{j+1} \rangle$, and that x_{j+1} has infinite order. But $H_0 \leq H_j$ contains the elements t and $t' = x_{j+1} t x_{j+1}^{-1}$. It follows that t' is an element of a factor in the free product $H_j \star \langle x_{j+1} \rangle$, which can also be written in a normal form of length 3 in the free product. This contradicts the uniqueness of normal form. Hence, H_{j+1} must have rank at most r . \square

5. Existence of log 7-semithick points

The goal of this section is to prove Theorem 1.4, which was stated in Section 1.

5.1. In this section, we use methods involving nerves of coverings that were first introduced in [12]. Recall that if $\mathcal{U} = (U_i)_{i \in I}$ is an indexed covering of a topological space by non-empty open subsets, then the *nerve* of \mathcal{U} is an abstract simplicial complex whose vertices are in bijective correspondence with the elements of the index set I . If we denote by v_i the vertex corresponding to the index i , then a k -simplex of the nerve is a collection $\{v_{i_0}, \dots, v_{i_k}\}$ of $k + 1$ vertices such that $U_{i_0} \cap \dots \cap U_{i_k} \neq \emptyset$.

5.2. Let Γ be a group and $(K, (C_v)_v)$ be a Γ -labeled complex. By a *labeling-compatible action of Γ on $(K, (C_v)_v)$* , we shall mean a simplicial action of Γ on K such that, for each vertex v of K , we have $C_{\gamma \cdot v} = \gamma C_v \gamma^{-1}$. (This concept was first introduced explicitly in [5], where the term ‘natural action’ was used. We are afraid this term may be confusing.)

5.3. It follows from the definition of a labeling-compatible action of a group Γ on a Γ -labeled complex K that, for every saturated subset A of $|K|$, and every element γ of Γ , we have $\Theta(\gamma \cdot A) = \gamma \Theta(A) \gamma^{-1}$.

In particular, if a saturated subset A of $|K|$ is invariant under an element γ of Γ , then $\Theta(A)$ is normalized by γ . In other words, the stabilizer in Γ of any saturated subset A of $|K|$ is contained in the normalizer of $\Theta(A)$.

5.4. Let Γ be a discrete, purely loxodromic subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and let $\lambda > 0$ be a number such that the indexed family of cylinders $\mathcal{Z} = (Z_\lambda(C))_{C \in \mathcal{C}_\lambda(\Gamma)}$ covers \mathbb{H}^3 . Let K denote the nerve of the covering \mathcal{Z} . According to the definition of the nerve, the vertices of K are in bijective correspondence with elements of $\mathcal{C}_\lambda(\Gamma)$. If C_v denotes the element of $\mathcal{C}_\lambda(\Gamma)$ corresponding to a vertex v , then C_v is by definition a maximal, and hence infinite, cyclic subgroup of Γ . Thus, $(K, (C_v)_v)$ is a Γ -labeled complex.

PROPOSITION 5.5. *Suppose that Γ is a discrete, torsion-free subgroup of $\text{Isom}_+(\mathbb{H}^3)$, and that $\lambda > 0$ is a number such that the indexed family of sets $\mathcal{Z} = (Z_\lambda(C))_{C \in \mathcal{C}_\lambda(\Gamma)}$ covers \mathbb{H}^3 . Let $(K, (C_v)_v)$ be the Γ -labeled complex defined as in Subsection 5.4. Then $(K, (C_v)_v)$ admits a labeling-compatible Γ -action.*

Proof. For any vertex v of K we have $C_v \in \mathcal{C}_\lambda(\Gamma)$, and hence $\gamma C_v \gamma^{-1} \in \mathcal{C}_\lambda(\Gamma)$ for every $\gamma \in \Gamma$. We may therefore define an action of Γ on the set of vertices of K by $C_{\gamma \cdot v} = \gamma C_v \gamma^{-1}$. If v_0, \dots, v_m are the vertices of an m -simplex of K , then, for every $\gamma \in \Gamma$, we have

$$\bigcap_{0 \leq i \leq m} Z_\lambda(\gamma C_{v_i} \gamma^{-1}) = \bigcap_{0 \leq i \leq m} \gamma \cdot Z_\lambda(C_{v_i}) = \gamma \cdot \bigcap_{0 \leq i \leq m} Z_\lambda(C_{v_i}) \neq \emptyset,$$

so that $\gamma \cdot v_0, \dots, \gamma \cdot v_m$ are the vertices of an m -simplex of K . Thus, the action of Γ on the vertex set extends to a simplicial action on K . It is immediate from the definitions that this is a labeling-compatible action on $(K, (C_v)_v)$. \square

LEMMA 5.6. *For any simplicial complex K , the inclusion $|K^3| - |K^0| \rightarrow |K| - |K^0|$ induces isomorphisms on π_0 and π_1 .*

Proof. The space $|K| - |K^0|$ is the direct limit of the system of subspaces $|L| - |K^0|$, where L ranges over all subcomplexes of K such that (1) $|K^3| \subset |L|$ and (2) $|L| - |K^3|$ contains only finitely many open simplices. Hence, it suffices to show that, for any subcomplex satisfying (1) and (2), the inclusion $|L| - |K^0| \rightarrow |K| - |K^0|$ induces isomorphisms on π_0 and π_1 . By induction on the number of simplices in $|L| - |K^3|$, this reduces to showing that if L and L_1 are subcomplexes of K satisfying (1) and (2), and $|L_1| = |L| \cup \sigma$ for some open simplex σ , then the inclusion $|L| - |K^0| \rightarrow |L_1| - |K^0|$ induces isomorphisms on π_0 and π_1 . If $d > 3$ denotes the dimension of σ , then $B = \bar{\sigma}$ is a closed topological d -ball, and the vertex set V of σ is a finite subset of the $(d-1)$ -sphere ∂B . As $d-1 > 2$, the set $(\partial B) - V$ is connected and simply connected. The set $B - V$ is contractible. As $|L_1| = |L| \cup (B - V)$ and $|L| \cap (B - V) = (\partial B) - V$, the inclusion $|L| - |K^0| \rightarrow |L_1| - |K^0|$ induces isomorphisms on π_0 and π_1 as required. \square

LEMMA 5.7. *Let H be a topological space that has the homotopy type of a CW-complex. Let $\mathcal{U} = (U_i)_{i \in I}$ be a covering of H by contractible open sets. Suppose that*

- (i) *for every finite non-empty subset $\{i_0, \dots, i_k\}$ of I , the set $U_{i_0} \cap \dots \cap U_{i_k} \subset H$ is either empty or contractible; and*
- (ii) *for every point $P \in H$ there exist distinct indices $i, j \in I$ such that $P \in U_i \cap U_j$.*

Let K denote the nerve of \mathcal{U} . Then the space $|K| - |K^0|$ is homotopy-equivalent to H .

Proof. According to Borsuk's Nerve Theorem (see, for example, [6, Theorem 6 and Remark 7]), Property (i) of the covering implies that the space $|K|$ is homotopy-equivalent to H . We shall complete the proof by showing that the inclusion $|K| - |K^0| \rightarrow |K|$ is a homotopy equivalence.

If $k \in I$ is any index, let us denote by J_k the set of all indices $i \in I$ such that $i \neq k$ but $U_i \cap U_k \neq \emptyset$. Condition (ii) implies that the indexed family $\mathcal{V}_k = (U_i \cap U_k)_{k \in J_k}$ is an open covering of the set U_k , to which we assign the subspace topology.

It follows from the definitions that the nerve of \mathcal{V}_k is simplicially isomorphic to the link of the vertex v_k in the nerve K of \mathcal{U} . (Although this observation is purely formal, it depends on defining the nerve via the index set J_k as in Subsection 5.1. Different indices in J_k may define the same set in \mathcal{V}_k even if they define different sets in \mathcal{U} .)

As \mathcal{U} satisfies condition (i), it is clear that (i) remains true when we replace \mathcal{U} by \mathcal{V}_k . Hence, by Borsuk's Nerve Theorem, the nerve of the covering \mathcal{V}_k is homotopy-equivalent to U_k , that is, it is contractible. This shows that the link in K of every vertex of K is contractible.

To show that the inclusion $|K| - |K^0| \rightarrow |K|$ is a homotopy equivalence, we first note that, as $|K|$ has the weak topology, we may regard $|K|$ as the topological direct limit of the subspaces $X_F = (|K| - |K^0|) \cup F$, where F ranges over the finite subsets of $|K^0|$. Hence, it suffices to prove that $|K| - |K^0| \rightarrow X_F$ is a homotopy equivalence for every finite $F \subset |K^0|$. By induction on the cardinality of F , this reduces to showing that if $F \subset |K^0|$ is finite, if $v \in |K^0| - F$, and if we set $F' = F \cup \{v\}$, then the inclusion $X_F \rightarrow X_{F'}$ is a homotopy equivalence.

We recall from Subsection 4.1 that K' denotes the first barycentric subdivision of K . For general reasons, $X_{F'} - \text{st}_{K'}(v)$ is a deformation retract of X_F . On the other hand, $X_{F'} - \text{st}_{K'}(v)$ is a deformation retract of $X_{F'}$, because $\text{lk}_{K'}(v)$ is homeomorphic to $\text{lk}_K(v)$ and is therefore contractible. \square

DEFINITION 5.8. Let K be a simplicial complex, and let A and B be disjoint saturated subsets of $|K|$. We say that A and B are *adjacent* if there are open simplices $\sigma \subset A$ and $\tau \subset B$ such that either σ is a face of τ or τ is a face of σ .

If K is a simplicial complex and if X_0 and X_1 are disjoint, saturated subsets of $|K|$, then we define a simplicial complex $\mathcal{G}(X_0, X_1)$ of dimension at most 1 as follows. The vertices of $\mathcal{G}(X_0, X_1)$ are the elements of $\mathcal{W}_0 \cup \mathcal{W}_1$, where \mathcal{W}_i denotes the set of connected components of X_i . A 1-simplex is a pair $\{W_0, W_1\}$, where $W_i \in \mathcal{W}_i$ for $i = 0, 1$, and W_0 and W_1 are adjacent. For $W \in \mathcal{W}_0 \cup \mathcal{W}_1$, we denote by v_W the vertex of $\mathcal{G}(X_0, X_1)$ corresponding to W .

DEFINITION 5.9. Let \mathcal{G} be a graph. A simplicial action of a group Γ on \mathcal{G} will be said to have *no inversions* if, for each edge e of \mathcal{G} and for each element γ of Γ such that $\gamma \cdot e = e$, we have $\gamma \cdot v = v$ for each endpoint v of e .

5.10. Suppose that a group Γ acts simplicially on a simplicial complex K , and that X_0 and X_1 are disjoint, saturated subsets of $|K|$, each of which is invariant under the action of Γ . Then, for $i = 0, 1$, an arbitrary element of Γ maps each connected component of X_i onto a connected component of X_i , and so the action of Γ defines an action on the vertex set of $\mathcal{G}(X_0, X_1)$. Furthermore, if W_i is a component of X_i for $i = 0, 1$, and W_0 and W_i are adjacent, then $\gamma \cdot W_0$ and $\gamma \cdot W_1$ are adjacent for every $\gamma \in \Gamma$, as the action of Γ on K is simplicial. Thus, the action of Γ on the vertex set of $\mathcal{G}(X_0, X_1)$ extends to an action on $\mathcal{G}(X_0, X_1)$. Note that this action has the property that the stabilizer in Γ of any vertex of $\mathcal{G}(X_0, X_1)$ is the stabilizer of some component of X_0 or X_1 under the action of Γ on K . Note also that, for $i = 0, 1$, the

set of vertices of $\mathcal{G}(X_0, X_1)$ corresponding to components of X_i is Γ -invariant. In particular, Γ acts without inversions on $\mathcal{G}(X_0, X_1)$.

DEFINITION 5.11. Let X and Y be topological spaces. We define a *homotopy-retraction* from X to Y to be a map $r : X \rightarrow Y$ that admits a right homotopy inverse. We shall say that Y is a *homotopy-retract* of X if there exists a homotopy retraction from X to Y .

LEMMA 5.12. *Suppose that K is a simplicial complex and that X_0 and X_1 are saturated subsets of $|K|$. Then $|\mathcal{G}(X_0, X_1)|$ is a homotopy-retract of the saturated subset $X_0 \cup X_1$ of $|K|$.*

Proof. We set $\mathcal{G} = \mathcal{G}(X_0, X_1)$, and we use the notation of Definition 5.8. For $m = 0, 1$, we denote by V_m the set of vertices of \mathcal{G} corresponding to elements of \mathcal{W}_m .

Recall from Subsection 4.1 that the first barycentric subdivision of K is denoted by K' , and that we identify $|K'|$ with $|K|$. For each open simplex σ in K we shall denote by b_σ the barycenter of σ , which is a vertex of K' .

For each $m \in \{0, 1\}$, we let B_m denote the set of all vertices of K' that have the form b_σ for some open simplex σ of K such that $\sigma \subset X_m$. If $b = b_\sigma \in B_m$, then we denote by W_b the component of X_m containing σ , and we set $v_b = v_{W_b}$. We make the following claim.

5.12.1. If $m \in \{0, 1\}$ and $b, c \in B_m$ are given, and if b and c are joined by an edge of K' , then $v_b = v_c$.

To prove this, we write $b = b_\sigma$ and $c = b_\tau$, where the open simplices σ and τ of K are contained in X_m . The hypothesis that b and c are joined by an edge of K' means that one of the simplices σ, τ is a face of the other. By symmetry we may assume that τ is a face of σ . Then $\tau \subset \bar{\sigma}$, and hence σ and τ are contained in the same component of X_m . In the notation introduced above, this means that $W_b = W_c$, and Paragraph 5.12.1 follows.

Next we make the following claim.

5.12.2. If $b_0 \in B_0$ and $b_1 \in B_1$ are joined by an edge of K' , then v_{b_0} and v_{b_1} are joined by an edge of \mathcal{G} .

To prove this, we write $b_m = b_{\sigma_m}$ for $m = 0, 1$, where the open simplex σ_m of K is contained in X_m . Here again we may assume by symmetry that σ_0 is a face of σ_1 . Then $\sigma_0 \subset \bar{\sigma}_1$, and hence $W_{b_0} \subset W_{b_1}$. By the definition of the graph $\mathcal{G} = \mathcal{G}(X_0, X_1)$, the conclusion of Paragraph 5.12.2 follows.

Now let S denote the set of all simplices s of K' such that $s \subset X_0 \cup X_1$. For each $s \in S$ we let \mathcal{B}_s denote the set of all vertices of s that lie in $B_0 \cup B_1$. Note that if σ denotes the open simplex of K containing s , then $\sigma \subset X_0 \cup X_1$ and hence $b_\sigma \in \mathcal{B}_s$; in particular, $\mathcal{B}_s \neq \emptyset$ for any $s \in S$. For each $s \in S$ we set

$$\mathcal{V}_s = \{v_b : b \in \mathcal{B}_s\}.$$

It follows from Paragraph 5.12.1 that \mathcal{V}_s contains at most one vertex from each of the sets B_0 and B_1 , and it follows from Paragraph 5.12.2 that if \mathcal{V} contains a vertex in B_0 and a vertex in B_1 , then these vertices are joined by an edge. Hence, \mathcal{V}_s is the vertex set of a simplex of dimension 0 or 1 in \mathcal{G} . We denote the corresponding open simplex of $|\mathcal{G}|$ by α_s . Thus, α_s is either a vertex or an open edge of $|\mathcal{G}|$.

It is immediate from the definition of \mathcal{V}_s that if $t, s \in S$ and if t is a face of s , then $\mathcal{V}_t \subset \mathcal{V}_s$. Hence, we make the following claim.

5.12.3. If $t, s \in S$ and if t is a face of s , then α_t is a face of α_s in the simplicial complex \mathcal{G} . (In other words, either $\alpha_t = \alpha_s$, or α_s is an edge and α_t is one of its endpoints.)

Using Paragraph 5.12.3, we may construct, recursively for $k \geq -1$, a continuous map $r_k : (K')^k \cap (X_0 \cup X_1) \rightarrow |\mathcal{G}|$ such that $r(s)$ is contained in (the open simplex) α_s for every simplex $s \in S$ of dimension at most k . We take r_{-1} to be the empty map, and construct r_{k+1} as an extension of r_k . If r_k has been defined, and $s \in S$ is $(k+1)$ -dimensional, then every face t of s that belongs to S is mapped by r_k into α_t , which by Paragraph 5.12.3 is a face of α_s . This allows us to extend $r_k|_{(\partial s) \cap (X_0 \cup X_1)}$ to a continuous map of $\bar{s} \cap (X_0 \cup X_1)$ into $\bar{\alpha}_s$ that maps s into α_s , and thus to complete the recursive definition. As r_{k+1} is an extension of r_k , we may define a map $r : X_0 \cup X_1 \rightarrow |\mathcal{G}|$ by setting $r|(K')^k \cap (X_0 \cup X_1) = r_k$ for each k . The construction gives the following claim.

5.12.4. Each simplex $s \in S$ is mapped by r into α_s .

We now make the following claim.

5.12.5. For each $m \in \{0, 1\}$ and each component W of X_m , the set $r(W)$ is contained in $\text{st}_{\mathcal{G}}(v_W)$.

To prove this, we consider any open simplex s of K' that is contained in W . As W is saturated in K , the open simplex σ of K containing s is also contained in W . We have $v_W = b_\sigma \in \mathcal{B}_s$, so that α_s is contained in $\text{st}_{\mathcal{G}}(v_W)$. By Paragraph 5.12.4 it follows that $r(s) \subset \text{st}_{\mathcal{G}}(v_W)$, and Paragraph 5.12.5 follows.

Now we construct a map $i : |\mathcal{G}| \rightarrow X_0 \cup X_1$ as follows. For each vertex $v = v_W$ of \mathcal{G} , we take $i(|v|)$ to be a point of W (chosen arbitrarily). If e is an edge of \mathcal{G} with endpoints v_0 and v_1 , where $v_m = v_{W_m} \in V_m$ for $m = 0, 1$, then the components W_0 and W_1 of X_0 and X_1 are adjacent. In particular, one of them meets the closure of the other, and hence $W_0 \cup W_1$ is connected. Hence, $i(v_0)$ and $i(v_1)$ are joined by a path in $W_0 \cup W_1$, and we may use this path to extend i to $|e|$. The construction of the map i gives the following claim.

5.12.6. For each vertex $v = v_W$ of \mathcal{G} we have $i(|v|) \subset W$, and for each edge e of \mathcal{G} with endpoints v_{W_0} and v_{W_1} we have $i(|e|) \subset W_0 \cup W_1$.

From Paragraphs 5.12.5 and 5.12.6, we immediately deduce the following property of the composition $r \circ i : |\mathcal{G}| \rightarrow |\mathcal{G}|$.

5.12.7. Each vertex v of $|\mathcal{G}|$ is mapped by $r \circ i$ into $\text{st}_{\mathcal{G}}(v)$ and each edge of $|\mathcal{G}|$ is mapped into the union of the open stars of its endpoints.

From Paragraph 5.12.7 we can deduce that $r \circ i$ is homotopic to the identity. First we construct a homotopy $H^{(0)}$ from the vertex set $|\mathcal{G}^{(0)}|$ into $|\mathcal{G}|$ such that $H_0^{(0)} = r \circ i$, $H_1^{(0)}$ is

the inclusion, and $H^{(0)}(\{|v|\} \times [0, 1])$ is contained in $\text{st}_{\mathcal{G}}(v)$ for every vertex v ; this is possible by Paragraph 5.12.7 and the connectedness of the open stars. Then we consider an arbitrary edge $|e|$ of $|\mathcal{G}|$, and let Y denote the union of the open stars of its endpoints. According to Paragraph 5.12.7 and the construction of $H^{(0)}$, we have $H^{(0)}((|e|) \times [0, 1]) \cup r \circ i(|e|) \subset W$. As W is simply connected, we may extend $H^{(0)}|_{(\partial e) \times [0, 1]}$ to a homotopy $H^e : |e| \times [0, 1] \rightarrow W$ such that H_1^e is the inclusion map from $|e|$ to W .

This shows that r is a homotopy retraction from $X_0 \cup X_1$ to \mathcal{G} . \square

LEMMA 5.13. *Let M be a closed, orientable, aspherical 3-manifold. Then $\pi_1(M)$ does not admit a simplicial action without inversions on a tree T with the property that the stabilizer in $\pi_1(M)$ of every vertex of T is a locally free subgroup of $\pi_1(M)$.*

Proof. We will prove this from the point of view used in [10]. Assume an action with the stated properties exists. It follows from the Bass–Serre theory (see [24, Theorem 7]) that $\pi_1(M)$ is then isomorphic to a graph of groups in which each vertex group or edge group is, respectively, isomorphic to the stabilizer of a vertex or edge of T . If the graph of groups is non-trivial in the sense that the full group $\pi_1(M)$ is not a vertex group, then it follows from [10, Proposition 2.3.1] that there is an incompressible surface $F \subset M$ such that the image (defined up to conjugacy) of the inclusion $\pi_1(F) \rightarrow \pi_1(M)$ is contained in an edge group. As M is simple and closed, this image is the fundamental group of a surface of genus greater than 1. This is impossible because any edge group is contained in a vertex group, and is therefore locally free by our assumption. If the graph of groups is trivial, then $\pi_1(M)$ is itself locally free, and as M is compact, $\pi_1(M)$ is in fact free. But as M is closed, orientable, and aspherical, we have $H_3(\pi_1(M); \mathbb{Z}) \cong \mathbb{Z}$, whereas a free group has trivial homology in dimensions greater than 1. Thus, this case is impossible as well. \square

Proof Theorem 1.4. Assume that the conclusion is false, so that every point of M is $(\log 7)$ -doubly thin. Then it follows from Proposition 3.5 that

$$\mathbb{H}^3 = \bigcup_{\substack{C, C' \in \mathcal{C}_{\log 7}(\Gamma) \\ C \neq C'}} Z_{\log 7}(C) \cap Z_{\log 7}(C').$$

Thus, the indexed family of sets $\mathcal{Z} = (Z_{\log 7}(C))_{C \in \mathcal{C}_{\log 7}(\Gamma)}$ is an open covering of \mathbb{H}^3 , and condition (ii) of Lemma 5.7 holds with $H = \mathbb{H}^3$ and $\mathcal{U} = \mathcal{Z}$. Condition (i) of Lemma 5.7 follows from the convexity of the cylinders $Z_{\log 7}(C)$. Hence, according to Lemma 5.7, if we let K denote the nerve of (U_i) , the space $|K| - |K^0|$ is homotopy-equivalent to \mathbb{H}^3 and is therefore contractible. It then follows by Lemma 5.6 that the set $|K^3| - |K^0|$ is connected and simply connected.

According to the observation in Subsection 5.4, with $\lambda = \log 7$, we may define a Γ -labeled complex $(K, (C_v)_v)$ by taking C_v to be the element of $\mathcal{C}_\lambda(\Gamma)$ corresponding to the vertex v .

Let σ be any open simplex contained in $|K^3| - |K^0|$, and consider the group $\Theta(\sigma)$, in the notation of Subsection 4.2. As σ has dimension at most 3, the rank of $\Theta(\sigma)$ is at most 4.

Suppose that $\Theta(\sigma)$ has rank 4. Then σ must be three-dimensional. Let v_0, \dots, v_3 denote the vertices of σ , and for $i = 0, \dots, 3$, set $C_i = C_{v_i}$ and $Z_i = Z_{\log 7}(C_i)$. Choose a generator x_i of each C_i . The definition of the nerve implies that the intersection $Z_0 \cap Z_1 \cap Z_2 \cap Z_3$ is non-empty. We choose a point z in this intersection.

For $i = 0, \dots, 3$, as $z \in Z_i = Z_{\log 7}(C_i)$, there exists an integer $m_i \neq 0$ such that

$$\text{dist}(z, x_i^{m_i} \cdot z) < \log 7 \quad \text{for } i = 0, \dots, 3. \quad (5.1)$$

As $\Theta(\sigma)$ is assumed to have rank 4, and as $\Gamma \cong \pi_1(M)$ is 4-free by hypothesis, $\Theta(\sigma)$ is freely generated by x_0, \dots, x_3 . Hence, $x_0^{m_0}, \dots, x_3^{m_3}$ also freely generate a rank-4-free group. But it follows from ([5, Theorem 6.1], together with the main result of Agol [1] or Calegari and Gabai [9]), that if $\xi_1, \dots, \xi_k \in \text{Isom}_+(\mathbb{H}^3)$ freely generate a discrete subgroup of $\text{Isom}_+(\mathbb{H}^3)$, then for any $z \in \mathbb{H}^3$, we have

$$\max_{1 \leq i \leq k} \text{dist}(z, \xi_i \cdot z) \geq \log(2k - 1).$$

Taking $k = 4$ and $\xi_i = x_{i+1}^{m_{i+1}}$, we obtain a contradiction. This shows that $\Theta(\sigma)$ has rank at most 3 for any open simplex $\sigma \subset |K^3| - |K^0|$.

On the other hand, for any open simplex $\sigma \subset |K^3| - |K^0|$, there are at least two distinct vertices, say v_C and $v_{C'}$, in $\bar{\sigma}$. Here C and C' are by definition distinct maximal cyclic subgroups of Γ , and hence $\Theta(\sigma)$, which contains C and C' , is non-abelian. This shows that $\Theta(\sigma)$ must have rank at least 2.

Thus, for any open simplex $\sigma \subset |K^3| - |K^0|$, the group $\Theta(\sigma)$ has rank 2 or 3. We may therefore write $|K^3| - |K^0|$ as a disjoint union

$$|K^3| - |K^0| = X_2 \dot{\cup} X_3,$$

where X_k is the union of all open simplices $\sigma \subset |K^3|$ for which $\Theta(\sigma)$ has rank k .

We make the following claim.

5.13.1. For any $m \in \{2, 3\}$ and for any component W of X_m , the local rank of $\Theta(W)$ is at most m .

This is an application of Proposition 4.4. In the case $m = 2$, hypothesis (ii) of Proposition 4.4 clearly holds, and hence $\Theta(W)$ has local rank at most 2. If $m = 3$, then $\Theta(\sigma)$ has rank 3 for every open simplex $\sigma \subset W$. If $d = \dim \sigma$, then $\Theta(\sigma)$ is generated by $d + 1$ cyclic groups, and hence $d \geq 2$. But $d \leq 3$ as $\sigma \subset X_3 \subset |K^3| - |K^0|$. Hence, in this case, hypothesis (i) of Proposition 4.4 holds with $n = 3$, and hence $\Theta(W)$ has local rank at most 3.

Next we make the following claim.

5.13.2. If W is a component of X_2 or X_3 , then the normalizer of $\Theta(W)$ in Γ has local rank at most 3.

To prove Paragraph 5.13.2, we let r denote the local rank of $\Theta(W)$. By Paragraph 5.13.1 we have $r \leq 3$. If $r \leq 1$, then $\Theta(W)$ is locally cyclic, and is therefore cyclic as Γ is a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$. But this is impossible because $\Theta(\sigma) \leq \Theta(W)$ has rank 2 or 3 for every open simplex $\sigma \subset W$. It follows that r is 2 or 3.

Now if Δ denotes the normalizer of $\Theta(W)$ in Γ , then Δ is 4-free and has the normal subgroup $\Theta(W)$ of local rank r . As r is 2 or 3, it follows from Proposition 4.5 that Δ has local rank at most r . This establishes Paragraph 5.13.2.

Let us now set $T = \mathcal{G}(X_2, X_3)$ (with the notation of Definition 5.8). According to Lemma 5.12, T is a homotopy-retract of $X_2 \cup X_3 = |K^3| - |K^0|$. As we have seen that $|K^3| - |K^0|$ is connected and simply connected, T is a tree.

According to Proposition 5.5, Γ admits a labeling-compatible action on K . It follows from Subsection 5.3 that, for any $\gamma \in \Gamma$ and for any simplex σ of K , the groups $\Theta(\sigma)$ and $\Theta(\gamma \cdot \sigma)$ are conjugate in Γ and hence have the same rank. It follows that each of the saturated sets X_2 and X_3 is invariant under the action of Γ .

Hence, according to Subsection 5.10, there is an induced action, without inversions, of Γ on T ; and under this induced action, if v is any vertex of T , the stabilizer Γ_v of v in Γ is the stabilizer of some component W of X_2 or X_3 under the action of Γ on $|K|$. Hence, by Subsection 5.3, Γ_v is contained in the normalizer of $\Theta(W)$. By Paragraph 5.13.1 this normalizer has local rank at most 3. In particular, Γ_v has local rank at most 3, and as Γ is in particular 3-free, it follows that Γ_v is locally free.

Thus, we have constructed a simplicial action, without inversions, of $\Gamma \cong \pi_1(M)$ on the tree T with the property that the stabilizer in Γ of every vertex of T is a locally free subgroup of Γ . This contradicts Lemma 5.13. \square

Theorem 1.4 will be used in what follows via the following corollary.

COROLLARY 5.14. *Suppose that M is a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 4-free. Then either*

- (i) M contains an embedded hyperbolic ball of radius $(\log 7)/2$; or
- (ii) there is a point $P \in \mathfrak{G}_M$ with $\mathfrak{s}_M(P) = \log 7$ (see Subsection 3.8).

Proof. According to Theorem 1.4, there is a $\log 7$ -semithick point $P_0 \in M$. By Proposition 3.1 and the definition of \mathfrak{N}_M in Notation 3.2, we have $\mathfrak{N}_M(P_0) \geq \log 7$. If it happens that the function $\mathfrak{N}_M(P)$ takes values at least $\log 7$ everywhere in M , then, by Proposition 3.7, the number $\log 7$ is a Margulis number for M . According to the discussion in Subsection 3.6, M then contains a $\log 7$ -thick point, and hence conclusion (i) of the corollary holds. Now suppose that $\mathfrak{N}_M(P)$ takes a value less than $\log 7$ somewhere in M . As \mathfrak{N}_M is continuous according to Proposition 3.3, it follows that there is a point $P \in M$ such that $\mathfrak{N}_M(P) = \log 7$. According to Proposition 3.9, either P is a $\log 7$ -thick point of M , in which case conclusion (i) of the corollary holds; or $P \in \mathfrak{G}_M$ and $\mathfrak{s}_M(P) = \log 7$, which gives conclusion (ii) of the corollary. \square

6. Caps

The term ‘cap’ refers to the intersection of a closed ball with a half-space whose interior does not contain the center of the ball. We begin by introducing some notation for describing caps.

6.1. Let z_0 be a point of \mathbb{H}^3 . For each positive real number R we shall denote by $S(R, z_0)$ the sphere of radius R centered at z_0 . Thus, $S(R, z_0)$ is the boundary of the closed ball $\overline{N}(z_0, R) \doteq \overline{N}(z_0, R)$ of radius R about z_0 . For each point $\zeta \in S(R, z_0)$ we denote by η_ζ the ray originating at z_0 and passing through ζ . We endow $S(R, z_0)$ with the spherical metric in which the distance between two points $\zeta, \zeta' \in S(R, z_0)$ is the angle between η_ζ and $\eta_{\zeta'}$.

For each point $\zeta \in S(R, z_0)$ and each number $w \geq 0$, we denote by $\Pi(z_0, \zeta, w)$ the plane that meets η_ζ perpendicularly at a distance w from z_0 , and by $H(z_0, \zeta, w)$ the closed half-space that is bounded by $\Pi(z_0, \zeta, w)$ and has unbounded intersection with η_ζ . We set

$$K(R, z_0, \zeta, w) = \overline{N}(z_0, R) \cap H(z_0, \zeta, w).$$

Thus, $K(R, z_0, \zeta, w)$ is a cap in the closed ball of radius R cut out by a plane at distance w from the center. Note that $K(R, z_0, \zeta, w) = \emptyset$ when $w \geq R$.

Let $B(r)$ denote the volume of a ball of radius r in \mathbb{H}^3 . The following result was discussed in Section 1.

PROPOSITION 6.2. *Let M be a closed, orientable hyperbolic 3-manifold, write $M = \mathbb{H}^3/\Gamma$ where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete and torsion-free, and set $q = q_\Gamma$. Let λ be a positive number, and suppose that P is a point of \mathfrak{G}_M with $\mathfrak{s}_M(P) \geq \lambda$. Let \tilde{P} be a point of $q^{-1}(P)$, let $j : \pi_1(M, P) \rightarrow \Gamma$ denote the isomorphism determined by the base point $\tilde{P} \in \mathbb{H}^3$ (see Subsection 2.6), and let x denote a generator of $j(C_P)$. For each integer $n \neq 0$, set $d_n = \text{dist}(\tilde{P}, x^n \cdot \tilde{P})/2$, and let ζ_n denote the point of intersection of $S(\lambda/2, \tilde{P})$ with the ray originating at \tilde{P} and passing through $x^n \cdot \tilde{P}$. Then*

$$\text{vol}(N(P, \lambda/2)) = B(\lambda/2) - \text{vol} \left(\bigcup_{0 \neq n \in \mathbb{Z}} K(\lambda/2, \tilde{P}, \zeta_n, d_n) \right).$$

Proof. For each $\gamma \in \Gamma - \{1\}$ let $\zeta_\gamma \in S(\lambda/2, \tilde{P})$ denote the intersection of $S(\lambda/2, \tilde{P})$ with the ray starting at \tilde{P} , which contains $\gamma\tilde{P}$, and let $d_\gamma = \text{dist}(\tilde{P}, \gamma\tilde{P})/2$. Let \mathcal{D} denote the Dirichlet domain for Γ centered at the point \tilde{P} . Then, by definition, we have

$$\mathcal{D} = \bigcap_{1 \neq \gamma \in \Gamma} H(\tilde{P}, \zeta_\gamma, d_\gamma).$$

As $\mathfrak{s}_M(P) \geq \lambda$, we have $d_\gamma \geq \lambda/2$ unless $\gamma = x^n$ for some $n \in \mathbb{Z}$. Thus,

$$\begin{aligned} (\text{int } \mathcal{D}) \cap \overline{N}(\tilde{P}, \lambda/2) &= \overline{N}(\tilde{P}, \lambda/2) - \bigcup_{1 \neq \gamma \in \Gamma} K(\lambda/2, \tilde{P}, \zeta_\gamma, d_\gamma) \\ &= \overline{N}(\tilde{P}, \lambda/2) - \bigcup_{0 \neq n \in \mathbb{Z}} K(\lambda/2, \tilde{P}, \zeta_n, d_n). \end{aligned}$$

The set $(\text{int } \mathcal{D}) \cap \overline{N}(\tilde{P}, \lambda/2)$ has the same volume as $(\text{int } \mathcal{D}) \cap N(\tilde{P}, \lambda/2)$, which in turn is isometric to an open subset of full measure in $N(P, \lambda/2)$. Thus, the result follows. \square

6.3. Now let z_0 be a point of \mathbb{H}^3 , and let w and R be real numbers with $0 < w < R$. We may regard $\overline{K}(R, z_0, \zeta, w) \cap S(R, z_0)$ as a metric ball about ζ in $S(R, z_0)$, with respect to the spherical metric. The radius Θ of this metric ball is the angle formed at z_0 between η_ζ and a segment joining z_0 to any point of the boundary of the topological disk $\overline{K}(R, z_0, \zeta, w) \cap S(R, z_0)$. We have $0 < \Theta < \pi/2$. Note that Θ is an angle in a hyperbolic right triangle with hypotenuse R , in which the other side adjoining the angle has length w . Hence,

$$\cos \Theta = \frac{\tanh w}{\tanh R}.$$

It follows that $\Theta = \Theta(w, R)$, where $\Theta(w, R)$ is the real-valued function with domain

$$\{(w, R) : 0 < w < R\} \subset \mathbb{R}^2$$

defined by

$$\Theta(w, R) = \arccos \left(\frac{\tanh w}{\tanh R} \right),$$

which takes values in $(0, \pi/2)$.

6.4. Let z_0 be a point in \mathbb{H}^3 , let $R > 0$ be a real number, let ζ and ζ' be points of $S(R, z_0)$, let α denote the spherical distance from ζ to ζ' , and let w and w' be positive numbers less than R . Set $K = K(R, z_0, \zeta, w)$ and $K' = K(R, z_0, \zeta', w')$. If $\alpha \leq \Theta(w, R) - \Theta(w', R)$, the spherical triangle inequality implies that $\overline{K'} \cap S(R, z_0) \subset \overline{K} \cap S(R, z_0)$. It follows easily that $K' \subset K$.

6.5. We define a function κ on $(0, \infty) \times (0, \infty) \subset \mathbb{R}^2$ by defining $\kappa(R, w)$ to be the volume of $K(R, z_0, \zeta, w)$, where z_0 and ζ are points in \mathbb{H}^3 separated by a distance R . Note that if we fix any $R > 0$, then the function $\kappa(R, \cdot)$ is monotone decreasing (in the weak sense) on $(0, \infty)$, and takes the value 0 when $w \geq R$.

Next we define functions ι and σ , which give, respectively, the volume of the intersection and the union of two caps. More precisely, we set

$$\begin{aligned} \iota(R, w, w', \alpha) &= \text{vol}(K(R, z_0, \zeta, w) \cap K(R, z_0, \zeta', w')), \\ \sigma(R, w, w', \alpha) &= \text{vol}(K(R, z_0, \zeta, w) \cup K(R, z_0, \zeta', w')), \end{aligned}$$

where z_0 is a point in \mathbb{H}^3 and ζ and ζ' are points in $S(R, z_0)$ separated by a spherical distance α . We regard ι and σ as real-valued functions with domain $(0, \infty)^3 \times [0, \pi] \subset \mathbb{R}^4$. Note that ι is symmetric in the second and third variables, and that

$$\sigma(R, w, w', \alpha) = \kappa(R, w) + \kappa(R, w') - \iota(R, w, w', \alpha), \tag{6.1}$$

for any $(R, w, w', \alpha) \in \mathcal{D}$.

In the Appendix (Section 13), we give a formula for the function κ and a numerical procedure for calculating ι and σ . These will be used in Sections 12 and 13.

The main result of this section, Proposition 6.7, gives some monotonicity properties of the function σ that will be needed in Section 7. The proof of Proposition 6.7 will involve the following technical lemma.

LEMMA 6.6. *Let z be a point of \mathbb{H}^3 , let R be a positive number, and let ζ_0, ζ_1 , and ζ_2 be points lying on a great circle of $S(R, z)$. Let w and w' be numbers such that $0 < w \leq w' < R$. For $i = 1, 2$, let α_i denote the spherical distance from ζ_0 to ζ_i . Suppose that $0 \leq \alpha_1 < \alpha_2 \leq \pi$ and that the spherical distance from ζ_1 to ζ_2 is $\alpha_2 - \alpha_1$. Assume that $\alpha_2 - \alpha_1 < \Theta(w', R)$, where Θ is the function defined in Subsection 6.3. Set $K_0 = K(R, z, \zeta_0, w)$, and $K_i = K(R, z, \zeta_i, w')$ for $i = 1, 2$. Set $Y = K_1 \cap K_2$, and $X_i = K_i - Y$ for $i = 1, 2$. Then either $X_1 \subset K_0$ or $X_2 \cap K_0 = \emptyset$.*

Proof. We set $\theta = \Theta(w, R)$ and $\theta' = \Theta(w', R)$, where Θ is the function defined in Subsection 6.3. As $0 < w \leq w' < R$, we have $0 < \theta' \leq \theta < \pi/2$. According to the hypothesis, we have $0 < \alpha_2 - \alpha_1 < \theta'$.

We set $B = N(R, z)$ and $S = S(R, z)$. As the points ζ_0, ζ_1 , and ζ_2 all lie on a great circle, the points z, ζ_0, ζ_1 , and ζ_2 lie on a plane W . For $i = 0, 1, 2$, the planes $\Pi_i = \Pi(z, \zeta_i, w')$ are all perpendicular to W . We reduce the proof of the lemma to a two-dimensional argument by considering the intersections of various sets with W .

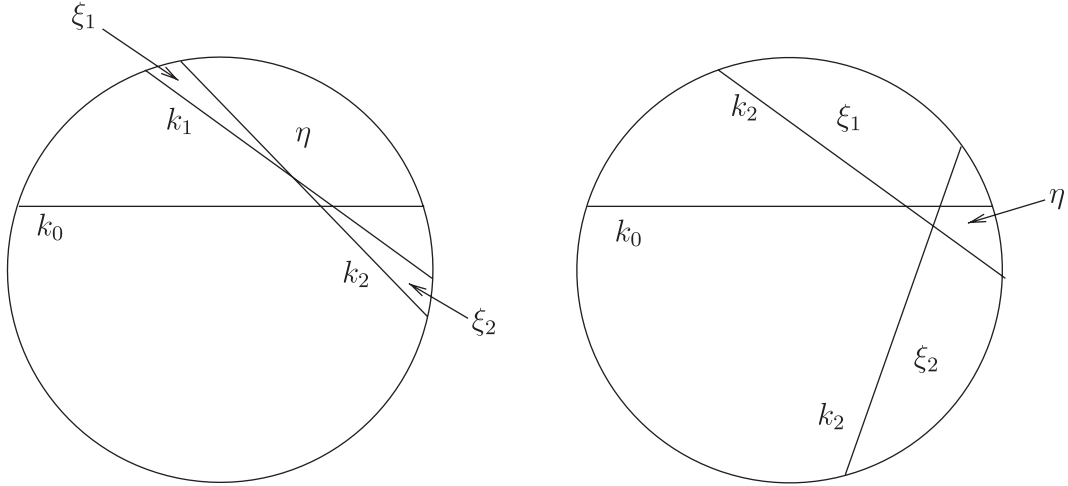
We set $\Delta = B \cap W$, $C = S \cap W$ and, for $i = 0, 1, 2$, we set $k_i = K_i \cap W$ and $\pi_i = \Pi_i \cap W$. Let $p : \mathbb{H}^3 \rightarrow W$ denote the perpendicular projection. Then we have $K_i = B \cap p^{-1}(k_i)$ for $i = 0, 1, 2$. We also set $\eta = k_1 \cap k_2$, and $\xi_i = k_i - \eta$ for $i = 1, 2$; then we have $X_i = B \cap p^{-1}(\xi_i)$ for $i = 1, 2$.

Let us orient the circle C in such a way that, for $i = 1, 2$, the clockwise angle from ζ_0 to ζ_1 is α_i . For $i = 0, 1, 2$, let A_i denote the arc $k_i \cap C$ and let χ_i denote the chord $\pi_i \cap \Delta$. Note that the arc A_0 subtends an angle of $2\theta < \pi$, whereas A_i subtends an angle $2\theta' < \pi$ for $i = 1, 2$. For $i = 0, 1, 2$, let l_i and r_i denote the endpoints of A_i , where l_i is the initial endpoint when A_i is described in the clockwise direction.

The clockwise angle from l_1 to r_2 is $\delta \doteq \alpha_2 - \alpha_1 + 2\theta' < 4\theta'$, so the arcs A_1 and A_2 overlap in a single non-degenerate subarc, and $A_1 \cup A_2$ is an arc that subtends an angle δ . When the arc $A_1 \cup A_2$ is described in the clockwise direction, the points of $T \doteq \partial A_1 \cup \partial A_2$ appear in the order l_1, l_2, r_1, r_2 .

The clockwise angle from l_0 to l_1 is $\gamma \doteq \theta + \alpha_1 - \theta'$, and we have $\gamma + \delta = \theta + \alpha_2 + \theta' < 2\pi$. Hence, if we describe C in the clockwise direction starting at l_0 , the points of $T \cup \{l_0\}$ appear in the order l_0, l_1, l_2, r_1, r_2 . In particular, the set $A_0 \cap T$ consists of the terms of an initial subsequence of (l_1, l_2, r_1, r_2) . It follows that the set $(C - A_0) \cap T$ consists of the terms of a final subsequence of (l_1, l_2, r_1, r_2) .

As the arcs A_1 and A_2 overlap, the chords χ_1 and χ_2 cross in a point Q in the interior of Δ . We distinguish two cases in the proof of the lemma, depending on whether or not Q lies in k_0 . The two cases are illustrated in the figure below.



If $Q \in k_0$, then each of the chords χ_1, χ_2 has at least one endpoint in A_0 . In particular, the set $A_0 \cap T$, which consists of the terms of an initial subsequence of (l_1, l_2, r_1, r_2) , has cardinality at least 2. Hence, the arc A_0 contains l_1 and l_2 , which are the endpoints of the arc $\beta_1 = \xi_1 \cap C$. Moreover, as the arc β_1 subtends an angle $\alpha_2 - \alpha_1 < \theta' < \pi/2$ whereas the arc A_0 subtends an angle $2\theta < \pi$, the arc A_0 cannot contain $C - \beta_1$. It therefore follows that $\beta_1 \subset A_0$. As k_0 is convex and as ξ_1 is the convex hull of $\beta_1 \cup \{Q\}$, we have

$$X_1 = B \cap p^{-1}(\xi_1) \subset B \cap p^{-1}(k_0) = K_0,$$

and the lemma is proved in this case. If $Q \notin k_0$, then each of the chords χ_1, χ_2 has at least one endpoint in $C - A_0$. In particular, the set $(C - A_0) \cap T$, which consists of the terms of a final subsequence of (l_1, l_2, r_1, r_2) , has cardinality at least 2. Hence, the endpoints r_1 and r_2 of the arc $\beta_2 = \xi_2 \cap C$ are contained in $C - A_0$. The arc $C - \beta_2$ subtends an angle $2\pi - (\alpha_2 - \alpha_1) > 2\pi - \theta$, whereas the arc $C - A_0$ subtends an angle $2\pi - 2\theta < 2\pi - \theta$. It follows that $C - \beta_2$ cannot be contained in $C - A_0$, and hence that β_2 is contained in $C - A_0$.

As $\Delta - k_0$ is convex, and as ξ_2 is the convex hull of $\beta_2 \cup \{Q\}$, we have $\xi_2 \subset \Delta - k_0$. We now have

$$X_2 = B \cap \pi^{-1}(\xi_2) \subset B \cap \pi^{-1}(\Delta - k_0) = B - K_0,$$

and the lemma is proved in this case as well. □

PROPOSITION 6.7. *For any $R, w > 0$ and $\alpha \in [0, \pi]$, the function σ is monotone decreasing in its third variable and monotone increasing in its fourth variable (in the sense of Subsection 2.3).*

Proof. The first assertion is easy, because if $0 < w'_1 < w'_2$, if ζ and ζ' are points of $S(R, z_0)$ whose spherical distance is α , and if we set $K = K(R, z_0, \zeta, w)$, and $K'_i = K(R, z_0, \zeta, w'_i)$

for $i = 1, 2$, then we have $K'_2 \subset K'_1$ and hence $K'_2 \cup K \subset K'_1 \cup K$, so that $\sigma(R, w, w'_2, \alpha) \leq \sigma(R, w, w'_1, \alpha)$.

To prove the second assertion, we must show that $\sigma(R, w, w', \cdot)$ is monotone increasing on $[0, \pi]$ for any $R, w, w' \in (0, \infty)$. We may assume that $w \leq w'$, as σ is obviously symmetric in its second and third arguments. In view of (6.1), it suffices to prove that $\iota(R, w, w', \cdot)$ is monotone decreasing. We may assume that $w' < R$, as otherwise the function $\iota(R, w, w', \cdot)$ is identically zero.

We set $\theta = \Theta(w, R)$ and $\theta' = \Theta(w', R)$, where Θ is the function defined in Subsection 6.3. It clearly suffices to show that if α_1 and α_2 satisfy $0 \leq \alpha_1 < \alpha_2 \leq \pi$ and $\alpha_2 - \alpha_1 < \theta'$, then $\iota(R, w, w', \alpha_2) \leq \iota(R, w, w', \alpha_1)$.

Let z_0 be a point of \mathbb{H}^3 , and let ζ_0, ζ_1 , and ζ_2 be points of $S(R, z_0)$ such that the spherical distance from ζ_0 to ζ_i is α_i for $i = 1, 2$, and such that ζ_1 lies on the great circular arc of length α_1 with endpoints ζ_0 and ζ_2 . We set $K_0 = K(R, z_0, \zeta_0, w)$ and $K_i = K(R, z_0, \zeta_i, w')$ for $i = 1, 2$. Then, for $i = 1, 2$, we have

$$\text{vol } K_i = \kappa(R, w')$$

and

$$\text{vol}(K_0 \cap K_i) = \iota(R, w, w', \alpha_i).$$

Let us also set $Y = K_1 \cap K_2$, and $X_i = K_i - Y$ for $i = 1, 2$. Then K_i is set-theoretically the disjoint union of X_i with Y . Hence,

$$\begin{aligned} \iota(R, w, w', \alpha_1) - \iota(R, w, w', \alpha_2) &= \text{vol}(K_0 \cap K_1) - \text{vol}(K_0 \cap K_2) \\ &= \text{vol}(K_0 \cap X_1) + \text{vol}(K_0 \cap Y) \\ &\quad - \text{vol}(K_0 \cap X_2) - \text{vol}(K_0 \cap Y), \end{aligned}$$

that is,

$$\iota(R, w, w', \alpha_1) - \iota(R, w, w', \alpha_2) = \text{vol}(K_0 \cap X_1) - \text{vol}(K_0 \cap X_2). \tag{6.2}$$

On the other hand, for $i = 1, 2$, we have

$$\text{vol } X_i = \text{vol } K_i - \text{vol } Y = \kappa(R, w') - \text{vol } Y.$$

In particular,

$$\text{vol } X_1 = \text{vol } X_2. \tag{6.3}$$

Now, according to Lemma 6.6, we have either $X_1 \subset K_0$ or $X_2 \cap K_0 = \emptyset$. If $X_1 \subset K_0$, then, from (6.2) and (6.3), we find that

$$\begin{aligned} \iota(R, w, w', \alpha_1) - \iota(R, w, w', \alpha_2) &= \text{vol}(X_1) - \text{vol}(K_0 \cap X_2) \\ &= \text{vol}(X_2) - \text{vol}(K_0 \cap X_2) \\ &\geq 0. \end{aligned}$$

On the other hand, if $X_2 \cap K_0 = \emptyset$, then, from (6.2), we find that

$$\iota(R, w, w', \alpha_1) - \iota(R, w, w', \alpha_2) = \text{vol}(K_0 \cap X_1) \geq 0.$$

Thus, in both cases we have $\iota(R, w, w', \alpha_2) \leq \iota(R, w, w', \alpha_1)$, as required. \square

7. Nearby volume

Suppose that M is a closed orientable hyperbolic 3-manifold. The goal of this section is to prove a technical result, Lemma 7.6, which gives a lower bound for the volume of the metric neighborhood $N(P, \lambda/2) \subset M$, where λ is a positive real number, P is a point of \mathfrak{G}_M such that $\mathfrak{s}_M(P) \geq \lambda$, and certain inequalities are satisfied. Proposition 6.2 expresses $\text{vol}(N(P, \lambda/2))$ in

terms of the volume of a union of caps associated to certain elements of the cyclic group C_P . The hypotheses of Lemma 7.6 ensure that the caps associated to powers of a generator of C_P with exponents greater than three are empty, which leads to an estimate for $\text{vol } N(P, \lambda/2)$ involving the first three powers of the generator. An additional hypothesis implies that the caps corresponding to the third powers are contained in the union of those associated to the first and second powers. This leads to an estimate for $\text{vol } N(P, \lambda/2)$ that involves only first and second powers of generators of C_P .

We begin by introducing some conventions that will be used for the statement and subsequent applications of Lemma 7.6.

7.1. For each integer $n \geq 1$ we define a function Φ_n on the domain

$$\{(\delta, D) : 0 < \delta \leq D\} \subset \mathbb{R}^2$$

by

$$\Phi_n(\delta, D) = \text{arccosh} \left(\cosh(n\delta) + \frac{(\cosh(n\delta) - 1)(\cosh D - \cosh \delta)}{\cosh \delta + 1} \right).$$

LEMMA 7.2. For each integer $n \geq 1$ the function Φ_n is increasing in each of the variables δ and D (in the sense of Subsection 2.3).

Proof. It is clear that Φ is increasing in the variable D . To prove that it is monotone increasing in δ , we fix an $n \geq 1$ and a $D > 0$, and set $A = \cosh D$. For any $\delta \in (0, D)$ we have $\Phi_n(\delta, D) = \text{arccosh}(\frac{1}{2}f(e^\delta))$, where f is the function defined on $(1, e^D)$ by

$$f(u) = u^n + u^{-n} + \frac{(u^n + u^{-n} - 2)(2A - u - u^{-1})}{u + u^{-1} + 2}.$$

It therefore suffices to show that f is monotone increasing on $(1, e^D)$.

Simplifying, we find that

$$f(u) = \frac{(2A + 2)(u^n + u^{-n} - 2)}{u + u^{-1} + 2} + 2.$$

This gives

$$f'(u) = \frac{2A + 2}{(u + u^{-1} + 2)^2} g(u),$$

where

$$g(u) = (n - 1)(u^n - u^{-n-2}) + (n + 1)(u^{n-2} - u^{-n}) + 2n(u^{n-1} - u^{-n-1}) + 2(1 - u^{-2}).$$

In this expression for $g(u)$, it is clear that each term is positive when $u > 1$ and $n \geq 1$. Hence, f' is positive on $(1, \infty)$. \square

LEMMA 7.3. Let δ be a positive number, and γ be a loxodromic isometry of \mathbb{H}^3 whose translation length is at least δ . Then, for any point $z \in \mathbb{H}^3$ and each integer $n \geq 1$, we have

$$\text{dist}(z, \gamma^n \cdot z) \leq \Phi_n(\delta, \text{dist}(z, \gamma \cdot z)).$$

Proof. Let R denote the distance from z to the axis of γ and set $D = \text{dist}(z, \gamma \cdot z)$. If γ has translation length l and twist angle θ , then $R = \omega(l, \theta, D)$ (see Subsection 2.4). In particular, we have

$$\sinh^2 R \geq \frac{\cosh D - \cosh l}{\cosh l + 1}.$$

Now, as γ^n has translation length nl and twist angle $n\theta$, we have

$$\begin{aligned} \cosh(\text{dist}(z, \gamma^n \cdot z)) &= \cosh(nl) + (\cosh(nl) - \cos(n\theta)) \sinh^2 R \\ &\geq \cosh(nl) + (\cosh(nl) - 1) \sinh^2 R \\ &\geq \cosh(nl) + \frac{(\cosh(nl) - 1)(\cosh D - \cosh l)}{\cosh l + 1} \\ &= \cosh \Phi_n(l, D) \\ &\geq \cosh \Phi_n(\delta, D), \end{aligned}$$

where the last inequality follows from Lemma 7.2. □

LEMMA 7.4. *If n is a positive integer and if δ and D are real numbers with $0 < \delta \leq D$, then $n\delta \leq \Phi_n(\delta, D) \leq nD$.*

Proof. The inequality $n\delta \leq \Phi_n(\delta, D)$ is immediate from the definition. To prove the other inequality, note that, as $0 < \delta \leq D$, there exists a loxodromic isometry γ of \mathbb{H}^3 with translation length δ , and a point $z \in \mathbb{H}^3$ such that $\text{dist}(z, \gamma \cdot z) = D$. The triangle inequality implies that

$$\text{dist}(z, \gamma^n \cdot z) \leq nD. \tag{7.1}$$

As Lemma 7.3 gives $\Phi_n(\delta, \text{dist}(z, \gamma \cdot z)) \leq \text{dist}(z, \gamma^n \cdot z)$, it follows that

$$\Phi_n(\delta, D) \leq nD. \tag{7.2}$$

□

7.5. If x and y are real numbers with $0 < y \leq 2x$, then we have

$$0 < (\coth x)(\coth y - \text{cosech } y) \leq (\coth x)(\coth 2x - \text{cosech } 2x) = 1.$$

Hence, on the domain

$$\{(x, y) : 0 < y \leq 2x\} \subset \mathbb{R}^2,$$

we may define a function Ψ , with values in $[0, \pi/2]$, by

$$\Psi(x, y) = \arccos((\coth x)(\coth y - \text{cosech } y)).$$

As $\coth x$ is monotone decreasing for $0 < x < \infty$, and $\coth y - \text{cosech } y$ is monotone increasing for $0 < y < \infty$, the function Ψ is monotone increasing in its first argument and monotone decreasing in its second.

If an isosceles hyperbolic triangle has base y and has its other two sides equal to x , the triangle inequality gives $y \leq 2x$, and the hyperbolic law of cosines shows that the base angles are equal to $\Psi(x, y)$.

The statement of the following lemma involves the function Θ , which was defined in Subsection 6.3, as well as the functions κ and σ , which were defined in Subsection 6.5.

LEMMA 7.6. *Let M be a closed, orientable hyperbolic 3-manifold and let P be a point of \mathfrak{G}_M . Suppose that δ and λ are constants with $0 < \lambda < 4\delta$. Assume that*

- (1) $\mathfrak{s}_M(P) \geq \lambda$; and that
- (2) the conjugacy class of a generator of C_P is represented by a closed geodesic of length at least δ .

Set $D = D_M(P)$. Then $T_n = \Phi_n(\delta, D)$ is defined for every $n \geq 1$. Furthermore, (D, T_2) lies in the domain of Ψ , and we have

$$\text{vol } N \left(P, \frac{\lambda}{2} \right) \geq B \left(\frac{\lambda}{2} \right) - 2\sigma \left(\frac{\lambda}{2}, \frac{D}{2}, \frac{T_2}{2}, \Psi(D, T_2) \right) - 2\kappa \left(\frac{\lambda}{2}, \frac{T_3}{2} \right).$$

If in addition we have $D < T_3 < \lambda$, so that, in particular, the quantities $\Theta(D/2, \lambda/2)$ and $\Theta(T_3/2, \lambda/2)$ are defined, and if

$$\cos \left(\Theta \left(\frac{D}{2}, \frac{\lambda}{2} \right) - \Theta \left(\frac{T_3}{2}, \frac{\lambda}{2} \right) \right) < \frac{\cosh D \cosh T_3 - \cosh 2D}{\sinh D \sinh T_3}, \quad (7.3)$$

then

$$\text{vol } N \left(P, \frac{\lambda}{2} \right) \geq B \left(\frac{\lambda}{2} \right) - 2\sigma \left(\frac{\lambda}{2}, \frac{D}{2}, \frac{T_2}{2}, \Psi(D, T_2) \right).$$

Proof. We write $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete, torsion-free, and cocompact. Set $q = q_\Gamma$ (2.6), and choose $\tilde{P} \in q^{-1}(P)$. We use the base point $\tilde{P} \in \mathbb{H}^3$ to identify $\pi_1(M, P)$ with Γ (see Subsection 2.6). In particular, C_P is identified with a subgroup of Γ . We fix a generator x of C_P .

The hypothesis implies that x has translation length at least δ . In particular, we have $D \geq \delta$, so that $T_n = \Phi_n(\delta, D)$ is defined for every $n \geq 1$.

According to Lemma 7.4, we have $T_2 = \Phi_2(\delta, D) \leq 2D$. Thus, (D, T_2) lies in the domain of Ψ .

For each integer $n \neq 0$ we set $d_n = \text{dist}(\tilde{P}, x^n \cdot \tilde{P})$. We observe that $d_{-n} = d_n$ for each $n \neq 0$ and that $d_1 = D$. Moreover, as x has translation length at least δ , it follows from Lemmas 7.4 and 7.3, and the triangle inequality, that

$$|n|\delta \leq \Phi_{|n|}(\delta, D) \leq d_n \leq |n|D, \quad (7.4)$$

for every integer $n \neq 0$, and in particular,

$$d_n \geq T_n \quad \text{for } n > 0. \quad (7.5)$$

We let ζ_n denote the point of intersection of $S(\lambda/2, \tilde{P})$ with the ray originating at \tilde{P} and passing through $x^n \cdot \tilde{P}$. We set

$$\mathcal{K}_n = K(\lambda/2, \tilde{P}, \zeta_n, d_n/2),$$

for each integer $n \neq 0$,

$$\mathcal{S} = \bigcup_{0 \neq n \in \mathbb{Z}} K(\lambda/2, \tilde{P}, \zeta_n, d_n/2)$$

and

$$\mathcal{S}_N = \bigcup_{0 < |n| \leq N} K(\lambda/2, \tilde{P}, \zeta_n, d_n/2),$$

for each integer $N > 0$. As $\mathfrak{s}_M(P) \geq \lambda$, it follows from Proposition 6.2 that

$$\text{vol}(N(P, \lambda/2)) = B(\lambda/2) - \text{vol } \mathcal{S}. \quad (7.6)$$

The estimates in the conclusion of the lemma will be deduced via (7.6) from suitable estimates for the volume of \mathcal{S} . As a first step, we shall estimate the volume of \mathcal{S}_2 . Note that

$$\mathcal{S}_2 = \mathcal{T}_{+1} \cup \mathcal{T}_{-1}, \quad (7.7)$$

where

$$\mathcal{T}_\epsilon = K(\lambda/2, \tilde{P}, \zeta_\epsilon, D/2) \cup K(\lambda/2, \tilde{P}, \zeta_{2\epsilon}, d_2/2),$$

for $\epsilon = \pm 1$.

For $\epsilon = \pm 1$ we consider the hyperbolic triangle with vertices \tilde{P} , $x^\epsilon \cdot \tilde{P}$, and $x^{2\epsilon} \cdot \tilde{P}$. The sides adjacent to the vertex \tilde{P} have lengths $d_1 = D$ and d_2 . The side opposite \tilde{P} has length $\text{dist}(x^\epsilon \cdot \tilde{P}, x^{2\epsilon} \cdot \tilde{P}) = \text{dist}(\tilde{P}, x^\epsilon \cdot \tilde{P}) = D$. Let α denote the angle at the vertex \tilde{P} . According to the discussion in Subsection 7.5 we have $d_2 \leq 2D$ and $\alpha = \Psi(D, d_2)$. By (7.5) we have $d_2 \geq T_2$. Hence, the monotonicity properties of Ψ pointed out in Subsection 7.5 give

$$\alpha \leq \Psi(D, T_2).$$

Our definitions of α and of the ζ_n imply that α is the spherical distance between ζ_ϵ and $\zeta_{2\epsilon}$. In view of the definition of \mathcal{T}_ϵ , and the definition of the function σ given in Subsection 6.5, it follows that $\text{vol}(\mathcal{T}_\epsilon) = \sigma(\lambda/2, D/2, d_2/2, \alpha)$ for $\epsilon = \pm 1$. As $\alpha \leq \Psi(D, T_2)$, and as $d_2 \geq T_2$, it follows from Proposition 6.7 that

$$\text{vol}(\mathcal{T}_\epsilon) \leq \sigma(\lambda/2, D/2, T_2/2, \Psi(D, T_2)),$$

for $\epsilon = \pm 1$. But, from (7.7), we have

$$\text{vol } \mathcal{S}_2 = \text{vol}(\mathcal{T}_{+1} \cup \mathcal{T}_{-1}) \leq \text{vol}(\mathcal{T}_{+1}) + \text{vol}(\mathcal{T}_{-1}),$$

and hence

$$\text{vol } \mathcal{S}_2 \leq 2\sigma(\lambda/2, D/2, T_2/2, \Psi(D, T_2)). \tag{7.8}$$

We now turn to the estimation of $\text{vol } \mathcal{S}$. As by hypothesis, we have $4\delta > \lambda$, it follows from (7.4) that $d_n > \lambda$ for any $n \geq 4$. In view of the remarks in Subsection 6.1, it follows that

$$\mathcal{K}_n = K(\lambda/2, \tilde{P}, \zeta_n, d_n/2) = \emptyset, \quad \text{for any } n \text{ with } |n| \geq 4.$$

Hence,

$$\mathcal{S} = \mathcal{S}_3. \tag{7.9}$$

It follows from (7.9) that

$$\text{vol}(\mathcal{S}) = \text{vol}(\mathcal{S}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_{-3}) \leq \text{vol}(\mathcal{S}_2) + \text{vol } \mathcal{K}_3 + \text{vol } \mathcal{K}_{-3}. \tag{7.10}$$

By the definition of the function κ given in Subsection 6.5, and the monotonicity observed there, together with (7.5), we find that

$$\text{vol } \mathcal{K}_3 + \text{vol } \mathcal{K}_{-3} = 2\kappa(\lambda/2, d_3/2) \leq 2\kappa(\lambda/2, T_3/2). \tag{7.11}$$

Combining (7.10) with (7.11) and (7.8), we deduce that

$$\text{vol}(\mathcal{S}) \leq 2\sigma(\lambda/2, D/2, T_2/2, \Psi(D, T_2)) + 2\kappa(\lambda/2, T_3/2). \tag{7.12}$$

The first assertion of the lemma follows immediately from (7.6) and (7.12).

For the proof of the second assertion, we shall begin by showing that if (7.3) holds, then

$$K(\lambda/2, \tilde{P}, \zeta_{3\epsilon}, d_3/2) \subset K(\lambda/2, \tilde{P}, \zeta_\epsilon, D/2) \quad \text{for } \epsilon = \pm 1. \tag{7.13}$$

If $d_3 \geq \lambda$, then, by the remarks in Subsection 6.1, we have $K(\lambda/2, \tilde{P}, \zeta_{3\epsilon}, d_3/2) = \emptyset$ for $\epsilon = \pm 1$, so that (7.13) is true in this case. Now suppose that $d_3 < \lambda$. Note that this implies that $\Theta(d_3/2, \lambda/2)$ is defined.

For $\epsilon = \pm 1$ we consider the hyperbolic triangle with vertices \tilde{P} , $x^\epsilon \cdot \tilde{P}$, and $x^{3\epsilon} \cdot \tilde{P}$. The sides adjacent to the vertex \tilde{P} have lengths $d_1 = D$ and d_3 . The side opposite \tilde{P} has length $\text{dist}(x^\epsilon \cdot \tilde{P}, x^{3\epsilon} \cdot \tilde{P}) = \text{dist}(\tilde{P}, x^{2\epsilon} \cdot \tilde{P}) = d_2$. From the hyperbolic law of cosines, it follows that the angle γ at the vertex \tilde{P} is determined by

$$\cos \gamma = \frac{\cosh D \cosh d_3 - \cosh d_2}{\sinh D \sinh d_3}.$$

We have $d_2 \leq 2D$ by (7.4). Thus,

$$\cos \gamma \geq \frac{\cosh D \cosh d_3 - \cosh 2D}{\sinh D \sinh d_3}.$$

If we set $f(D, d_3)$ equal to the right-hand side of the inequality above, then

$$\begin{aligned} \frac{\partial f}{\partial d_3} &= \frac{\cosh D \sinh D \sinh^2 d_3 - \cosh D \sinh D \cosh^2 d_3 + \sinh D \cosh 2D \cosh d_3}{\sinh^2 D \sinh^2 d_3} \\ &= \frac{\sinh D (\cosh(2D) \cosh d_3 - \cosh D)}{\sinh^2 D \sinh^2 d_3}. \end{aligned}$$

As $\cosh(2D) \cosh d_3 - \cosh D > \cosh(2D) - \cosh D > 0$, the function f is increasing in d_3 . By (7.5) we have $d_3 \geq T_3$. Hence,

$$\cos \gamma \geq \frac{\cosh D \cosh T_3 - \cosh 2D}{\sinh D \sinh T_3}.$$

In view of (7.3), it follows that

$$\cos(\Theta(D/2, \lambda/2) - \Theta(T_3/2, \lambda/2)) < \cos \gamma. \quad (7.14)$$

It is clear from the definition given in Subsection 6.3 that Θ is monotone decreasing in the first variable. As Θ takes values in $(0, \pi/2)$, and as $D < T_3$ by hypothesis, we have

$$\pi/2 > \Theta(D/2, \lambda/2) - \Theta(T_3/2, \lambda/2) > 0. \quad (7.15)$$

As $0 < \gamma < \pi$, it follows from (7.14) and (7.15) that

$$\gamma < \Theta(D/2, \lambda/2) - \Theta(T_3/2, \lambda/2).$$

As Θ is monotone decreasing in the first variable and as $T_3 \leq d_3$ by (7.5), we deduce that

$$\gamma < \Theta(D/2, \lambda/2) - \Theta(d_3/2, \lambda/2). \quad (7.16)$$

Our definitions of γ and of the ζ_n imply that γ is the spherical distance between ζ_ϵ and $\zeta_{3\epsilon}$. It therefore follows from Subsection 6.4 and from (7.16) that $K(\lambda/2, \tilde{P}, \zeta_{3\epsilon}, d_3/2) \subset K(\lambda/2, \tilde{P}, \zeta_\epsilon, D/2)$. This shows that (7.13) holds in this case as well.

Now it follows from (7.9) and (7.13) that, under the assumption (7.3), we have

$$\mathcal{S} = \mathcal{S}_3 = \mathcal{S}_2. \quad (7.17)$$

Combining (7.17) with (7.8), we deduce that

$$\text{vol } \mathcal{S} \leq \sigma(\lambda/2, D/2, T_2/2, \Psi(D, T_2)). \quad (7.18)$$

The second assertion of the lemma follows immediately from (7.6) and (7.18). \square

8. Distant points

The purpose of this section is to adapt some results proved in [15] to the context of the present paper.

8.1. We define a set $\mathcal{X} \subset \mathbb{R}^2$ by

$$\mathcal{X} = \left\{ (D, \lambda) : \frac{1}{1 + e^D} + \frac{1}{1 + e^\lambda} < \frac{1}{2} \right\}.$$

For each integer $k > 2$ we define a real-valued function ρ_k on \mathcal{X} by

$$\rho_k(D, \lambda) = \frac{1}{2} \log \left(\frac{k - 2}{1/2 - 1/(1 + e^D) - 1/(1 + e^\lambda)} - 1 \right).$$

Thus, for any $(D, \lambda) \in \mathcal{X}$, we have

$$\frac{k-2}{1+e^{2\rho_k(D,\lambda)}} + \frac{1}{1+e^D} + \frac{1}{1+e^\lambda} = \frac{1}{2}.$$

We define a real-valued function Σ on $[0, \infty)^3$ by

$$\Sigma(h, R_1, R_2) = \operatorname{arccosh}(\sinh R_1 \sinh R_2 + \cosh R_1 \cosh R_2 \cosh h).$$

According to [16, p. 89], given a line A in \mathbb{H}^2 and points z_1 and z_2 of \mathbb{H}^2 , which lie in different components of $\mathbb{H}^2 - A$, if R_i is the distance from z_i to A , and if h is the distance between the perpendicular projections of z_1 and z_2 to A , then $\operatorname{dist}(z_1, z_2) = \Sigma(h, R_1, R_2)$.

8.2. It follows that if z_1 and z_2 are points in \mathbb{H}^3 , if A is a line in \mathbb{H}^3 , if R_i is the distance from z_i to A , and h is the distance between the perpendicular projections of z_1 and z_2 to A , then $\operatorname{dist}(z_1, z_2) \leq \Sigma(h, R_1, R_2)$.

LEMMA 8.3. *Let $k > 2$ be an integer, let M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is k -free, and let μ be a Margulis number for M . Let P be a point of \mathfrak{G}_M , and set $\lambda = \mathfrak{s}_M(P)$ and $D = D_M(P)$. Then we have $(D, \lambda) \in \mathcal{X}$, and there is a point $Q \in M_{\text{thick}}(\mu)$ such that $\operatorname{dist}(P, Q) \geq \rho_k(D, \lambda)$.*

Proof. It follows from the definitions given in Subsection 3.8 that a generator x_0 of C_P is represented by a loop at P of length D , and that some element x_1 of $\Gamma - C_P$ is represented by a loop at P of length λ .

As $x_1 \notin C_P$, the elements x_0 and x_1 do not commute. As $\Gamma \cong \pi_1(M)$ is in particular 2-free, x_0 and x_1 are independent (in the sense of Subsection 2.1).

As Γ is k -free, and as there are two independent elements of $\pi_1(M, P)$ represented by loops of length D and λ , it follows from the case $m = 2$ of Culler and Shalen [15, Corollary 6.2] that there is a point $Q \in M_{\text{thick}}(\mu)$ such that $\rho = \operatorname{dist}_M(P, Q)$ satisfies

$$\frac{k-2}{1+e^{2\rho}} + \frac{1}{1+e^D} + \frac{1}{1+e^\lambda} \leq \frac{1}{2}. \tag{8.1}$$

This implies that $(D, \lambda) \in \mathcal{X}$, and that $\operatorname{dist}(P, Q) \geq \rho_k(D, \lambda)$. □

PROPOSITION 8.4. *Let $k > 2$ be an integer, let M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is k -free, and let μ be a Margulis number for M . Let P be a point of \mathfrak{G}_M , and set $\lambda = \mathfrak{s}_M(P)$ and $D = D_M(P)$. Then we have $(D, \lambda) \in \mathcal{X}$. Furthermore, if s is a real number such that $\lambda/2 \leq s \leq \rho_k(D, \lambda)$, then there is a point $Y_s \in M_{\text{thick}}(\mu)$ such that $\operatorname{dist}(P, Y_s) = s$.*

Note that the second assertion of Proposition 8.4 is vacuous if $\lambda/2 > \rho_k(D, \lambda)$.

Proof of Proposition 8.4. The first assertion, that $(D, \lambda) \in \mathcal{X}$, is included in Lemma 8.3.

To prove the second assertion, we first define a continuous function $\Delta : M_{\text{thick}}(\mu) \rightarrow \mathbb{R}$ by $\Delta(Y) = \operatorname{dist}_M(Y, P)$. As μ is a Margulis number for M , the set $M_{\text{thick}}(\mu)$ is connected by Subsection 3.6. As Δ is continuous, the set $J = \Delta(\mathcal{Y}) \subset \mathbb{R}$ is an interval.

According to Lemma 8.3, there is a point $Q \in M_{\text{thick}}(\mu)$ such that $\operatorname{dist}(P, Q) \geq \rho_k(D, \lambda)$. If we set $d = \Delta(Q)$, then it follows that $d \geq \rho_k(D, \lambda)$ and that $d \in J$.

Now recall from Subsection 3.8 that there is a loop γ_0 of length $\ell_P < \mathfrak{s}_M(P) = \lambda$ based at P ; that $[\gamma_0]$ lies in the maximal cyclic subgroup C_P of $\pi_1(M, P)$; and that there is a loop

γ_1 of length $\mathfrak{s}_M(P) = \lambda$ based at P such that $[\gamma_1] \notin C_P$. In particular, $[\gamma_0]$ and $[\gamma_1]$ do not commute. Hence, if we denote by c_i the support of γ_i (that is, $c_i = \gamma_i([0, 1])$), and by K the path-connected set $c_0 \cup c_1 \subset M$, then the inclusion homomorphism $\pi_1(K, P) \rightarrow \pi_1(M, P)$ has a non-abelian image. As μ is a Margulis number for M , each component of $M_{\text{thin}}(\mu)$ has an abelian fundamental group, and hence $K \not\subset M_{\text{thin}}(\mu)$.

Let us fix a point $Q' \in K \cap M_{\text{thick}}(\mu)$, and set $d' = \Delta(Q') \in J$. The definition of K implies that Q' lies in the support of a loop of length at most λ based at P , and hence that $d' = \text{dist}(P, Q') \leq \lambda/2$.

Now if s is a real number such that $\lambda/2 \leq s \leq \rho_k(D, \lambda)$, then, in particular, we have $d' \leq s \leq d$. As the interval J contains d and d' , it also contains s . This means that there is a point $Y_s \in M_{\text{thick}}(\mu)$ such that $s = \Delta(Y_s) = \text{dist}(P, Y_s)$. \square

9. The volume of a metric ball

We have

$$B(r) = \pi(\sinh(2r) - 2r). \quad (9.1)$$

9.1. For $n \geq 2$ and for $R > 0$, we shall denote by $h_n(R)$ the distance from the barycenter to a vertex of the regular hyperbolic tetrahedron $\Delta(R)$ with sides of length $2R$. It is easy to verify, using hyperbolic trigonometry, that

$$\tanh h_2(R) = \frac{2 \sinh^2 R}{\sqrt{\cosh^2(2R) - \cosh^2 R}}$$

and

$$\tanh h_3(R) = \frac{2 \sinh^2 R}{\sqrt{\cosh^2(2R) - \cosh^2 h_2(R)}}. \quad (9.2)$$

9.2. Let R be any positive number. Consider an arbitrary sphere-packing in \mathbb{H}^3 by spheres of radius R . Let \mathcal{D} denote the Dirichlet domain for this packing, centered at a point $z \in \mathbb{H}^3$. The main result of Böröczky's paper [7] states that

$$\text{vol } \mathcal{D} \geq B(R)/d(R), \quad (9.3)$$

where d is a function, the definition of which will be reviewed below.

Now let M be a hyperbolic 3-manifold, and suppose that Y is a point in M that is the center of a hyperbolic ball of radius R (that is, Y is a $2R$ -thick point). Let us write $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}(\mathbb{H}^3)$ is discrete and torsion-free. Set $q = q_\Gamma$ (2.6), and let \tilde{Y} denote a point in $q^{-1}(Y)$. Then $q^{-1}(Y)$ is the set of center points for a sphere-packing in \mathbb{H}^3 by spheres of radius R . Applying (9.3) with $z = \tilde{Y}$, we find that $\text{vol } M \geq B(R)/d(R)$.

Böröczky's proof of (9.3) actually gives a stronger conclusion. It is shown in the proof that, in fact,

$$\text{vol}(\mathcal{D} \cap N(z, h_3(R))) \geq B(R)/d(R). \quad (9.4)$$

Although (9.4) may not give improved estimates for the density of a sphere-packing, it does have a very natural interpretation for a sphere-packing defined as above in terms of a hyperbolic 3-manifold M and a point $Y \in M$: it is equivalent to the statement that $\text{vol } N(Y, h_3(R)) \geq B(R)/d(R)$.

The main result of this section, Proposition 9.7, will give a stronger lower bound for the volume of $N(Y, h_3(R))$ under the stronger hypothesis that there exists a point $Q \in M$ that is sufficiently far away from Y .

9.3. We now give the definition of Böröczky's density function d . Let

$$\beta(r) = \operatorname{arcsec}(\operatorname{sech}(2r) + 2)$$

denote the dihedral angle of $\Delta(r)$. Let

$$\tau(r) = 3 \int_{\beta(r)}^{\operatorname{arcsec} 3} \operatorname{arcsech}(\sec t - 2) dt$$

denote the volume of $\Delta_3(r)$. Then

$$d(r) = (3\beta(r) - \pi)(\sinh(2r) - 2r)/\tau(r).$$

9.4. Böröczky's inequality (9.4) applies in particular to the case of a sphere-packing in \mathbb{H}^3 consisting of a single sphere of radius R centered at a point z . In this case, we have $\mathcal{D} = \mathbb{H}^3$, so that $\operatorname{vol}(\mathcal{D} \cap N(z, h_3(R))) = \operatorname{vol}(N(z, h_3(R))) = B(h_3(R))$. Thus, (9.4) becomes

$$B(h_3(R)) \geq B(R)/d(R), \tag{9.5}$$

which therefore holds for every $R > 0$.

The statement of Proposition 9.7 involves a function $V_{\text{Bör}}$ that we shall now define.

DEFINITION 9.5. Let R and ρ be positive real numbers such that $\rho > h_3(R)$. We define

$$\begin{aligned} \phi_1(R, \rho) &= \arcsin \left(\frac{\sqrt{\cosh^2 \rho - \cosh^2 R}}{\sinh \rho \cosh R} \right), \\ \phi_2(R) &= \arcsin \left(\frac{\sqrt{\cosh^2 h_3(R) - \cosh^2 R}}{\sinh h_3(R) \cosh R} \right), \\ \phi(R, \rho) &= \phi_1(R, \rho) - \phi_2(R) \end{aligned}$$

and

$$V_{\text{Bör}}(R, \rho) = \left(\frac{1 - \cos \phi(R, \rho)}{2} \right) B(h_3(R)) + \left(\frac{1 + \cos \phi(R, \rho)}{2} \right) B(R)/d(R).$$

REMARK 9.6. For any fixed $R > 0$ the function $\phi(R, \cdot)$ is positive-valued and monotone increasing on $(h_3(R), \infty)$. In view of the inequality (9.5), it follows that $V_{\text{Bör}}(R, \cdot)$ is also monotone increasing on $(h_3(R), \infty)$.

PROPOSITION 9.7. Let M be a hyperbolic 3-manifold, let R be a positive number, and suppose that Y is a $2R$ -thick point in M . Suppose that there exists a point $P \in M$ such that $\rho \doteq \operatorname{dist}_M(Y, P) > h_3(R)$. Then

$$\operatorname{vol} N(Y, h_3(R)) \geq V_{\text{Bör}}(R, \rho). \tag{9.6}$$

Proof of Proposition 9.7. We set $q = q_{\Gamma}$ (2.6), and we choose a point $\tilde{Y} \in q^{-1}(Y)$. As $\ell_Y \geq 2R$, there is a sphere-packing in \mathbb{H}^3 consisting of spheres of radius R centered at the

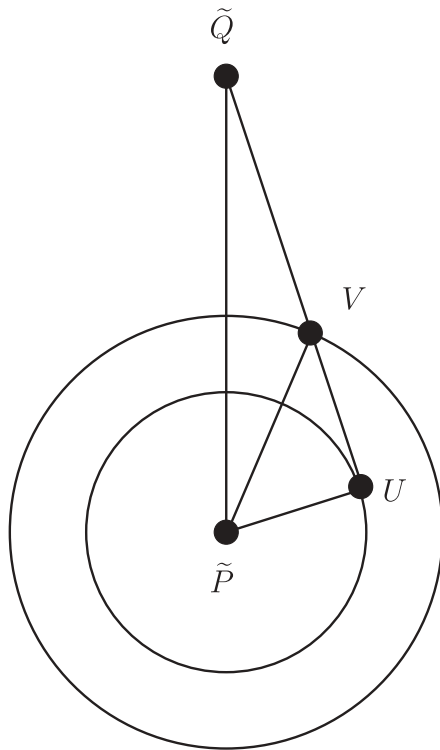
points $q^{-1}(Y)$. Let \mathcal{D} be a Dirichlet domain for the sphere centered at \tilde{Y} . Let \mathcal{B} denote the ball of radius R centered at \tilde{Y} and let \mathcal{B}' denote the ball of radius $h_3(R)$ centered at \tilde{Y} . If X is a subset of \mathbb{H}^3 , then we let $C(X)$ denote the union of all rays from \tilde{Y} that contain a point of X .

The construction given in [7, Section 5], when restricted to the three-dimensional case, begins by decomposing \mathcal{D} as the union of $D_0 = C(\mathcal{D} \cap \partial\mathcal{B}')$ and D_1 , where D_1 is the union of the sets of the form $C(F \cap \mathcal{B}')$ as F runs over the closed two-dimensional faces of \mathcal{D} . The set D_1 is then further subdivided to obtain a decomposition of \mathcal{D} into the union of D_0 and a certain family of three-dimensional convex cells. For each cell E it is shown that $\text{vol}(\mathcal{B} \cap E) / \text{vol}(\mathcal{B}' \cap E) \leq d(R)$. Moreover, as D_0 is the cone based at \tilde{Y} on the subset of $\partial\mathcal{B}'$ that is contained in \mathcal{D} , we have $\text{vol}(\mathcal{B} \cap D_0) / \text{vol}(\mathcal{B}' \cap D_0) = \text{vol} \mathcal{B} / \text{vol} \mathcal{B}' < d(R)$. Thus, $\text{vol}(\mathcal{B} \cap \mathcal{D}) / \text{vol}(\mathcal{B}' \cap \mathcal{D}) \leq d(R)$; as $\mathcal{B} \subset \mathcal{D}$, this implies Böröczky's stated result that $\text{vol} \mathcal{B} / \text{vol} \mathcal{D} \leq d(R)$.

We may summarize the discussion above as follows. If we set $t = \text{vol} D_0 / \text{vol} \mathcal{B}'$, so that $\text{vol} D_0 = tB(h_3(R))$, then Böröczky's argument implies that

$$\text{vol} \mathcal{D} = \text{vol} D_0 + \text{vol}(\mathcal{D} - D_0) \geq tB(h_3(R)) + (1 - t)B(R)/d(R). \tag{9.7}$$

(It is shown in [7, Lemma 12] that the vertices of \mathcal{D} lie outside \mathcal{B}' , so the set D_0 is always non-empty; that is, we have $t > 0$ in (9.7). This observation does not strengthen Böröczky's theorem about general sphere-packings, as one has no a priori information about the size of the set D_0 . Our goal here is to quantify the improvement given by (9.7) in terms of $\text{dist}(Y, P)$.)



In order to establish (9.6), it suffices to show that the quantity $\text{vol} D_0 / \text{vol} \mathcal{B}'$, which is denoted by t in (9.7), is greater than $(1 - \cos \phi) / 2$, where ϕ is the angle defined in the statement. For this, it suffices to show that D_1 contains $C \cap \mathcal{B}'$, where C is the convex region bounded by a circular cone with apex at \tilde{Y} , such that the angle between the axis and a generating line of ∂C is at least ϕ .

Let \tilde{P} be a point of \mathcal{D} such that $q(\tilde{P}) = P$. We know that \mathcal{D} contains the ball \mathcal{B} . As \mathcal{D} is convex, the entire convex hull H of $\mathcal{B} \cup \{\tilde{P}\}$ is contained in \mathcal{D} . We will show that H contains a conical region C of angle ϕ .

Consider a line through \tilde{P} that is tangent to $\partial\mathcal{B}$ at a point U , and let V be the point where the segment $[\tilde{P}, U]$ meets $\partial\mathcal{B}'$.

We have $\text{dist}(\tilde{Y}, \tilde{P}) = \rho$, $\text{dist}(\tilde{Y}, V) = h_3(R)$, and $\text{dist}(\tilde{Y}, U) = R$. The angle $\angle\tilde{Y}U\tilde{P}$ is a right angle. Hence, it follows easily from the hyperbolic Pythagorean Theorem and the hyperbolic law of sines that the measure of $\angle\tilde{P}\tilde{Y}U$ is ϕ_1 and the measure of $\angle V\tilde{Y}U$ is ϕ_2 , where ϕ_1 and ϕ_2 are as defined in the statement. It is clear that H contains $\mathcal{B} \cap C$, where C is the convex region bounded by the circular cone with apex at \tilde{Y} , whose axis contains \tilde{P} and whose boundary contains V . As $\phi = \phi_1 - \phi_2$ is the angle between the axis and a generator of the boundary cone of C , the result follows. \square

10. Margulis numbers and diameter

LEMMA 10.1. *Let m be a positive integer and let H be a finitely generated group that is m -free and has rank $\geq m$. Let S be a finite generating set for H and let $T_0 \subset S$ be an independent set (see Subsection 2.1). Then there exists an independent set T with $T_0 \subset T \subset S$ and $|T| = m$.*

Proof. Among all independent sets T such that $T_0 \subset T \subset S$, we choose one, say T_1 , that is maximal with respect to inclusion. Assume, for a contradiction, that $|T_1| < m$.

We are given that H has rank at least m . Hence, among all sets T , such that $T_1 \subset T \subset S$ and $\langle T \rangle$ has rank at least m , we may choose one, say T_2 , that is minimal with respect to inclusion. As $\langle T_1 \rangle$ has rank less than m , we have $T_2 \neq T_1$. We choose an element $x_0 \in T_2 - T_1$. The minimality of T_2 implies that the group $J = \langle T_2 - \{x_0\} \rangle$ has rank less than m . As $\langle T_2 \rangle = \langle J \cup \{x_0\} \rangle$ has rank at least m , the rank of J must be $m - 1$ and T_2 must have rank m . We fix a generating set $\{x_1, \dots, x_{m-1}\}$ for J . Then $\{x_0, \dots, x_{m-1}\}$ is a generating set for $\langle T_2 \rangle$. But the rank- m group $\langle T_2 \rangle$ is free as H is m -free, and so $\{x_0, \dots, x_{m-1}\}$ is a basis for $\langle T_2 \rangle$ (cf. Subsection 2.1). Hence, $\langle T_2 \rangle$ is a free product $J \star \langle x_0 \rangle$, where $\langle x_0 \rangle$ is infinite cyclic. In particular, the group $\langle T_1 \cup \{x_0\} \rangle$ is a free product $\langle T_1 \rangle \star \langle x_0 \rangle$, and hence the set $T_1 \cup \{x_0\}$ is independent. This contradicts the maximality of T_1 . \square

PROPOSITION 10.2. *Let k and m be integers with $2 \leq m < k$, and let M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is k -free. Let μ be a Margulis number for M , let λ be a positive real number such that*

$$\frac{m-1}{1+e^\lambda} + \frac{1}{1+e^\mu} \geq \frac{1}{2},$$

and let Δ denote the extrinsic diameter (2.7) of $M_{\text{thick}}(\mu)$ in M . Suppose that

$$\frac{m}{1+e^\lambda} + \frac{k-m}{1+e^{2\Delta}} \geq \frac{1}{2}.$$

Let P be any point in M , and let H denote the subgroup of $\pi_1(M, P)$ generated by all elements that are represented by loops of length less than λ . Then H has rank less than m .

Proof. Let us write $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete, cocompact, and torsion-free. We set $q = q_\Gamma$ (2.6) and choose a point $z \in q^{-1}(P)$. We use the base point $z \in \mathbb{H}^3$ to identify $\pi_1(M, P)$ with Γ (see Subsection 2.6). We may then regard H as the subgroup generated by the

set S consisting of all elements $\xi \in \Gamma$ such that $\text{dist}(\xi \cdot z, z) < \lambda$. The discreteness of Γ implies that S is finite. As Γ is m -free, $H \leq \Gamma$ is in particular m -free; this will allow applications of Lemma 10.1 to H .

We distinguish two cases, depending on whether $P \in M_{\text{thick}}(\mu)$ or $P \in M_{\text{thin}}(\mu)$.

First, suppose that $P \in M_{\text{thick}}(\mu)$. In this case, we apply Lemma 10.1, taking $T_0 = \emptyset$. In order to show that $\text{rank } H < m$, it suffices to show that there is no independent set $T \subset S$ such that $|T| = m$. Suppose that $T_0 = \{\xi_1, \dots, \xi_m\}$ is such a set. We let $d_i < \lambda$ denote the minimal length of a loop based at P and representing ξ_i .

As ξ_1, \dots, ξ_m are independent and $\pi_1(M)$ is k -free, we may apply [15, Corollary 6.2] to obtain a point $Q \in M_{\text{thick}}(\mu)$ such that $\rho = \text{dist}_M(P, Q)$ satisfies

$$\frac{k-m}{1+e^{2\rho}} + \sum_{i=1}^m \frac{1}{1+e^{d_i}} \leq \frac{1}{2}. \quad (10.1)$$

Hence, $\rho \leq \Delta$. As $\xi_i \in S$, we have $d_i < \lambda$ for $i = 1, \dots, m$. It therefore follows from (10.1) that

$$\frac{m}{1+e^\lambda} + \frac{k-m}{1+e^{2\Delta}} < \frac{1}{2}.$$

This contradicts the hypothesis.

Now suppose that $P \in M_{\text{thin}}(\mu)$. We fix an element $\eta \neq 1$ of Γ such that $\text{dist}(z, \eta \cdot z) < \mu$. In this case, we apply Lemma 10.1, letting $S \cup \{\eta\}$ play the role of S in Lemma 10.1, and taking $T_0 = \{\eta\}$. In order to show that $\text{rank } H < m$, it suffices to show that there is no independent set T with $\eta \in T \subset S \cup \{\eta\}$ such that $|T| = m$. Suppose that $T_0 = \{\xi_1, \dots, \xi_m\}$ is such a set, with $\xi_1 = \eta$ and $\xi_2, \dots, \xi_m \in S$. We write $d_i = \text{dist}(z, \xi_i \cdot z)$ for $i = 1, \dots, m$. As ξ_1, \dots, ξ_m are independent, it follows from [5, Theorem 6.1], together with the main result of Agol [1] or Calegari and Gabai [9], that

$$\sum_{i=1}^m \frac{1}{1+e^{d_i}} \leq \frac{1}{2}. \quad (10.2)$$

As $\xi_1, \dots, \xi_m \in S$, we have $d_i \leq \lambda$ for $i = 2, \dots, m$. Our choice of $\xi_1 = \eta$ gives $d_1 < \mu$. Hence, (10.2) gives

$$\frac{m-1}{1+e^\lambda} + \frac{1}{1+e^\mu} < \frac{1}{2}.$$

This contradicts the hypothesis. □

COROLLARY 10.3. *Let $k > 2$ be an integer and M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is k -free. Let μ be a Margulis number for M , let λ be a positive real number such that*

$$\frac{1}{1+e^\lambda} + \frac{1}{1+e^\mu} \geq \frac{1}{2},$$

and let Δ denote the extrinsic diameter (2.7) of $M_{\text{thick}}(\mu)$ in M . Suppose that

$$\frac{2}{1+e^\lambda} + \frac{k-2}{1+e^{2\Delta}} \geq \frac{1}{2}.$$

Then λ is itself a Margulis number for M .

Proof. This is the case $m = 2$ of Proposition 10.2. □

LEMMA 10.4. *Suppose that M is a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 4-free and $\text{vol } M \leq 3.468$. Then 1.119 is a Margulis number for M .*

Proof. Set $\lambda = 1.119$ and $\mu = 1.078$. By direct computation, we find that

$$\frac{1}{e^\lambda + 1} + \frac{1}{e^\mu + 1} > \frac{1}{2}. \tag{10.3}$$

As $\pi_1(M)$ is 2-free, it follows from [3, Corollary 4.2] that $\log 3$ is a Margulis number for M , so that $\mu < \log 3$ is also a Margulis number for M .

As $\pi_1(M)$ is 3-free, it follows from [2, Corollary 9.3] that some point $P \in M$ is the center of a hyperbolic ball of radius $(\log 5)/2$.

Let Δ denote the extrinsic diameter of the compact subset $M_{\text{thick}}(\mu)$ of M , and set

$$\rho = \max_{x \in M_{\text{thick}}(\mu)} \text{dist}(P, x).$$

The triangle inequality implies that $\Delta \leq 2\rho$.

Choose a point $Q \in M_{\text{thick}}(\mu)$ such that $\text{dist}(P, Q) = \rho$. Set $V = N(Q, 0.444)$ and $W = N(P, h_3((\log 5)/2))$. As $0.444 < \mu/2$, the set V is (intrinsically) isometric to a hyperbolic ball of radius 0.444, and hence $\text{vol } V = B(0.444) = 0.381\dots$. On the other hand, as P is the center of a ball of radius $(\log 5)/2$, it follows from (9.3) that $\text{vol } W \geq B((\log 5)/2)/d((\log 5)/2) = 3.087\dots$. Hence, $\text{vol } V + \text{vol } W > 3.468 \geq \text{vol } M$, and therefore $V \cap W \neq \emptyset$. The triangle inequality therefore implies that

$$\rho = \text{dist}(P, Q) \leq h_3\left(\frac{\log 5}{2}\right) + 0.444 < 1.392.$$

Hence, $\Delta < 2.784$. We therefore have

$$\frac{2}{1 + e^\lambda} + \frac{2}{1 + e^{2\Delta}} > \frac{2}{1 + e^{1.119}} + \frac{2}{1 + e^{5.568}} > \frac{1}{2}. \tag{10.4}$$

It follows from (10.3) and (10.4) that the hypotheses of Corollary 10.3 hold with $k = 4$, and with λ and μ defined as above. Hence, λ is a Margulis number for M . \square

11. Distant volume

11.1. We set

$$\mu_0 = 1.119$$

and

$$h = h_3\left(\frac{\mu_0}{2}\right) = 0.67\dots,$$

where h_3 is the function defined in Subsection 9.1 and is calculated using (9.2).

We shall define a function $V_{\text{far}}(D, \lambda)$ on the set \mathcal{X} that was defined in Subsection 8.1. The definition will use the function ρ_4 that was defined in Subsection 8.1, the function B defined by (9.1), and the function $V_{\text{Bör}}$ given by Definition 9.5. We set $Z(D, \lambda) = \rho_4(D, \lambda) - \lambda/2$ and define

$$V_{\text{far}}(D, \lambda) = \begin{cases} V_{\text{Bör}}\left(\frac{\mu_0}{2}, h + \frac{\lambda}{2}\right) \\ \quad + B\left(\min\left(\frac{\mu_0}{2}, \frac{1}{2}(Z(D, \lambda) - h)\right)\right) & \text{if } Z(D, \lambda) > h, \\ B\left(\min\left(\frac{\mu_0}{2}, Z(D, \lambda)\right)\right) & \text{if } 0 < Z(D, \lambda) \leq h, \\ 0 & \text{if } Z(D, \lambda) \leq 0. \end{cases}$$

(Note that $V_{\text{Bör}}(\mu_0/2, h + \lambda/2)$ is defined for every $\lambda > 0$, as $h + \lambda/2 > h = h_3(\mu_0/2)$.)

LEMMA 11.2. *The function V_{far} is monotone decreasing in its first argument.*

Proof. Suppose $D_1 < D_2$. Set $\rho^{(i)} = \rho_4(D_i, \lambda)$ for $i = 1, 2$. It is clear from the definition given in Subsection 8.1 that ρ_4 is monotone decreasing in its first argument, and hence

$$\rho^{(1)} \geq \rho^{(2)}. \quad (11.1)$$

It is clear from the definition of V_{far} that $V_{\text{far}}(D_1, \lambda)$ and $V_{\text{far}}(D_2, \lambda)$ are both non-negative. In the case where $\rho^{(2)} \leq \lambda/2$, we have $V_{\text{far}}(D_2, \lambda) = 0$, and hence

$$V_{\text{far}}(D_1, \lambda) \geq V_{\text{far}}(D_2, \lambda). \quad (11.2)$$

If $\rho^{(2)} > \lambda/2$, then by (11.1) we have $\rho^{(1)} > \lambda/2$.

If $\rho^{(1)}$ and $\rho^{(2)}$ both lie in the interval $(\lambda/2, h + \lambda/2]$, then (11.2) follows from (11.1). If $\rho^{(1)}$ and $\rho^{(2)}$ both lie in the interval $(h + \lambda/2, \infty)$, then (11.2) follows from (11.1) and Remark 9.6. Finally, suppose that $\lambda/2 < \rho^{(2)} \leq h + \lambda/2$ and that $\rho^{(1)} > h + \lambda/2$. As the definition of $V_{\text{Bör}}$ immediately implies that $V_{\text{Bör}}(R, \rho) \geq B(R)$ for any $R > 0$ and any $\rho > h_3(R)$, we have $V_{\text{far}}(D_1, \lambda) \geq V_{\text{Bör}}(\mu_0/2, h + \lambda/2) \geq B(\mu_0/2) \geq V_{\text{far}}(D_2, \lambda)$, so that (11.2) holds in this case as well. \square

LEMMA 11.3. *Let M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 4-free. Suppose that μ_0 is a Margulis number for M . Suppose that P is a point of \mathfrak{G}_M , and set $\lambda = \mathfrak{s}_M(P)$. Then $(D_M(P), \lambda) \in \mathcal{X}$, so that $V_{\text{far}}(D_M(P), \lambda)$ is defined, and*

$$\text{vol}(M - N(P, \lambda/2)) \geq V_{\text{far}}(D_M(P), \lambda).$$

Proof. It follows from Lemma 8.3 that $(D_M(P), \lambda) \in \mathcal{X}$.

Let us set $D = D_M(P)$ and $\rho = \rho_4(D, \lambda)$.

The lemma is trivial if $\rho \leq \lambda/2$, as we have $V_{\text{far}}(D, \lambda) = 0$ in that case. We shall therefore assume that $\lambda/2 < \rho$. It then follows from Proposition 8.4 that there is a point $Y_\rho \in M_{\text{thick}}(\mu_0)$ such that

$$\text{dist}(P, Y_\rho) = \rho. \quad (11.3)$$

Consider the case in which $\lambda/2 < \rho \leq h + \lambda/2$. In this case, we set $r = \min(\mu_0/2, \rho - \lambda/2) > 0$, so that according to the definition in Subsection 11.1 we have $V_{\text{far}}(D) = B(r)$. As $r + \lambda/2 \leq \rho \leq \text{dist}(P, Y_\rho)$, we have $N(P, \lambda/2) \cap N(Y_\rho, r) = \emptyset$. Hence,

$$\text{vol}(M - N(P, \lambda/2)) \geq \text{vol} N(Y_\rho, r). \quad (11.4)$$

On the other hand, as $Y_\rho \in M_{\text{thick}}(\mu_0)$ and $0 < r \leq \mu_0/2$, the set $N(Y_\rho, r)$ is intrinsically isometric to a hyperbolic ball of radius r , and so

$$\text{vol} N(Y_\rho, r) = B(r) = V_{\text{far}}(D, \lambda). \quad (11.5)$$

In this case, the conclusion of the lemma follows from (11.4) and (11.5).

Now consider the case in which $\rho > h + \lambda/2$. In this case, we set $\nu = \frac{1}{2}(\rho - (h + (\lambda/2))) > 0$. We then have

$$\rho = h + \frac{\lambda}{2} + 2\nu. \quad (11.6)$$

If we set $t = \lambda/2 + \nu$, then we therefore have $\lambda/2 < t < \rho$, and it follows from Proposition 8.4 that there is a point $Y_t \in M_{\text{thick}}(\mu_0)$ such that

$$\text{dist}(Y_t, P) = t = \frac{\lambda}{2} + \nu. \quad (11.7)$$

From (11.3), (11.6), (11.7), and the triangle inequality, it follows that

$$\text{dist}(Y_t, Y_\rho) \geq h + \nu. \tag{11.8}$$

From (11.7), (11.8), and the triangle inequality, we deduce that

$$N(P, \lambda/2) \cap N(Y_t, \nu) = \emptyset = N(Y_\rho, h) \cap N(Y_t, \nu). \tag{11.9}$$

On the other hand, as $\rho > h + \lambda/2$, it follows from (11.3) that

$$\text{dist}(Y_\rho, P) > h + \lambda/2,$$

so that the triangle inequality gives

$$N(P, \lambda/2) \cap N(Y_\rho, h) = \emptyset. \tag{11.10}$$

From (11.9) and (11.10), we deduce that

$$\text{vol}(M - N(P, \lambda/2)) \geq \text{vol} N(Y_t, \nu) + \text{vol} N(Y_\rho, h). \tag{11.11}$$

We now apply Proposition 9.7, taking $R = \mu_0/2$, taking for Y the μ_0 -thick point Y_ρ , and defining P and ρ as above. This gives $\text{vol} N(Y, h_3(R)) \geq V_{\text{Bör}}(\mu_0/2, \rho)$. As $\rho > h + \lambda/2$, it follows from Remark 9.6 that $V_{\text{Bör}}(\mu_0/2, \rho) > V_{\text{Bör}}(\mu_0/2, h + (\lambda/2))$. Hence,

$$\text{vol} N(Y_\rho, h) > V_{\text{Bör}}\left(\frac{\mu_0}{2}, h + \frac{\lambda}{2}\right). \tag{11.12}$$

Set $m = \min(\mu_0/2, \nu)$. As $m \leq \mu_0/2$ and $Y_t \in M_{\text{thick}}(\mu_0)$, the set $N(Y_t, m)$ is intrinsically isometric to a hyperbolic ball of radius m . Hence,

$$\text{vol} N(Y_t, \nu) \geq B(m). \tag{11.13}$$

From (11.11)–(11.13), it follows immediately that

$$\text{vol}(M - N(P, \lambda/2)) > V_{\text{Bör}}\left(\frac{\mu_0}{2}, h + \frac{\lambda}{2}\right) + B(m) = V_{\text{far}}(D),$$

which gives the conclusion of the lemma in this case. □

12. The case where there is no short geodesic

We set $\delta_0 = 0.58$ and $\lambda_0 = \log 7$.

LEMMA 12.1. *Let D be a number with $\delta_0 \leq D \leq 0.7$, and set $T_3 = \Phi_3(\delta_0, D)$. Then we have $D < T_3 < \lambda_0$, so that, in particular, $\Theta(D/2, \lambda_0/2)$ and $\Theta(T_3/2, \lambda_0/2)$ are defined, and*

$$\cos\left(\Theta\left(\frac{D}{2}, \frac{\lambda_0}{2}\right) - \Theta\left(\frac{T_3}{2}, \frac{\lambda_0}{2}\right)\right) < \frac{\cosh D \cosh T_3 - \cosh 2D}{\sinh D \sinh T_3}. \tag{12.1}$$

Proof. Let ϕ denote the function defined on $[\delta_0, \infty)$ by $\phi(x) = \Phi_3(\delta_0, x)$. Let θ denote the function defined on $(0, \lambda_0/2)$ by $\theta(x) = \Theta(x, \lambda_0/2)$. From the definition of Φ_n given in Subsection 7.1 and the definition of Θ given in Subsection 6.3, it is clear that ϕ is monotone increasing on its domain and that θ is monotone decreasing on its domain.

If D is a point of $[\delta_0, 0.7]$, then the monotonicity of ϕ implies that

$$D \leq 0.7 < 3\delta_0 = \phi(\delta_0) \leq \phi(D) \leq \phi(0.7) = 1.766 \dots < \lambda_0.$$

As $T_3 = \phi(D)$, this proves the first assertion.

To prove the second assertion, we consider an arbitrary subinterval $[a, b]$ of $[\delta_0, 0.7]$. From the definition given in Subsection 6.3, it is clear that the function Θ is monotone decreasing in

its first argument, and hence, for any $D \in [a, b]$, we have

$$\theta(D/2) \geq \theta(b/2) \geq \theta(0.7/2) = 1.1 \dots$$

and

$$\theta(\phi(D)/2) \leq \theta(\phi(a)/2) \leq \theta(\phi(\delta_0)/2) = 0.36 \dots$$

It follows that

$$\theta(D/2) - \theta(\phi(D)/2) \geq \theta(b/2) - \theta(\phi(a)/2) > 0.$$

As Θ takes values in $(0, \pi/2)$, we have $\theta(D/2) - \theta(\phi(D)/2) < \pi/2$. Hence,

$$\cos(\theta(D/2) - \theta(\phi(D)/2)) \leq \cos(\theta(b/2) - \theta(\phi(a)/2)), \quad (12.2)$$

for any $D \in [a, b]$.

On the other hand, for any $D \in [a, b]$, using the monotonicity of the hyperbolic cotangent, we find that

$$\begin{aligned} \frac{\cosh D \cosh T_3 - \cosh 2D}{\sinh D \sinh T_3} &= \coth D \coth \phi(D) - \frac{\cosh 2D}{\sinh D \sinh \phi(D)} \\ &\geq \coth b \coth \phi(b) - \frac{\cosh 2b}{\sinh a \sinh \phi(a)}. \end{aligned} \quad (12.3)$$

If, for all a and b with $\delta_0 \leq a < b \leq 0.7$, then we set

$$\Delta(a, b) = \cos(\theta(b/2) - \theta(\phi(a)/2)) - \left(\coth b \coth \phi(b) - \frac{\cosh 2b}{\sinh a \sinh \phi(a)} \right),$$

then it follows from (12.2) and (12.3) that, for every interval $[a, b] \subset [\delta_0, 0.7]$ and every point $D \in [a, b]$, we have

$$\cos(\theta(D/2) - \theta(\phi(D)/2)) - \left(\frac{\cosh D \cosh \phi(D) - \cosh 2D}{\sinh D \sinh \phi(D)} \right) \leq \Delta(a, b).$$

In particular, (12.1) will hold for $D \in [a, b]$ provided that $\Delta(a, b) < 0$. But by direct computation we find that $\Delta(0.58, 0.63)$, $\Delta(0.63, 0.67)$, $\Delta(0.67, 0.68)$, $\Delta(0.68, 0.69)$, and $\Delta(0.69, 0.7)$ are all negative. This completes the proof. \square

12.2. We define functions $V_{\text{near}}^*(D)$ and $V_{\text{near}}(D)$ on $[\delta_0, \infty)$ as follows. For $n = 2, 3$, we set $T_n = T_n(D) = \Phi_n(\delta_0, D)$. According to Lemma 7.4, we have $T_2 = \Phi_2(\delta_0, D) \leq 2D$, so that (D, T_2) is contained in the domain of Ψ . We may therefore define

$$V_{\text{near}}^*(D) = B \left(\frac{\lambda_0}{2} \right) - 2\sigma \left(\frac{\lambda_0}{2}, \frac{D}{2}, \frac{T_2}{2}, \Psi(D, T_2) \right).$$

Finally, we define

$$V_{\text{near}}(D) = \begin{cases} V_{\text{near}}^*(D) & \text{for } \delta_0 \leq D < 0.7, \\ V_{\text{near}}^*(D) - 2\kappa(\lambda_0/2, T_3/2) & \text{for } D \geq 0.7. \end{cases}$$

LEMMA 12.3. *Let M be a closed, orientable hyperbolic 3-manifold that contains no closed geodesic of length less than δ_0 . Suppose that P is a point of \mathfrak{G}_M such that $\mathfrak{s}_M(P) \geq \lambda_0$ (see Subsection 3.8). Then $D_M(P) \geq \delta_0$, so that $V_{\text{near}}(D_M(P))$ is defined, and*

$$\text{vol } N(P, \lambda_0/2) \geq V_{\text{near}}(D_M(P)).$$

Proof. We set $D = D_M(P)$.

We shall apply Lemma 7.6, taking $\lambda = \lambda_0$ and $\delta = \delta_0$. By direct computation, we find that $3\delta_0 < \lambda_0 < 4\delta_0$. The hypothesis that M contains no closed geodesic of length less than δ_0 implies that a generator x of C_P (see Subsection 3.8) has translation length at least δ_0 . As we have assumed that $\mathfrak{s}_M(P) \geq \lambda_0$, the hypotheses needed for the first assertion of Lemma 7.6 are satisfied.

It follows from the first assertion of Lemma 7.6 and the definition of V_{near}^* that

$$\text{vol } N \left(P, \frac{\lambda_0}{2} \right) \geq V_{\text{near}}^* - 2\kappa \left(\frac{\lambda_0}{2}, \frac{T_3}{2} \right).$$

In view of the definition of V_{near} , it follows that $\text{vol } N(P, \lambda_0/2) \geq V_{\text{near}}(D)$ if $D \geq 0.7$.

Now suppose that $D < 0.7$. In this case, it follows from Lemma 12.1 that $\max(D, T_3) < \lambda_0$, so that $\Theta(D/2, \lambda_0/2)$ and $\Theta(T_3/2, \lambda_0/2)$ are defined, and that

$$\cos \left(\Theta \left(\frac{D}{2}, \frac{\lambda_0}{2} \right) - \Theta \left(\frac{T_3}{2}, \frac{\lambda_0}{2} \right) \right) < \frac{\cosh D \cosh T_3 - \cosh 2D}{\sinh D \sinh T_3}.$$

It therefore follows from the second assertion of Lemma 7.6 and the definition of V_{near}^* that $\text{vol } N(P, \lambda_0/2) \geq V_{\text{near}}^*(D)$. As $V_{\text{near}}(D) = V_{\text{near}}^*(D)$ in this case, the present lemma is now proved in all cases. \square

LEMMA 12.4. *For any $D \in [\delta_0, \infty)$ we have $(D, \lambda_0) \in \mathcal{X}$, so that $V_{\text{far}}(D, \lambda_0)$ is defined, and*

$$V_{\text{near}}(D) + V_{\text{far}}(D, \lambda_0) > 3.44. \tag{12.4}$$

Proof. If $D \geq \delta_0$, then we have

$$\frac{1}{1 + e^D} + \frac{1}{1 + e^{\lambda_0}} \leq \frac{1}{1 + e^{\delta_0}} + \frac{1}{8} = 0.483\dots < \frac{1}{2},$$

so that $(D, \lambda_0) \in \mathcal{X}$.

Before turning to the proof of (12.4), we shall summarize the monotonicity properties of various functions that will be used in the proof. It is clear from the definition given in Subsection 7.1 that the function Φ_n is monotone increasing in its second argument for each n . We pointed out in Subsection 6.5 that κ is monotone decreasing in its second argument. We pointed out in Subsection 7.5 that Ψ is monotone increasing in its first argument and monotone decreasing in its second. According to Lemma 11.2, the function V_{far} is monotone decreasing in its first argument. Finally, as κ is monotone decreasing in its second argument, the function $x \mapsto \kappa(\lambda_0/2, \Phi_3(\delta_0, x)/2)$ is monotone decreasing on its domain.

We now turn to the proof of (12.4). We first consider the case $D \geq \lambda_0$. In this case, we have $\kappa(\lambda_0/2, D/2) = 0$, and hence

$$\begin{aligned} V_{\text{near}}^*(D) &= B \left(\frac{\log 7}{2} \right) - 2\sigma(\lambda_0/2, D/2, \Phi_2(\delta_0, D)/2, \Psi(D, \Phi_2(\delta_0, D))) \\ &= B \left(\frac{\log 7}{2} \right) - 2\kappa(\lambda_0/2, \Phi_2(\delta_0, D)/2) \\ &\geq B \left(\frac{\lambda_0}{2} \right) - 2\kappa(\lambda_0/2, \Phi_2(\delta_0, \lambda_0)/2) \\ &= 4.015\dots \end{aligned}$$

Furthermore, we have

$$\Phi_3(\delta_0, D/2) \geq \Phi_3(\delta_0, \lambda_0/2) = 2.307\dots > \lambda_0.$$

Hence, $\kappa(\lambda_0/2, \Phi_3(\delta_0, D)/2) = 0$, and so $V_{\text{near}}(D) = V_{\text{near}}^*(D) = 4.015\dots$. As $V_{\text{far}}(D, \lambda_0) = 0$ in this case, (12.4) holds.

For the rest of the proof, we shall restrict attention to the case $\delta_0 \leq D < \lambda_0$.

Let us define a *useful interval* to be a half-open interval $I = [a, b) \subset [\delta_0, \lambda_0)$ whose interior (a, b) does not contain 0.7.

For any useful interval $[a, b)$, we have $\Phi_2(\delta_0, a) \leq 2a \leq 2b$ according to Subsection 12.2. Hence, $(b, \Phi_2(\delta_0, a)) \in \mathbb{R}^2$ lies in the domain of Ψ . In view of the monotonicity properties pointed out above, for any $D \in [a, b)$ we have

$$\begin{aligned} V_{\text{near}}^*(D) &= B\left(\frac{\lambda_0}{2}\right) - 2\sigma(\lambda_0/2, D/2, \Phi_2(\delta_0, D)/2, \Psi(D, \Phi_2(\delta_0, D))) \\ &\geq B\left(\frac{\lambda_0}{2}\right) - 2\sigma(\lambda_0/2, a/2, \Phi_2(\delta_0, D)/2, \Psi(b, \Phi_2(\delta_0, D))) \\ &\geq B\left(\frac{\lambda_0}{2}\right) - 2\sigma(\lambda_0/2, a/2, \Phi_2(\delta_0, a)/2, \Psi(b, \Phi_2(\delta_0, a))). \end{aligned}$$

Thus, if for every useful interval $I = [a, b)$, we set

$$m_{\text{near}}^*(I) = B\left(\frac{\lambda_0}{2}\right) - 2\sigma(\lambda_0/2, a/2, \Phi_2(\delta_0, a)/2, \Psi(b, \Phi_2(\delta_0, a))),$$

we have

$$V_{\text{near}}^*(D) \geq m_{\text{near}}^*(I) \quad \text{whenever } D \in I. \quad (12.5)$$

Let us associate a number $m_{\text{near}}(I)$ to any useful interval $I = [a, b)$ by setting $m_{\text{near}}(I) = m_{\text{near}}^*(I)$ if $I \subset [\delta_0, 0.7)$ and $m_{\text{near}}(I) = m_{\text{near}}^*(I) - \kappa(\lambda_0/2, \Phi_3(\delta_0, a)/2)$ if $I \subset [0.7, \lambda_0)$.

It follows from (12.5), the definition of V_{near} (see Subsection 12.2), and the monotonicity of $x \mapsto \kappa(\lambda_0/2, \Phi_3(\delta_0, x)/2)$ that, for any useful interval I , we have

$$V_{\text{near}}(D) \geq m_{\text{near}}(I) \quad \text{for every } D \in I. \quad (12.6)$$

As V_{far} is monotone decreasing in its first argument, for any useful interval $I = [a, b)$ we have

$$V_{\text{far}}(D, \lambda_0) \geq V_{\text{far}}(b, \lambda_0) \quad \text{for every } D \in I = [a, b). \quad (12.7)$$

It follows from (12.6) and (12.7) that, for any useful interval $I = [a, b)$, we have

$$V_{\text{near}}(D) + V_{\text{far}}(D, \lambda_0) \geq m_{\text{near}}(I) + V_{\text{far}}(b, \lambda_0) \quad \text{for every } D \in I. \quad (12.8)$$

Hence, in order to complete the proof of the lemma, it suffices to write $[\delta_0, \lambda_0)$ as a union of a family \mathcal{I} of useful intervals such that $m_{\text{near}}(I) + V_{\text{far}}(b, \lambda_0) > 3.44$ for every $I = [a, b) \in \mathcal{I}$.

The proof is completed by direct calculation. A separate calculation was done on each of the intervals $[\delta_0 = 0.58, 0.598)$, $[0.598, 0.608)$, $[0.608, 0.618)$, $[0.618, 0.7)$, $[0.7, \log 7 = \lambda_0)$. Each of these intervals was further subdivided into 20 equal-sized subintervals $I_i = [x_n, x_{n+1})$ for $n = 0, \dots, 19$. The subintervals I_n are useful, and it was verified that $m_{\text{near}}(I_n) + V_{\text{far}}(x_n, \lambda_0) > 3.44$ for each n .

The minimum value computed in this way was $3.4409\dots$, which arises for the subinterval $[0.5971, 0.598)$. Note that the calculation of m_{near} requires calculating a value of the function ι . See Subsection A.7 for an explanation of the methods that were used to make this calculation. \square

LEMMA 12.5. *Let M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 4-free. Suppose that M contains no closed geodesic of length less than δ_0 . Then $\text{vol } M > 3.44$.*

Proof. We may assume that $\text{vol } M \leq 3.468$, as otherwise there is nothing to prove. Then, by Lemma 10.4, $\mu_0 = 1.119$ is a Margulis number for M .

As $\pi_1(M)$ is 4-free, we may apply Corollary 5.14. If alternative (i) of Corollary 5.14 holds, that is, if M contains an embedded hyperbolic ball of radius $\lambda_0/2$, then we have $\text{vol } M \geq B(\lambda_0/2) = 4.65\dots$ (Recall that we have set $\lambda_0 = \log 7$ in this section.)

Now suppose that alternative (ii) of Corollary 5.14 holds, that is, that there is a point $P \in \mathfrak{G}_M$ with $\mathfrak{s}_M(P) = \lambda_0$. We set $D = D_M(P)$. As in particular, $\mathfrak{s}_M(P) \geq \lambda_0$, and as M contains no closed geodesic of length less than δ_0 , it follows from the definition of D_M (see Subsection 3.8) that $D \geq \delta_0$, and it follows from Lemma 12.3 that

$$\text{vol } N(P, \lambda_0/2) \geq V_{\text{near}}(D). \tag{12.9}$$

On the other hand, as $\mathfrak{s}_M(P) = \lambda_0$, and as μ_0 is a Margulis number for M , Lemma 11.3 gives

$$\text{vol}(M - N(P, \lambda_0/2)) \geq V_{\text{far}}(D, \lambda_0). \tag{12.10}$$

From (12.9) and (12.10), it follows that

$$\text{vol } M \geq V_{\text{near}}(D) + V_{\text{far}}(D, \lambda_0).$$

The conclusion of the present lemma now follows from Lemma 12.4. □

13. The case where there is a short geodesic

As in Section 12, we set $\delta_0 = 0.58$. As in Section 11, we set $\mu_0 = 1.119$.

PROPOSITION 13.1. *Let M be a closed, orientable hyperbolic 3-manifold, and μ be a Margulis number for M . Suppose that c is a closed geodesic in M of length $l < \mu$, and P be any point of c . Then $P \in \mathfrak{G}_M$ and $D_M(P) = l$. Furthermore, we have*

$$\text{vol } N(P, \mathfrak{s}_M(P)/2) = B(\mathfrak{s}_M(P)/2) - 2\kappa(\mathfrak{s}_M(P)/2, l/2).$$

Proof. Let $C \leq \pi_1(M, P)$ denote the image of $\pi_1(c, P)$ under the inclusion homomorphism, and let x denote a generator of C . As c is a closed geodesic of length l , the cyclic subgroup C of $\pi_1(M, P)$ is maximal and, for any integer $n \neq 0$, the minimal length of a loop based at P and representing x^n is nl . Now, if α is a loop based at P that represents an element of $\pi_1(M, P) - C$, then $[\alpha]$ does not commute with x ; and as $l < \mu$, and μ is a Margulis number for M , the length of α is at least μ , and in particular greater than l . It follows that $C = C_M(P)$, that $P \in \mathfrak{G}_M$, and that $D_M(P) = l$.

The second assertion is an application of Proposition 6.2. Let us write $M = \mathbb{H}^3/\Gamma$ where $\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete and torsion-free, and set $q = q_\Gamma$. Set $\lambda = \mathfrak{s}_M(P)$. Let \tilde{P} be a point of $q^{-1}(P)$, and let us identify $\pi_1(M, P)$ with Γ via the isomorphism determined by the base point $\tilde{P} \in \mathbb{H}^3$ (see Subsection 2.6). As c is a closed geodesic, the component \tilde{c} of $q^{-1}(c)$ containing \tilde{P} is the axis of C . Let ρ_{+1} and ρ_{-1} be the closed rays emanating from \tilde{P} and contained in \tilde{c} . We may suppose them to be labeled in such a way that $x^{\epsilon n} \cdot \tilde{P} \in \rho_\epsilon$ for every $n > 0$ and for $\epsilon = \pm 1$.

For each integer $n \neq 0$, we have $\text{dist}(\tilde{P}, x^n \cdot \tilde{P}) = |n|l$. For $\epsilon = \pm 1$ let ζ_ϵ denote the point of intersection of $S(\lambda/2, \tilde{P})$ with ρ_ϵ . According to Proposition 6.2, we have

$$\text{vol}(N(P, \lambda/2)) = B(\lambda/2) - \text{vol} \left(\bigcup_{n>0, \epsilon=\pm 1} K(\lambda/2, \tilde{P}, \zeta_\epsilon, nl/2) \right). \tag{13.1}$$

In the notation of Subsection 6.1, the plane $\Pi(\tilde{P}, \zeta_\epsilon, nl)$ is orthogonal to \tilde{c} for each $n > 0$ and for $\epsilon = \pm 1$. Hence, the half-spaces $H(z_0, \zeta_{+1}, l/2)$ and $H(z_0, \zeta_{-1}, l/2)$ are disjoint, and

$$H(z_0, \zeta_\epsilon, nl/2) \subset H(z_0, \zeta_\epsilon, l/2),$$

for each $n > 1$ and for $\epsilon = \pm 1$. It follows that

$$K(\lambda/2, \tilde{P}, \zeta_{+1}, l/2) \cap K(\lambda/2, \tilde{P}, \zeta_{-1}, l/2) = \emptyset, \quad (13.2)$$

and that

$$K(\lambda/2, \tilde{P}, \zeta_\epsilon, nl/2) \subset K(\lambda/2, \tilde{P}, \zeta_\epsilon, l/2), \quad (13.3)$$

for each $n > 0$ and for $\epsilon = \pm 1$. From (13.1)–(13.3), we find that

$$\begin{aligned} \text{vol}(N(P, \lambda/2)) &= B(\lambda/2) - \text{vol}(K(\lambda/2, \tilde{P}, \zeta_{+1}, l/2)) + \text{vol}(K(\lambda/2, \tilde{P}, \zeta_{-1}, l/2)) \\ &= B(\lambda/2) - 2\kappa(\lambda/2, l/2). \end{aligned} \quad \square$$

We define a function W with domain $\mathcal{X} \subset \mathbb{R}^2$ (see Subsection 8.1) by

$$W(l, \lambda) = V_{\text{far}}(l, \lambda) + B(\lambda/2) - 2\kappa(\lambda/2, l/2).$$

LEMMA 13.2. *Let k be an integer greater than 2, let M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is k -free, and suppose that μ_0 is a Margulis number for M . Suppose that c is a closed geodesic in M of length $l < \mu_0$, and let P be any point of c . Then $P \in \mathfrak{G}_M$ and $(l, \mathfrak{s}_M(P)) \in \mathcal{X}$. Furthermore, we have*

$$\text{vol } M \geq W(l, \mathfrak{s}_M(P)).$$

Proof. Set $\lambda = \mathfrak{s}_M(P)$. According to Lemma 13.1, we have $P \in \mathfrak{G}_M$, $D_M(P) = l$, and

$$\text{vol } N(P, \lambda/2) = B(\lambda/2) - 2\kappa(\lambda/2, l/2).$$

According to Lemma 11.3, we have $(D_M(P), \lambda) \in \mathcal{X}$, so that $V_{\text{far}}(D_M(P), \lambda)$ is defined, and

$$\text{vol}(M - N(P, \lambda/2)) \geq V_{\text{far}}(D_M(P), \lambda) = V_{\text{far}}(l, \lambda).$$

Hence,

$$\begin{aligned} \text{vol } M &\geq \text{vol } N(P, \lambda/2) + \text{vol}(M - N(P, \lambda/2)) \\ &\geq B(\lambda/2) - 2\kappa(\lambda/2, l/2) + V_{\text{far}}(l, \lambda) \\ &= W(l, \lambda). \end{aligned} \quad \square$$

LEMMA 13.3. *For any $(l, \lambda) \in \mathcal{X}$ with $0.003 \leq l \leq \delta_0$, we have $W(l, \lambda) > 3.44$.*

Proof. Let us define $\rho(l) = \frac{1}{2} \log((e^l + 3)/(e^l - 1))$, so that

$$\frac{1}{1 + e^l} + \frac{1}{1 + e^{2\rho(l)}} = \frac{1}{2}.$$

By definition the set \mathcal{X} consists of all points in \mathbb{R}^2 such that $\lambda \geq 2\rho(l)$; or equivalently, of all points of the form $(l, 2\rho(l) + y)$ with $l > 0$ and $y \geq 0$.

We define $W^*(l, y) = W(l, 2\rho(l) + y)$ for $l > 0$ and $y \geq 0$. It suffices to show that, for any $l \in [0.003, \delta_0]$ and any $y > 0$, we have $W^*(l, y) > 3.44$. We observe that the function ρ is monotone decreasing for $l > 0$.

For $(l, \lambda) \in \mathcal{X}$ we define

$$V_N(l, \lambda) = B(\lambda/2) - 2\kappa(\lambda/2, l/2), \quad (13.4)$$

so that $W = V_{\text{far}} + V_N$. We observe that V_N is decreasing in l and increasing in λ . For $l > 0$ and $y \geq 0$, we set $V_N^*(l, y) = V_N(l, 2\rho(l) + y)$ and $V_{\text{far}}^*(l, y) = V_{\text{far}}(l, 2\rho(l) + y)$, so that $W^* = V_{\text{far}}^* + V_N^*$. As ρ is monotone decreasing, $V_N^*(l, y)$ is decreasing in l and increasing in y . For $l \in [0.003, \delta_0]$ and $y > 0.5$, we have

$$V^*(l, y) > V^*(\delta_0, 0.5) = 3.557\dots > 3.44,$$

where we have computed $V^*(\delta_0, 0.5)$ using the formula for the function κ given by Proposition A.3. It therefore suffices to show that, for any (l, y) in the rectangle $R = [0.003, \delta_0] \times [0, 0.5]$, we have $W^*(l, y) > 3.44$.

We denote by \mathcal{S} the set of all subrectangles of R of the form $[l_0, l_1] \times [y_0, y_1]$ with $0.003 < l_0 < l_1 \leq \delta_0$ and $0 \leq y_0 < y_1 \leq 0.05$.

We define a continuous function χ on R by

$$\chi(l, y) = \rho_4(l, 2\rho(l) + y) - \left(h + \frac{2\rho(l) + y}{2} \right).$$

It follows from the definition of V_{far} given in Subsection 11.1 that we have

$$V_{\text{far}}^*(l, y) \geq B \left(\max \left(0, \left(\min \left(\frac{\mu_0}{2}, \rho_4(l, 2\rho(l) + y) - \frac{2\rho(l) + y}{2} \right) \right) \right) \right), \quad (13.5)$$

for every $(l, y) \in R$, with equality when $\chi(l, y) \leq 0$. Furthermore, when $\chi(l, y) \geq 0$, we have

$$\begin{aligned} V_{\text{far}}^*(l, y) &= V_{\text{Bör}} \left(\frac{\mu_0}{2}, h + \frac{2\rho(l) + y}{2} \right) \\ &\quad + B \left(\min \left(\frac{\mu_0}{2}, \frac{1}{2} \left(\rho_4(l, 2\rho(l) + y) - \left(h + \frac{2\rho(l) + y}{2} \right) \right) \right) \right). \end{aligned} \quad (13.6)$$

The function ρ is monotone decreasing, and the function ρ_4 is decreasing in each of its arguments. Hence, for each $S = [l_0, l_1] \times [y_0, y_1] \in \mathcal{S}$, the function $\chi|_S$ is bounded below by

$$\chi_S \doteq \rho_4(l_1, 2\rho(l_0) + y_1) - \left(h + \frac{2\rho(l_0) + y_1}{2} \right).$$

Suppose that $S = [l_0, l_1] \times [y_0, y_1]$ is a rectangle such that $\chi_S > 0$. Then, in particular, χ takes only positive values on S , and hence (13.6) holds for every $(l, y) \in S$. As ρ is a monotone decreasing function, ρ_4 is decreasing in each of its arguments, and B is increasing, and as $V_{\text{Bör}}$ is increasing in its second argument according to Remark 9.6, the quantity

$$\begin{aligned} V_+^S &\doteq V_{\text{Bör}} \left(\frac{\mu_0}{2}, h + \frac{2\rho(l_1) + y_0}{2} \right) \\ &\quad + B \left(\min \left(\frac{\mu_0}{2}, \frac{1}{2} \left(\rho_4(l_1, 2\rho(l_0) + y_1) - \left(h + \frac{2\rho(l_0) + y_1}{2} \right) \right) \right) \right) \end{aligned} \quad (13.7)$$

is a lower bound for $V_{\text{far}}^*|_S$ for any $S = [l_0, l_1] \times [y_0, y_1] \in \mathcal{S}$ with $\chi_S > 0$.

On the other hand, if $S = [l_0, l_1] \times [y_0, y_1]$ is an arbitrary rectangle in \mathcal{S} , then (13.5) holds for every $(l, y) \in S$. Hence,

$$V_-^S \doteq B \left(\max \left(0, \left(\min \left(\frac{\mu_0}{2}, \rho_4(l_1, 2\rho(l_0) + y_1) - \frac{2\rho(l_0) + y_1}{2} \right) \right) \right) \right) \quad (13.8)$$

is a lower bound for $V_{\text{far}}^*|_S$, for any $S = [l_0, l_1] \times [y_0, y_1] \in \mathcal{S}$.

We have observed that $W^* = V_{\text{far}}^* + V_N^*$, and that $V_N^*(l, y)$ is decreasing in l and increasing in y . It follows that $V_N^S \doteq V_N^*(l_1, y_0)$ is a lower bound for $V_N^*|_S$ for any $S = [l_0, l_1] \times [y_0, y_1] \in \mathcal{S}$. Hence, $V_N^S + V_+^S$ is a lower bound for $W^*|_S$ for every $S = [l_0, l_1] \times [y_0, y_1] \in \mathcal{S}$ with $\chi_S > 0$, and $V_N^S + V_-^S$ is a lower bound for $W^*|_S$ for every $S = [l_0, l_1] \times [y_0, y_1] \in \mathcal{S}$.

Hence, in order to complete the proof of the lemma, it suffices to specify subsets \mathcal{S}_+ and \mathcal{S}_0 of \mathcal{S} with $\mathcal{S}_+ \subset \mathcal{S}_0$, such that

- (i) R is the union of the rectangles in \mathcal{S}_0 ;
- (ii) $\chi_S > 0$ for every $S \in \mathcal{S}_+$;
- (iii) $V_N^S + V_+^S > 3.44$ for every $S \in \mathcal{S}_+$; and
- (iv) $V_N^S + V_-^S > 3.44$ for every $S \in \mathcal{S}_0 - \mathcal{S}_+$.

To define \mathcal{S}_0 we begin with the rectangles $R_1 = [0.003, 0.103] \times [0, 0.5]$, $R_2 = [0.1, 0.5] \times [0, 0.5]$, and $R_3 = [0.5, 0.58] \times [0, 0.5]$. We subdivide each of the R_i into equal-sized subrectangles, where the subrectangles are chosen to form a 40×100 grid on R_1 , a 50×100 grid on R_2 , and an 80×100 grid on R_3 . We define \mathcal{S}_0 to be the union of the rectangles in these three grids.

To specify $\mathcal{S}_+ \subset \mathcal{S}_0$, we compute χ_S numerically for each $S \in \mathcal{S}_0$ and take \mathcal{S}_+ to consist of those rectangles for which the computed value of χ_S exceeds 0.1. Numerical computation of χ_S involves evaluation of elementary functions and arithmetic. As the round-off errors in the arithmetic and the errors inherent in the standard approximations of elementary functions by rational functions combine to give a margin of error much less than 0.1, we indeed have $\chi_S > 0$ for every $S \in \mathcal{S}_+$.

We set $W_S = V_N^S + V_+^S$ if $S \in \mathcal{S}_+$, and $W_S = V_N^S + V_-^S$ if $S \in \mathcal{S}_0 - \mathcal{S}_+$. We computed W_S numerically for each $S \in \mathcal{S}_0$, using one of the formulas 13.7 or 13.8 and using Proposition A.3 to compute the value of the function κ which appears in formula 13.4. The minimum value of W_S obtained in this manner is 13.4 . . . , which arises for $S = [0.579, 0.58] \times [0.145, 0.15]$. This shows that W^* is bounded below by 3.44 on R , as required. \square

LEMMA 13.4. *Let M be a closed, orientable hyperbolic 3-manifold such that $\pi_1(M)$ is 4-free. Suppose that M contains a closed geodesic of length less than δ_0 . Then $\text{vol } M > 3.44$.*

Proof. We may assume that $\text{vol } M \leq 3.468$, as otherwise there is nothing to prove. Then, by Lemma 10.4, $\mu_0 = 1.119$ is a Margulis number for M .

Let c be a closed geodesic in M of length $l < \delta_0$. Let P be any point of c . As $\delta_0 < \mu_0$, we may apply Lemma 13.2 to deduce that $P \in \mathfrak{G}_M$, that $(l, \mathfrak{s}_M(P)) \in \mathcal{X}$, and that

$$\text{vol } M \geq W(l, \mathfrak{s}_M(P)).$$

If $l \geq 0.003$, it follows from Lemma 13.3 that $W(l, \mathfrak{s}_M(P)) > 3.44$, and hence that $\text{vol } M > 3.44$.

There remains the case in which $l < 0.003$. In this case, let T denote the maximal embedded tube about c . It follows from [5, Corollary 10.5] that

$$\text{vol } T \geq V(0.003), \tag{13.9}$$

where V is the function defined in [5, Section 10]. Computing $V(0.003)$ from the definition given in [5], we find that

$$V(0.003) = 3.1345 \dots \tag{13.10}$$

On the other hand, it follows from a result of Przeworski's [21, Corollary 4.4] on the density of cylinder-packings that

$$\text{vol } T < 0.91 \text{ vol } M. \tag{13.11}$$

From (13.9) to (13.11) it follows that

$$\text{vol } M > \frac{3.134}{0.91} > 3.44. \tag{13.12} \quad \square$$

Proof of Theorem 1.6. The theorem is an immediate consequence of Lemmas 12.5 and 13.4. \square

The proof of Theorem 1.7 will involve combining Theorem 1.6 with the results of Culler and Shalen [14]. We refer the reader to [14, Section 6] for the definition of a fibroid. As in [14], we use a result due to Agol, Storm, and Thurston from [4]. The information from [4] that we need is summarized in [2, Theorem 9.4], which states that if M is a closed orientable hyperbolic 3-manifold containing a connected incompressible closed surface that is not a fibroid, then $\text{vol}(M) > 3.66$.

Proof of Theorem 1.7. Assume that $\dim_{\mathbb{Z}_2} H_1(M; \mathbb{Z}_2) \geq 8$. Then according to [14, Proposition 7.1], either $\pi_1(M)$ is 4-free, or M contains a closed incompressible surface of genus at most 3 that is not a fibroid. If $\pi_1(M)$ is 4-free, then it follows from Theorem 1.6 that $\text{vol}(M) > 3.44$. If M contains a closed incompressible surface that is not a fibroid, then it follows from [2, Theorem 9.4] that $\text{vol}(M) > 3.66$. In either case the hypothesis is contradicted. \square

Appendix. Computations with caps

In this section, we describe the methods used for numerical computation of particular values of the functions $\iota(R, w, w', \alpha)$ and $\kappa(R, D)$, which were needed for the proofs of Lemmas 12.4 and 13.3. The main results are Propositions A.3–A.6. In Subsection A.7, we will show how to combine these results to calculate ι and κ for any values of the arguments.

The following well-known special case of the distance formula in the upper half-space model of \mathbb{H}^2 will be needed.

LEMMA A.1. *Let γ be a geodesic in the hyperbolic plane modeled in the upper half-plane by a semicircle s with center $X = (x, 0)$ and radius ρ . Let $P = (x, \rho)$. Suppose that Q is a point of s such that the arc of s from P to Q subtends an angle θ in the Euclidean plane. Then $\cosh d_h(P, Q) = \sec \theta$.*

Proof. After applying a hyperbolic isometry that fixes ∞ , we may assume that $\rho = 1$, $x = 0$, $P = (0, 1)$, and $Q = (\sin \theta, \cos \theta)$. The arc of the unit circle from P to Q is a hyperbolic geodesic arc of length

$$\int_0^\theta \frac{d\theta}{\cos \theta} = \log |\sec \theta + \tan \theta|.$$

Thus, as $0 < \theta \leq \pi/2$, we have

$$\begin{aligned} \cosh d_h(P, Q) &= \frac{1}{2} \left(\sec \theta + \tan \theta + \frac{1}{\sec \theta + \tan \theta} \right) \\ &= \sec \theta. \end{aligned} \quad \square$$

We will also need the following fact from Euclidean geometry, of which the proof is an easy exercise.

LEMMA A.2. *Let S_1 and S_2 be two spheres in \mathbb{E}^3 such that $S_1 \cap S_2$ is a circle \mathcal{I} . Let r_i denote the radius of S_i , and let D denote the distance between the centers of S_1 and S_2 . Then the radius r of \mathcal{I} satisfies*

$$r^2 = r_1^2 - \frac{(r_1^2 + D^2 - r_2^2)^2}{4D^2}.$$

Furthermore, the distance between the centers of S_1 and \mathcal{I} is $(r_1^2 + D^2 - r_2^2)/(2D)$. \square

PROPOSITION A.3. *Let R and w be positive numbers with $0 < w < R$. Set*

$$\epsilon = e^{-w} \sqrt{1 - \frac{\cosh^2 w}{\cosh^2 R}}$$

and

$$L(r) = \cosh R - \sqrt{\sinh^2 R - r^2}.$$

Then

$$\kappa(R, w) = \pi \left(e^R \cosh R - \frac{\cosh R}{L(\epsilon)} - \log L(\epsilon) - R + \frac{1}{2} \log(e^{-2w} - \epsilon^2) + w \right).$$

Proof. We use the notation of Subsection 6.1. By definition $\kappa(R, w)$ is the volume of $K(R, Z, \zeta, w)$, where Z and ζ are points in \mathbb{H}^3 separated by a distance R . We may identify \mathbb{H}^3 conformally with the upper half-space $\mathbb{U}^3 = \mathbb{R}^2 \times (0, \infty) \subset \mathbb{R}^3$ in such a way that $Z = (0, 0, 1)$ and $\zeta = (0, 0, e^{-R})$. Then $\bar{N}(Z, R)$ is identified with the Euclidean ball B of radius $\sinh R$ centered at $(0, 0, \cosh R)$. The half-space $H(Z, \zeta, w)$ is identified with the intersection of \mathbb{U}^3 with the Euclidean ball B' of radius e^{-w} centered at $(0, 0, 0)$.

The boundaries of B and B' intersect in a circle \mathcal{I} . According to Lemma A.2, the square of the radius of \mathcal{I} is equal to

$$e^{-2w} - \frac{(e^{-2w} + \cosh^2 R - \sinh^2 R)^2}{4 \cosh^2 R},$$

which implies that the radius of \mathcal{I} is the quantity ϵ defined in the statement of the present proposition.

As Z and ζ lie on the vertical axis $\{(0, 0)\} \times \mathbb{R}$, the circle \mathcal{I} lies in a horizontal plane and has center on the vertical axis. It follows that the vertical projection $p : (x, y, t) \mapsto (x, y)$ maps $K(R, Z, \zeta, w) = \bar{N}(Z, R) \cap H(Z, \zeta, w)$ onto a disk Δ of radius ϵ about $(0, 0)$, and that, for every $P \in \Delta$, the set $p^{-1}(P)$ is a line segment whose lower and upper endpoints lie, respectively, in the lower hemisphere of ∂B and the upper hemisphere of $\partial B'$. If r denotes the distance from P to $(0, 0)$, then the definitions of the balls B and B' imply that the vertical coordinates of these endpoints are, respectively, equal to $L(r)$ and $U(r)$, where $L(r)$ is defined as in the statement of the proposition, and $U(r) = \sqrt{e^{-2w} - r^2}$.

Using cylindrical coordinates in \mathbb{R}^3 , we therefore find that

$$\text{vol } K(R, Z, \zeta, w) = \int_0^{2\pi} \int_0^\epsilon \int_{L(r)}^{U(r)} \frac{r}{t^3} dt dr d\theta.$$

We have

$$\begin{aligned} \int_0^{2\pi} \int_0^\epsilon \int_{L(r)}^{U(r)} \frac{r}{t^3} dt dr d\theta &= \pi \int_0^\epsilon \left(\frac{r}{L(r)^2} - \frac{r}{U(r)^2} \right) dr \\ &= \pi \int_0^\epsilon \frac{r dr}{L(r)^2} - \pi \int_0^\epsilon \frac{r dr}{e^{-2x} - r^2} \\ &= \pi \int_0^\epsilon \frac{r dr}{L(r)^2} + \pi \left(\frac{1}{2} \log(e^{-2x} - \epsilon^2) + x \right). \end{aligned}$$

To complete the proof, we observe that $r dr = (\cosh R - L(r)) dL(r)$, and $L(0) = e^{-R}$. Thus, we have

$$\begin{aligned} \int_0^\epsilon \frac{r dr}{L(r)^2} &= \int_{L(0)}^{L(\epsilon)} \frac{\cosh R - u}{u^2} du \\ &= -\cosh R \left(\frac{1}{L(\epsilon)} - \frac{1}{L(0)} \right) - \log L(\epsilon) + \log L(0) \\ &= e^R \cosh R - \frac{\cosh R}{L(\epsilon)} - \log L(\epsilon) - R, \end{aligned}$$

which completes the proof. \square

The following result, Proposition A.4, gives an integral formula that can be used to compute $\iota(R, w, 0, \alpha)$, when $\alpha \in (\pi/2, \pi)$. The subsequent results of this section, Propositions A.5 and A.6, will be used to reduce the general calculation of $\iota(R, w, w', \alpha)$ to the special case covered by Proposition A.4.

PROPOSITION A.4. *Let R and w be positive numbers with $0 < w < R$, and let $\alpha \in (\pi/2, \pi)$ be given. Set $\phi = \alpha - \pi/2 \in (0, \pi/2)$.*

- (1) *If we define quantities v and c by $v = (\sinh w + \cosh w \cos \phi)^{-1}$ and $c = v \sin \phi \cosh w$, then*

$$v^2 - \frac{(v^2 + c^2 + 1)^2}{4(c^2 + \cosh^2 R)} \geq 0.$$

- (2) *Let us define quantities μ , ρ , and m by*

$$\begin{aligned} \mu &= \sqrt{v^2 - \frac{(v^2 + c^2 + 1)^2}{4(c^2 + \cosh^2 R)}}, \quad \rho = \frac{\cosh R}{\sqrt{c^2 + \cosh^2 R}}, \\ \text{and } m &= c - \frac{c(v^2 + c^2 + 1)}{2(c^2 + \cosh^2 R)}. \end{aligned}$$

If $m > \rho\mu$, then $\iota(R, w, 0, \alpha) = 0$.

- (3) *Suppose that $m \leq \rho\mu$, and define a quantity θ_0 and a function $\epsilon(\theta)$ by*

$$\theta_0 = \arctan \left(\frac{\sqrt{\rho^2 \mu^2 - m^2}}{m\rho} \right) \quad \text{and} \quad \epsilon(\theta) = \frac{\rho\mu}{\sqrt{\cos^2 \theta + \rho^2 \sin^2 \theta}}.$$

Then $m \sec \theta \leq \epsilon(\theta)$ for every $\theta \in [-\theta_0, \theta_0]$. Furthermore, if we set

$$V = \{\theta, r \in [-\pi, \pi] \times \mathbb{R} : -\theta_0 \leq \theta \leq \theta_0 \text{ and } m \sec \theta \leq r \leq \epsilon(\theta)\},$$

then, for every $(\theta, r) \in V$, we have

$$\sinh^2 R - (r \cos \theta - m)^2 - r^2 \sin^2 \theta \geq 0$$

and

$$v^2 - (r \cos \theta + c - m)^2 - r^2 \sin^2 \theta \geq 0.$$

Finally, if we define functions $U(r, \theta)$ and $L(r, \theta)$ on V by

$$U(r, \theta) = \sqrt{v^2 - (r \cos \theta + c - m)^2 - r^2 \sin^2 \theta}$$

and

$$L(r, \theta) = \cosh R - \sqrt{\sinh^2 R - (r \cos \theta - m)^2 - r^2 \sin^2 \theta},$$

then $L(r, \theta) \leq U(r, \theta)$ for each $(r, \theta) \in V$, and we have

$$\iota(R, w, 0, \alpha) = \int_0^{\theta_0} \int_{m \sec \theta}^{\epsilon(\theta)} \left(\frac{r}{L(r, \theta)^2} - \frac{r}{U(r, \theta)^2} \right) dr d\theta. \quad (\text{A.1})$$

Proof. In this proof, it will be understood that $v, c, \mu, \rho, m, \theta_0, \epsilon(\theta), V, U(r, \theta)$, and $L(r, \theta)$ are defined as in the statement of the proposition. Many of these objects are defined only subject to certain assertions in the statement, and will not be mentioned until after these assertions have been proved.

We use the notation of Subsection 6.1. By definition we have

$$\iota(R, w, 0, \alpha) = \text{vol}(K(R, Z, \zeta_1, 0) \cap K(R, Z, \zeta_2, w)),$$

where Z is a point in \mathbb{H}^3 , and ζ_1 and ζ_2 are points in $S(R, Z)$ separated by a spherical distance α . For $i = 1, 2$, we set $\eta_i = \eta_{\zeta_i}$, and we let ℓ_i denote the line containing η_i . We set $\Pi_1 = \Pi(Z, \zeta_1, 0)$, $H_1 = H(Z, \zeta_1, 0)$, $\Pi_2 = \Pi(Z, \zeta_2, w)$, $H_2 = H(Z, \zeta_2, w)$. $K_1 = K(R, Z, \zeta_1, 0)$, and $K_2 = K(R, Z, \zeta_2, w)$.

As $\alpha > \pi/2$, we have $\zeta_1 \notin H_2$. In particular,

$$K_1 \not\subset K_2. \quad (\text{A.2})$$

As the rays η_1 and η_2 form an angle $\alpha \in (\pi/2, \pi)$ at Z , there is a unique plane $U \subset \mathbb{H}^3$ containing ℓ_1 and ℓ_2 . In particular, U is perpendicular to Π_1 , and the line $U \cap \Pi_1$ is perpendicular to ℓ_1 . Hence, there is a unique ray $\tau \subset U \cap \Pi_1$ with origin Z , which forms an angle $\phi = \alpha - \pi/2$ with η_2 .

As $0 < x < R$, the sphere S and the plane Π_2 meet in a circle \mathcal{I} .

Now let \mathbb{U}^3 denote the upper half-space in \mathbb{R}^3 . We denote the coordinates in \mathbb{R}^3 by x, y , and t , so that \mathbb{U}^3 is defined by $t > 0$. We identify \mathbb{H}^3 conformally with \mathbb{U}^3 in such a way that $Z = (0, 0, 1)$; Π_1, H_1 , and U are the subsets of \mathbb{U}^3 defined, respectively, by $x = 0$, $x \geq 0$ and $y = 0$; and τ is defined by $x = y = 0$ and $0 < t \leq 1$.

If X and Y are points of $\bar{\mathbb{U}}^3 \subset \mathbb{R}^3$, then we shall denote the Euclidean distance between X and Y by $|XY|$. When $X, Y \in \mathbb{U}^3$ we shall write $d_h(X, Y)$ for the hyperbolic distance between X and Y .

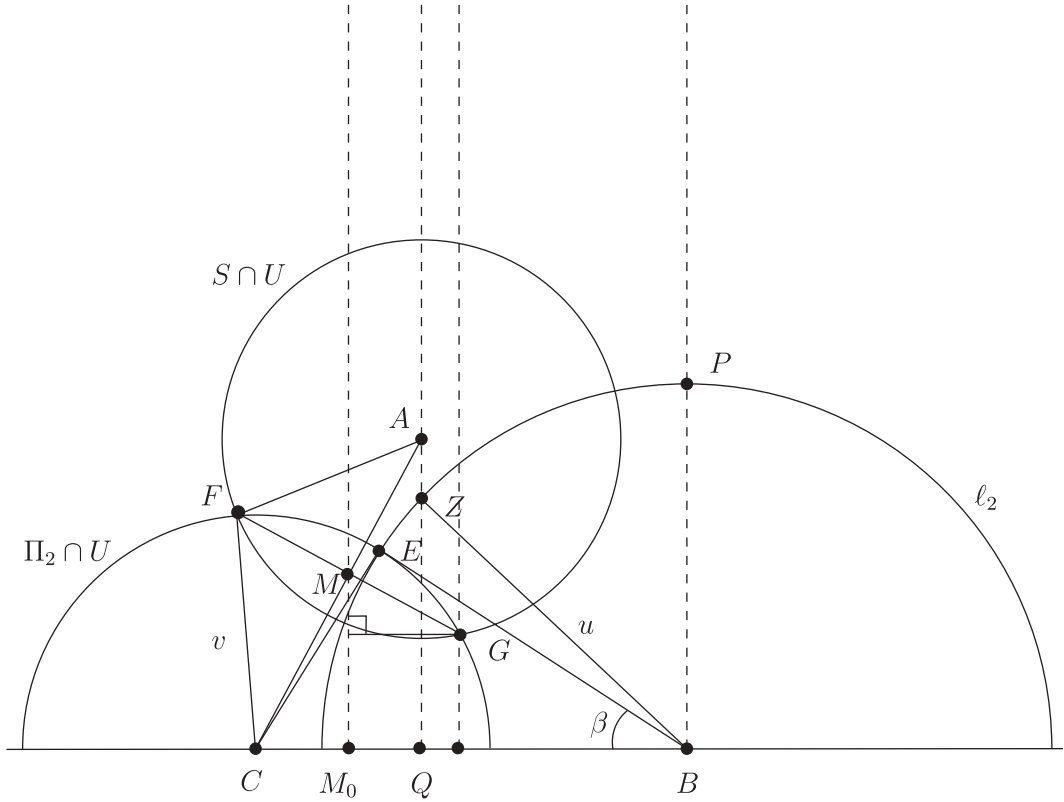
We treat the xz -plane, which contains U , as a Cartesian plane with coordinates x and z ; in particular, each Euclidean line in this plane has a well-defined *slope*.

As $Z = (0, 0, 1)$, the ball $\bar{N}(Z, R)$ is identified with the Euclidean ball \mathcal{B} of radius $\sinh R$ centered at $A = (0, 0, \cosh R)$. We set $S = \partial\mathcal{B}$.

As $\phi > 0$, we have $\ell_2 \neq U \cap \Pi_1$, so that the tangent vector to ℓ_2 at $Z = (0, 0, 1)$ is a non-vertical vector in \mathbb{R}^3 . Hence, the hyperbolic line ℓ_2 is identified with a semicircle in the half-plane U , whose center B lies on the x -axis. We write $B = (b, 0, 0)$. As η_2 forms an angle $\alpha = \phi + (\pi/2)$ with η_1 (which has tangent vector $(1, 0, 0)$) and forms an angle ϕ with τ (which has tangent vector $(0, 0, -1)$), the semicircle ℓ_2 has positive slope at Z , and hence $b > 0$.

If E denotes the point of intersection of ℓ_2 with Π_2 , then as $E \in \zeta_2$, the x -coordinate of E is negative. It follows that the tangent line to the semicircle ℓ_2 has positive slope at E and hence that the tangent line at E to the semicircle $\Pi_2 \cap U$, which meets ℓ_2 orthogonally at the point E , has negative slope. This implies that Π_2 is identified with a hemisphere whose center C lies

on the x -axis and has a negative x -coordinate.



We let u denote the radius of the semicircle ℓ_2 , and we set $\beta = \angle CBE$.

We have $\angle CBZ = \phi$. As the Euclidean distance from $(0, 0)$ to Z is 1, we have $u = \csc \phi$ and $b = \cot \phi$.

Let P denote the point of intersection of ℓ_2 with the vertical ray in U originating at B . By Lemma A.1, we have $\cosh(d_h(P, Z)) = \csc \phi$. As $d_h(P, E) = w + d_h(P, Z)$, Lemma A.1 implies that $\csc \beta = \cosh(w + \operatorname{arccosh} u)$. Hence,

$$\cot \beta = \sinh(w + \operatorname{arccosh} u) = \cosh w \sqrt{u^2 - 1} + u \sinh w = \cosh w \cot \phi + \sinh w \csc \phi$$

and

$$\cos \beta = \tanh(w + \operatorname{arccosh} u) = \frac{\tanh w + \sqrt{1 - u^{-2}}}{1 + \sqrt{1 - u^{-2}} \tanh w} = \frac{\tanh w + \cos \phi}{1 + \tanh w \cos \phi}.$$

As BEC is a Euclidean right triangle, we have

$$|CE| = u \tan \beta = \frac{\csc \phi}{\cosh w \cot \phi + \sinh w \csc \phi} = v.$$

Thus,

$$\operatorname{radius}(\Pi_2) = v. \tag{A.3}$$

The difference of the x -coordinates of B and C is $|BC| = u \sec \beta$. Hence, the x -coordinate of C is

$$b - u \sec \beta = \cot \phi - (\csc \phi) \frac{1 + \tanh w \cos \phi}{\tanh w + \cos \phi} = -\frac{\sin \phi}{\tanh w + \cos \phi} = -c,$$

that is,

$$C = (-c, 0, 0). \quad (\text{A.4})$$

We have seen that the center of the Euclidean sphere S is $(0, 0, \cosh R)$. It therefore follows from (A.4) that the Euclidean distance between the centers of the sphere S and the hemisphere Π_2 is $\sqrt{c^2 + \cosh^2 R}$. We have also seen that S has radius $\sinh R$, whereas Π_2 has radius v by (A.3). In the notation of Lemma A.2, taking $S_2 = S$ and taking S_1 to be the sphere containing Π_1 , we have $r_1 = v$, $r_2 = \sinh R$, and $D = \sqrt{c^2 + \cosh^2 R}$. The first assertion of the lemma gives

$$\text{radius}(\mathcal{I})^2 = v^2 - \frac{(v^2 + c^2 + 1)^2}{4(c^2 + \cosh^2 R)}.$$

This shows that $v^2 - (v^2 + c^2 + 1)^2 / (4(c^2 + \cosh^2 R)) > 0$, which is assertion (1) of the proposition. It also shows that if we define μ as in the statement of the proposition, then

$$\text{radius}(\mathcal{I}) = \mu. \quad (\text{A.5})$$

The second assertion of Lemma A.2 gives the distance between the center C of Π_2 and the center of \mathcal{I} , which we denote by M :

$$|CM| = \frac{v^2 + c^2 + 1}{2\sqrt{c^2 + \cosh^2 R}}. \quad (\text{A.6})$$

Let $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ denote the projection $(x, y, t) \mapsto (x, y)$. We have $p(M) = (-m_0, 0)$ for some $m_0 \in \mathbb{R}$. By similar triangles, we have $(c - m_0)/|CM| = c/|AC|$, so

$$m_0 = c \left(1 - \frac{|CM|}{|AC|} \right) = c - \frac{c(v^2 + c^2 + 1)}{2(c^2 + \cosh^2(R))} = m.$$

Hence,

$$p(M) = (-m, 0). \quad (\text{A.7})$$

As the sphere S and the hemisphere Π_2 are centered on the closed Euclidean half-plane \bar{U} , their circle of intersection \mathcal{I} meets U in two points F and G , which are diametrically opposite points of \mathcal{I} . Hence, M is the midpoint of the segment \overline{FG} . We write $p(F) = -f$ and $p(G) = g$, and we label F and G in such a way that $-f < g$. In view of (A.7), we have $m = (f - g)/2$.

Note that \mathcal{I} is contained in the subset of \mathbb{U}^3 defined by $-f \leq x \leq g$. Note also that AC , the line joining the centers of S and Π_1 , meets the diameter \overline{FG} of \mathcal{I} perpendicularly at the center M . Furthermore, as $A \notin \Pi_2$ and $M \in \Pi_2$, the point M lies on the segment \overline{AC} . Let δ denote the diameter of \mathcal{I} that is perpendicular to the line FG . Then δ is invariant under reflection about U and is therefore contained in a horizontal plane.

We now derive an expression for $g + m$. We define points $M_0, Q, Z \in U$ by setting $M_0 = (-m, 0, 0)$ and $Q = (0, 0, 0)$, and defining Z to be the intersection of the vertical line through M with the horizontal line in U through G . As the triangles ACQ and MCM_0 are similar, we have $\angle ACQ = \angle MCM_0 = (\pi/2) - \angle CMM_0 = \angle CMG - \angle CMM_0 = \angle GMM_0$. Hence, the right triangles ACQ and GMM_0 are similar, and so

$$|AC|(g + m) = |GM| \cosh R.$$

It follows from (A.5) that $|GM| = \mu$, and we have $|AC|^2 = \cosh^2 \mu + c^2$. Hence, if ρ is defined as in the statement of the proposition, we have

$$g + m = \rho\mu. \quad (\text{A.8})$$

We can now prove assertion (2) of the proposition. If $m > \rho\mu$, then by (A.8) we have $g < 0$. As we have observed that \mathcal{I} is contained in the subset of \mathbb{U}^3 defined by $-f \leq x \leq g$, it follows

that \mathcal{I} is contained in the open half-space $\mathbb{U}^3 - H_1$ of \mathbb{U}^3 . Hence, the set $\Pi_2 \cap B$, which is the hyperbolic convex hull of \mathcal{I} , is also contained in $\mathbb{U}^3 - H_1$, and is in particular disjoint from K_1 . But $H_2 \cap B$ is the frontier of K_2 relative to B , and so K_2 either contains K_1 or is disjoint from it. The former alternative is ruled out by (A.2); hence, $K_1 \cap K_2 = \emptyset$, and so $\iota(R, w, 0, \alpha) = 0$. This is assertion (2) of the proposition.

We now turn to the proof of assertion (3) of the proposition. Assume that $m \leq \rho\mu$. Then, by (A.8), we have $g \geq 0$. This means that $G \in H_1$, so that

$$\mathcal{I} \cap H_1 \neq \emptyset. \tag{A.9}$$

It is clear from the definition of m that $m > 0$. As $g \geq 0$ and $m = (f - g)/2$, it follows that $f > 0$.

Let $T \subset \mathbb{R}^3$ denote the half-space $z \leq \cosh R$, whose boundary plane contains A . As M lies on the segment \overline{AC} , we have $M \in T$.

As $0 < \alpha < \pi/2$, the definition of c implies that $c > 0$. In view of (A.4), it follows that the line $AC \subset U$ has positive slope, and that the segment \overline{AC} is disjoint from H_1 . In particular, we have $M \notin H_1$. As the line FG is perpendicular to AC , it has negative slope in U , and hence the ray originating at M and passing through G is contained in T . In particular, $FG \cap H_1 \subset T$.

Let W denote the plane containing \mathcal{I} . As W contains FG and is perpendicular to U , it now follows that $W \cap H_1 \subset T$.

In particular, if S_- and S_+ denote the lower and upper hemisphere of S , respectively, then we have

$$\mathcal{I} \cap H_1 \subset S_-. \tag{A.10}$$

The projection p maps \mathcal{I} onto an ellipse \mathcal{E} in \mathbb{R}^2 . As δ and FG are mutually perpendicular diameters of \mathcal{I} , and δ is contained in a horizontal plane, p maps δ and FG onto the major and minor axes of \mathcal{E} , respectively. In particular, the minor axis of \mathcal{E} is contained in the x -axis; the length of the semi-major axis of \mathcal{E} is the radius of \mathcal{I} , which by (A.5) is equal to μ ; and the semi-minor axis of \mathcal{E} has length $g + m$, which by (A.8) is equal to $\rho\mu$. The center of \mathcal{E} is $p(M) = (-m, 0)$.

We denote by $\hat{\mathcal{E}}$ the compact set bounded by \mathcal{E} . As $f > 0$ and $g \geq 0$, the line $x = 0$ meets $\hat{\mathcal{E}}$ in a possibly degenerate line segment ν . We let V_0 denote the intersection of $\hat{\mathcal{E}}$ with the half-plane $x \geq 0$.

The hemisphere S_{\pm} is the graph of the function $L_0^{\pm}(x, y) = \cosh R \pm \sqrt{\sinh^2 R - x^2 - y^2}$ on the disk $\mathcal{D} \subset \mathbb{R}^2$, which has radius $\sinh R$ and is centered at $(0, 0)$. Likewise, it follows from (A.3) and (A.4), $p(\Pi_2)$ is the graph of the function $U_0(x, y) = \sqrt{v^2 - (x + c)^2 - y^2}$ on the disk $\mathcal{D}' \subset \mathbb{R}^2$ that has radius v and is centered at $(-c, 0)$.

We have $\mathcal{E} = p(\mathcal{I}) = p(S \cap \Pi_2) \subset p(S) \cap p(\Pi_2) = \mathcal{D} \cap \mathcal{D}'$. As $\mathcal{D} \cap \mathcal{D}'$ is convex, we have $\hat{\mathcal{E}} \subset \mathcal{D} \cap \mathcal{D}'$.

The functions $\psi^{\pm} = U_0 - L_0^{\pm}$ have domain $\mathcal{D} \cap \mathcal{D}'$. Let \mathcal{R} denote the right half-plane in \mathbb{R}^2 , defined by $x \geq 0$. Set $\mathcal{W} = \mathcal{D} \cap \mathcal{D}' \cap \mathcal{R}$. It follows from (A.10) that the function ψ^- is identically zero on $\mathcal{R} \cap \mathcal{E} = \mathcal{W} \cap \mathcal{E}$, and is non-zero on $\mathcal{W} \setminus \mathcal{E}$, whereas the function ψ^+ is non-zero on \mathcal{W} . As $-\psi^-$ is clearly a convex function, and as $\mathcal{W} \cap \mathcal{E}$ is the frontier relative to \mathcal{W} of the convex set $\mathcal{R} \cap \hat{\mathcal{E}} = \mathcal{W} \cap \hat{\mathcal{E}}$, the function ψ^- must be non-negative on $\mathcal{W} \cap \hat{\mathcal{E}}$ and negative on $\mathcal{W} \setminus \hat{\mathcal{E}}$. The function ψ^+ is non-zero on the connected domain \mathcal{W} , and is bounded above by ψ^- . But ψ^- vanishes on the subset $\mathcal{W} \cap \mathcal{E}$ of \mathcal{W} , and this subset is non-empty by (A.9). Hence, ψ^+ is negative-valued on \mathcal{W} .

It follows that we have

$$L_0^-(x, y) \leq U_0(x, y) < L_0^+(x, y) \quad \text{when } (x, y) \in \mathcal{W} \cap \hat{\mathcal{E}} \tag{A.11}$$

and

$$L_0^-(x, y) > U_0(x, y) \quad \text{when } (x, y) \in \mathcal{W} \setminus \hat{\mathcal{E}}. \tag{A.12}$$

We have $K_1 \cap K_2 = B \cap H_1 \cap H_2$, and $p(B \cap H_2) \subset \mathcal{D} \cap \mathcal{D}'$. Set $q = p|(K_1 \cap K_2) : K_1 \cap K_2 \rightarrow \mathcal{D} \cap \mathcal{D}'$. It follows from (A.12) that $q^{-1}(x, y) = \emptyset$ when $(x, y) \in \mathcal{W} \setminus \hat{\mathcal{E}}$; and it follows from (A.11) that, when $(x, y) \in \mathcal{R} \cap \hat{\mathcal{E}} = \mathcal{W} \cap \hat{\mathcal{E}}$, we have $L_0^-(x, y) \leq U_0(x, y)$, and $q^{-1}(x, y)$ is a vertical line segment whose endpoints have t -coordinates $L_0^-(x, y)$ and $U_0(x, y)$. As the element of hyperbolic volume on \mathbb{U}^3 is $dt dA/t^3$, where dA is the Euclidean area element on \mathbb{R}^2 , we now find

$$\begin{aligned} \iota(R, w, 0, \alpha) &= \text{vol}(K_1 \cap K_2) \\ &= \int_{\mathcal{R} \cap \hat{\mathcal{E}}} \int_{L_0^-(x, y)}^{U_0(x, y)} \frac{dt dA}{t^3} \\ &= \int_{\mathcal{R} \cap \hat{\mathcal{E}}} \left(\frac{1}{2L_0^-(x, y)^2} - \frac{1}{2U_0(x, y)^2} \right) dA. \end{aligned} \tag{A.13}$$

We shall complete the proof by reinterpreting the facts proved above in terms of polar coordinates (r, θ) in \mathbb{R}^2 , taking the origin for the polar coordinates to be the point $(-m, 0)$. As \mathcal{E} is centered at $(-m, 0)$, has its minor axis contained in the x -axis, and has semi-major axis of length μ and semi-minor axis of length $g + m$, it is defined in these coordinates by the equation $r = \epsilon(\theta)$. The set $\hat{\mathcal{E}}$ is defined by $r \leq \epsilon(\theta)$. The half-plane \mathcal{R} is defined by $r \geq m \sec \theta$. The endpoints of the segment ν are the intersections of \mathcal{E} with the y -axis, whose polar equation is $r = m \sec \theta$. In view of the definitions of ϵ and θ_0 , it is clear that these intersection points are $(\pm\theta_0, m \sec \theta_0)$. As $\nu \subset \hat{\mathcal{E}}$, it follows that we have $m \sec \theta \leq \epsilon(\theta)$ whenever $-\theta_0 \leq \theta \leq \theta_0$.

It now follows that V is the set of polar coordinate pairs of points in $\mathcal{R} \cap \hat{\mathcal{E}}$. As $\mathcal{R} \cap \hat{\mathcal{E}} \subset \hat{\mathcal{E}} \subset \mathcal{D} \cap \mathcal{D}'$, we have $\sinh^2 R - (r \cos \theta - m)^2 - r^2 \sin^2 \theta \geq 0$ and

$$v^2 - (r \cos \theta + c - m)^2 - r^2 \sin^2 \theta \geq 0,$$

for every $(\theta, r) \in V$.

The transition to polar coordinates transforms $U_0(x, y)$ and $L_0^-(x, y)$ to the functions $U(r, \theta)$ and $L(r, \theta)$. It follows that $L(r, \theta) \leq U(r, \theta)$ for all $(r, \theta) \in V$. The area element is given by $r dr d\theta$, and (A.13) becomes

$$\begin{aligned} \iota(R, w, 0, \alpha) &= \int_{-\theta_0}^{\theta_0} \int_{m \sec \theta}^{\epsilon(\theta)} \left(\frac{r}{2L(r, \theta)^2} - \frac{r}{2U(r, \theta)^2} \right) dr d\theta \\ &= \int_0^{\theta_0} \int_{m \sec \theta}^{\epsilon(\theta)} \left(\frac{r}{L(r, \theta)^2} - \frac{r}{U(r, \theta)^2} \right) dr d\theta. \end{aligned} \quad \square$$

PROPOSITION A.5. *If $\alpha \in [0, \pi/2]$ and $w \leq 0$, then*

$$\iota(R, w, 0, \alpha) + \iota(R, w, 0, \pi - \alpha) = \kappa(R, w).$$

Proof. Let $Z \in \mathbb{H}^3$ be given, set $B = \overline{N(Z, R)}$, and let ζ_1 and ζ_2 be points in $S \doteq S(R, Z)$ separated by a spherical distance α . Let ζ_1^* denote the antipode of ζ_1 on S , so that the spherical distance between ζ_1^* and ζ_2 is $\pi - \alpha$. Then $K_1 \doteq K(R, Z, \zeta_1, 0)$ and $K_1^* \doteq K(R, Z, \zeta_1^*, 0)$ are the two half-balls bounded by the plane $H_1 \doteq H(Z, \zeta_1, 0)$, so that $K_1 \cup K_1^* = B$ and $K_1 \cap K_1^* = H_1$. In particular, setting $K_2 = K(R, Z, \zeta_2, w)$, we have $(K_1 \cap K_2) \cup (K_1^* \cap K_2) = K_2$ and $(K_1 \cap K_2) \cap (K_1^* \cap K_2) = K_2 \cap H_1$. Hence,

$$\begin{aligned} \iota(R, w, 0, \alpha) + \iota(R, w, 0, \pi - \alpha) &= \text{vol}(K_1 \cap K_2) + \text{vol}(K_1^* \cap K_2) \\ &= \text{vol } K_2 = \kappa(R, w). \end{aligned} \quad \square$$

PROPOSITION A.6. *Let Z be a point in \mathbb{H}^3 , let $R > 0$ be a real number, and let α, w_1 and w_2 be real numbers with $0 \leq \alpha \leq \pi$ and $0 < w_1 \leq w_2 < R$. Define quantities Ψ_1 and Ψ_2 by $\Psi_i = \arccos(\tanh w_i / \tanh R)$ (so that $0 < \Psi_2 \leq \Psi_1 < \pi/2$).*

- (1) *If $\alpha \leq \Psi_1 - \Psi_2$, then $\iota(R, w_1, w_2, \alpha) = \kappa(R, w_2)$.*
- (2) *If $\alpha > \Psi_1 + \Psi_2$, then $\iota(R, w_1, w_2, \alpha) = 0$.*
- (3) *If $\Psi_1 - \Psi_2 < \alpha \leq \Psi_1 + \Psi_2$, then there exists a unique pair (α_1, α_2) of numbers with $-\pi/2 \leq \alpha_i \leq \pi/2$ such that*

$$\alpha_1 + \alpha_2 = \alpha \tag{A.14}$$

and

$$\tanh w_1 \cos \alpha_2 = \tanh w_2 \cos \alpha_1. \tag{A.15}$$

Moreover,

$$\iota(R, w_1, w_2, \alpha) = \iota(R, w_1, 0, \alpha_1 + \pi/2) + \iota(R, w_2, 0, \alpha_2 + \pi/2). \tag{A.16}$$

Proof. Let Z be a point in \mathbb{H}^3 , and let ζ_1 and ζ_2 be points of $S \doteq S(R, Z)$ separated by a spherical distance α . We set $\mathcal{B} = N(R, Z)$. For $i = 1, 2$, set $\eta_i = \eta_{\zeta_i}$, $\Pi_i = \Pi(Z, \zeta_i, w_i)$, $H_i = H(Z, \zeta_i, 0)$, and $K_i = K(R, Z, \zeta_i, w_i)$.

Let Π be a hyperbolic plane containing Z , ζ_1 , and ζ_2 . We set $s = S(R, Z) \cap \Pi$, $D = \mathcal{B} \cap \Pi$, $\lambda_i = \Pi_i \cap \Pi$, and $k_i = K_i \cap \Pi$. We have $K_1 \subset K_2$ if and only if $k_1 \subset k_2$, and we have $K_1 \cap K_2 = \emptyset$ if and only if $k_1 \cap k_2 = \emptyset$.

Let A_i and B_i denote the two points where λ_i meets s . For $i = 1, 2$ let P_i denote the point $\eta_i \cap \lambda_i$. Using the right triangle $A_i P_i Z$, we find that $\cos \angle A_i Z \zeta_i = (\tanh Z P_i) / (\tanh Z A_i) = (\tanh w_i) / (\tanh R)$, so that

$$\Psi_i = \angle A_i Z \zeta_i = \angle \zeta_i Z B_i = \frac{1}{2} \angle A_i Z B_i. \tag{A.17}$$

As $0 < \Psi_2 \leq \Psi_1 < \pi/2$, there is a unique arc $\tau_i \subset s$, which has endpoints A_i and B_i and has length less than half the length of s .

Let $E : t \mapsto e^{2\pi t}$ denote the standard covering map $\mathbb{R} \rightarrow S^1$, and let $p : \mathbb{R} \rightarrow s$ denote the composition of E with a Euclidean similarity transformation of S^1 onto s . As the spherical distance between ζ_1 and ζ_2 is α , we may choose the similarity transformation defining p in such a way that $p(0) = \zeta_1$ and $p(\alpha) = \zeta_2$.

Note that if $t_1, t_2 \in \mathbb{R}$ are given, and we set $T_i = p(t_i)$ for $i = 1, 2$, then $\angle T_1 Z T_2 = |t_1 - t_2|$.

For $i = 1, 2$, it follows from (A.17) that p maps the unordered pair $\{\pm \Psi_1\}$ to $\{A_1, B_1\}$, and maps the unordered pair $\{\alpha \pm \Psi_2\}$ to $\{A_2, B_2\}$. After possibly relabeling the A_i and B_i , we may assume that $p(-\Psi_1) = A_1$, $p(\Psi_1) = B_1$, $p(\alpha - \Psi_2) = A_2$, and $p(\alpha + \Psi_2) = B_2$.

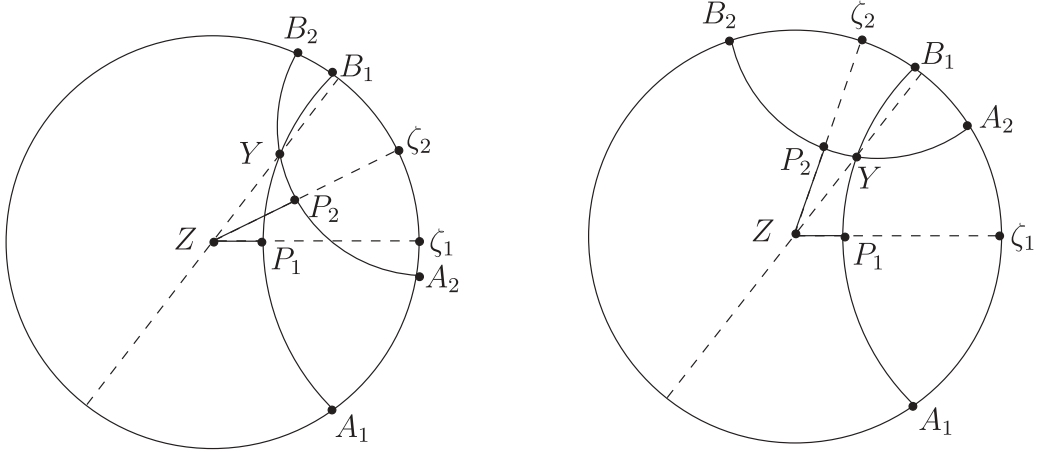
Let $J \subset \mathbb{R}$ denote the smallest closed interval containing the four numbers $\pm \Psi_1$ and $\alpha \pm \Psi_2$. As $0 \leq \Psi_i \leq \pi/2$ and $0 \leq \alpha \leq \pi$, we have $J \subset [\pi/2, \pi]$. Hence, p maps J homeomorphically onto an arc in s . As the intervals $[-\Psi_1, \Psi_1]$ and $[\alpha - \Psi_2, \alpha + \Psi_2]$ have length less than π , the map p sends $[-\Psi_1, \Psi_1]$ and $[\alpha - \Psi_2, \alpha + \Psi_2]$ onto τ_1 and τ_2 , respectively.

Consider the case in which $\alpha \leq \Psi_1 - \Psi_2$. In this case, we have $-\Psi_1 \leq \alpha - \Psi_2 < \alpha + \Psi_2 \leq \Psi_1$, so that $[\alpha - \Psi_2, \alpha + \Psi_2] \subset [-\Psi_1, \Psi_1]$ and hence $\tau_2 \subset \tau_1$. We therefore have $k_1 \subset k_2$ and hence $K_1 \subset K_2$, so that $\iota(R, w_1, w_2, \alpha) = \kappa(R, w_2)$. This proves (1).

Next suppose that $\alpha > \Psi_1 + \Psi_2$. In this case, we have $-\Psi_1 < \Psi_1 < \alpha - \Psi_2 < \alpha + \Psi_2$, so that $[\alpha - \Psi_2, \alpha + \Psi_2] \cap [-\Psi_1, \Psi_1] = \emptyset$ and hence $\tau_2 \cap \tau_1 = \emptyset$. We therefore have $k_1 \cap k_2 = \emptyset$ and hence $K_1 \cap K_2 = \emptyset$, so that $\iota(R, w_1, w_2, \alpha) = 0$. This proves (2).

We now turn to the case $\Psi_1 - \Psi_2 < \alpha \leq \Psi_1 + \Psi_2$. (In particular, α is then non-zero.) As $\Psi_1 \geq \Psi_2$, in this case we have $-\Psi_1 < \alpha - \Psi_2 < \Psi_1 < \alpha + \Psi_2$, so that $[\alpha - \Psi_2, \alpha + \Psi_2]$ and $[-\Psi_1, \Psi_1]$ overlap in the common subinterval $[\alpha - \Psi_2, \Psi_1]$, which is proper in both of them. Hence, the arcs τ_1 and τ_2 overlap in a common subarc τ , which has endpoints A_2 and B_1 and

is proper in both the τ_i . It follows that the lines λ_1 and λ_2 meet at a point Y lying in the disk D , and that the ray originating at Z and passing through Y meets the arc τ at some point X . We may write $X = p(\alpha_1)$ for some $\alpha_1 \in (\alpha - \Psi_2, \Psi_1)$. We set $\alpha_2 = \alpha - \alpha_1 \in [\alpha - \Psi_1, \Psi_2]$. In particular, we have $\alpha_1, \alpha_2 \in (-\pi/2, \pi/2)$, and (A.14) obviously holds with our choice of the α_i . The cases where $\alpha_2 > 0$ and where $\alpha_2 < 0$ are illustrated in the diagram.



As $p(\alpha_1) = X$ and $p(0) = \zeta_1$, we have $\angle P_1ZY = \angle \zeta_1ZX = |\alpha_1|$. As $p(\alpha_1) = X$ and $p(\alpha) = \zeta_2$, we have $\angle P_2ZY = \angle \zeta_2ZX = |\alpha - \alpha_1| = |\alpha_2|$. Thus, for $i = 1, 2$, we have

$$\angle P_iZY = |\alpha_i|. \tag{A.18}$$

For $i = 1, 2$, the hyperbolic line segment \overline{ZY} is the common hypotenuse of the two right triangles with vertices at Z, Y , and P_i . Hence, for $i = 1, 2$, the hyperbolic tangent of the length of this segment is $(\tanh w_i)/\cos(\angle P_iZY)$, which by (A.18) is equal to $(\tanh w_i)/\cos \alpha_i$. In particular, we have $\tanh w_1 \cos \alpha_2 = \tanh w_2 \cos \alpha_1$. This is (A.15).

For $i = 1, 2$, as Π_i meets Π perpendicularly in the line ℓ_i , we have $\Pi_1 \cap \Pi_2 = L$, where L denotes the line meeting Π perpendicularly at Y . Let ℓ_0 denote the hyperbolic line containing Z, Y , and X , and let Π_0 s denote the plane that meets Π perpendicularly in the line ℓ_0 . As $X \in \tau$, and as τ subtends an angle $\Psi_1 + \Psi_2 - \alpha < 2\Psi_2 < \pi$, the endpoints B_1 and A_2 of τ lie in different components of $\mathbb{H}^3 - \Pi_0$. We index the two half-spaces bounded by Π_0 as H_0^1 and H_0^2 in such a way that $B_1 \in H_0^1$ and $A_2 \in H_0^2$. We set $K_0^i = \mathcal{B} \cap H_0^i$ for $i = 1, 2$.

Let ℓ denote the line that is perpendicular to Π at Y . As the lines ℓ_0, ℓ_1 , and ℓ_2 meet at the point Y , the planes Π_0, Π_1 , and Π_2 meet in the line ℓ . Let h_0, h_1 , and h_2 denote the half-planes of Π_0, Π_1 , and Π_2 which are bounded by ℓ and contain X, B_1 , and A_2 , respectively. The definition of the half-spaces H_0^1 and H_0^2 implies that $H_0^1 \cap H_0^2$ has frontier $h_1 \cup h_2$ and contains h_0 . It follows that $H_1 \cap H_2 = (H_1 \cap H_0^1) \cup (H_2 \cap H_0^2)$. In particular, we have

$$K_1 \cap K_2 = (K_1 \cap K_0^1) \cup (K_2 \cap K_0^2).$$

As $(K_1 \cap K_0^1) \cap (K_2 \cap K_0^2) \subset \Pi_0$, it follows that

$$\iota(R, w_1, w_2, \alpha) = \text{vol}(K_1 \cap K_2) = \text{vol}(K_1 \cap K_0^1) + \text{vol}(K_2 \cap K_0^2). \tag{A.19}$$

Now set $\zeta_0^1 = p(\alpha_1 + \pi/2)$ and $\zeta_0^2 = p(\alpha_1 - \pi/2)$. For $i = 1, 2$, set $\eta_0^i = \eta_{\zeta_0^i}$. As $p(0) = \zeta_1$ and $p(\alpha_1 + \pi/2) = \zeta_0^1$, we have $\angle \zeta_1 Z \zeta_0^1 = |(\alpha_1 + \pi/2) - 0| = \alpha_1 + \pi/2$. As $p(\alpha) = \zeta_2$ and $p(\alpha_1 - \pi/2) = \zeta_0^2$, we have $\angle \zeta_2 Z \zeta_0^2 = |(\alpha_1 - \pi/2) - \alpha| = \alpha_2 + \pi/2$. Thus, for $i = 1, 2$, we have

$$\angle \zeta_i Z \zeta_0^i = \alpha_i + \pi/2. \tag{A.20}$$

As $p(\alpha_1) = X, p(\alpha_1 + \pi/2) = \zeta_0^1$, and $p(\alpha_1 - \pi/2) = \zeta_0^2$, the rays η_0^1 and η_0^2 are perpendicular to ℓ_0 and hence to Π_0 , and they point in opposite directions in the line through Z , which is

perpendicular to Π_0 . Hence, each of the half-spaces $H(Z, \zeta_0^i, 0)$ is bounded by Π_0 . It follows that $H(Z, \zeta_0^1, 0)$ and $H(Z, \zeta_0^2, 0)$ are equal to H_0^1 and H_0^2 in some order. But as $p(\alpha_1 + \pi/2) = \zeta_0^1$, $p(\Psi_1) = B_1$ and $\alpha_1 < \Psi_1$, we have $\angle \zeta_0^1 Z B_1 = |\alpha_1 + \pi/2 - \Psi_1| = \alpha_1 - \Psi_1 + \pi/2 > \pi/2$. It follows that $H(Z, \zeta_0^1, 0) = H_0^1$, and therefore that $H(Z, \zeta_0^2, 0) = H_0^2$. Hence,

$$K_0^i = K(R, Z, \zeta_0^i, 0) \quad \text{for } i = 1, 2. \tag{A.21}$$

From (A.20), (A.21), and the definition of ι , it follows that

$$\text{vol}(K_i \cap K_0^i) = \iota(R, w_i, 0, \alpha_i + \pi/2). \tag{A.22}$$

The equality (A.16) follows from (A.19) and (A.22).

It remains to show that $\alpha_1, \alpha_2 \in [-\pi/2, \pi/2]$ are uniquely determined by the conditions (A.14) and (A.15). For this purpose, it suffices to show that there is at most one number $x \in [-\pi/2, \pi/2]$ such that $\tanh w_1 \cos x = \tanh w_2 \cos(\alpha - x)$, that is, such that

$$\tanh w_1 \cos x = (\tanh w_2)(\cos \alpha \cos x + \sin \alpha \sin x). \tag{A.23}$$

As α and w_2 are non-zero, the equality (A.23) cannot hold with $x = \pm\pi/2$. If $\pi/2 < x < \pi/2$, then (A.23) is equivalent to $(\tanh w_2)(\sin \alpha \tan x + \cos \alpha) = \tanh w_1$, which can have at most one solution for $x \in (-\pi/2, \pi/2)$ as $\tan x$ increases monotonically on that interval. \square

A.7. The results of this appendix can be used to compute arbitrary values of the functions κ and ι . For any $R > 0$ it is clear that $\kappa(R, w) = 0$ for $w \geq R$ and that $\kappa(R, 0) = B(R)/2 = (\pi/2)(\sinh(2R) - 2R)$ (see (9.1)). When $0 < w < R$, we can calculate $\kappa(R, w)$ directly from Proposition A.3.

Similarly, it is clear that, for any $R > 0$ and any $\alpha \in [0, \pi]$, we have $\iota(R, w_1, w_2, \alpha) = 0$ whenever either w_1 or w_2 is at least R . If each of the w_i is less than R , then Proposition A.6 reduces the calculation of $\iota(R, w_1, w_2, \alpha)$ to the special case in which $w_2 = 0$. It is clear that $\iota(R, w, 0, 0) = \kappa(R, w)$ and that $\iota(R, w, 0, \pi) = 0$, and it follows from Proposition A.5 that $\iota(R, w, 0, \pi/2) = \kappa(R, w)/2$. If $\pi/2 < \alpha < \pi$, then we can compute $\iota(R, w, 0, \alpha)$ directly from Proposition A.4. If $0 < \alpha < \pi/2$, Proposition A.5 reduces the calculation of $\iota(R, w_1, w_2, \alpha)$ to the calculation of $\iota(R, w, \pi - \alpha)$ and $\kappa(R, w)$, which can be carried out using Propositions A.6 and A.3.

There are two points in the body of this paper where the formulas given in this appendix were used in rigorous sampling arguments to establish numerical bounds for certain functions. The proof of Lemma 12.4 required calculation of 100 different numerical values of $\iota(R, w, w', \alpha)$, and the proof of Lemma 13.3 required calculation of several thousand values of κ .

The evaluation of κ from Proposition A.3 is straightforward as it is given by a closed-form expression. In evaluating ι , there are two steps that require somewhat more elaborate numerical methods. First, the application of Proposition A.6 in the case where $\Psi_1 - \Psi_2 < \alpha \leq \Psi_1 + \Psi_2$ requires the numerical solution of the equations (A.14) and (A.15). We used the routine `hybrd`, which is included in the MINPACK library, to find approximate solutions to this system. The step involving Proposition A.4 seems to require numerical integration, as we do not know of a closed-form expression for the integral in (A.1). For this purpose, we relied on the adaptive Gaussian quadrature method, which is implemented in the python `scipy` module and uses the FORTRAN quadrature routines from the QUADPACK library.

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