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Dehn surgery on knots

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Introduction

We consider the problem of which Dehn surgeries on a knot can produce 3-manifolds with cyclic fundamental group. It is natural to work in the following setting. Let M be a compact, connected, irreducible, orientable 3-manifold such that ∂M is a torus. The unoriented isotopy class of a non-trivial simple closed curve in ∂M will be called its *slope*. For any slope r, a closed 3-manifold $M(r)$ may be constructed by attaching a solid torus J to M so that a curve of slope r bounds a disk in J.

If r and s are two slopes, we denote their (minimal) geometric intersection number by $\Delta(r, s)$.

The main result of this paper is the following theorem.

Cyclic Surgery Theorem. *Suppose that M is not a Seifert fibered space. If $\pi_1(M(r))$ and $\pi_1(M(s))$ are cyclic, then $\Delta(r, s) \leq 1$. Hence there are at most three slopes r such that $\pi_1(M(r))$ is cyclic.*

This result is sharp; Fintushel-Stern have shown (private communication) that 18- and 19-surgeries on the $(-2, 3, 7)$ pretzel knot yield lens spaces. Many examples of the same phenomenon have been produced by Berge (private communication).

For a detailed account of the manifolds obtained by attaching solid tori to a Seifert fibered space along some of its boundary components, see [Hei].

We give some corollaries which apply to Dehn surgery on knots K in S^3. In the proofs, M will denote the complement of an open tubular neighborhood of

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K. We use $\mathbf{Q} \cup \{1/0\}$ as in [R] to parametrize the slopes on ∂M, and denote $M(r)$ by $K(r)$. In particular, $K(1/0) = S^3$.

COROLLARY 1. *If K is not a torus knot and $r \in \mathbf{Q}$, then $\pi_1(K(r))$ can be cyclic only if r is an integer. Moreover, there are at most two such integers r, and if there are two then they must be successive.*

Proof. The manifold M is Seifert-fibered if and only if K is a torus knot. Also, $\Delta(a/b, 1/0) = |b|$, and if n_1 and n_2 are integers, then $\Delta(n_1, n_2) = |n_1 - n_2|$. □

The next corollary is a step towards establishing the conjecture that any nontrivial knot has Property P, in other words, that $K(r)$ is not simply-connected if $r \in \mathbf{Q}$.

COROLLARY 2. *If K is a non-trivial knot and $r \in \mathbf{Q}$ is not equal to* 1 *or* -1 *then $K(r)$ is not simply-connected. Moreover,* $K(1)$ *and* $K(-1)$ *cannot both be simply-connected.*

Proof. This follows from Corollary 1 and the fact that non-trivial torus knots have Property P [Ms]. □

COROLLARY 3. *Up to unoriented equivalence, there are at most two knots whose complements are of a given topological type.*

Proof. Suppose that there are three inequivalent knots whose complements are homeomorphic. Then for any one of these knots, say K, there exist two distinct rational numbers r and s such that $K(r) = K(s) = K(1/0) = S^3$. This contradicts Corollary 2. □

COROLLARY 4. *If K is a non-trivial amphicheiral knot and $r \in \mathbf{Q} - \{0\}$, then $\pi_1(K(r))$ is not cyclic. In particular, K has Property* P.

Proof. If K is amphicheiral, then there is an automorphism of M which takes r to $-r$ for all slopes $r \in \mathbf{Q} \cup \{1/0\}$. Since $\Delta(r, -r) \geq 2$ if $r \in \mathbf{Q} - \{0\}$, and since torus knots are not amphicheiral, the result follows. □

COROLLARY 5. *Knots of Arf invariant* 1 *are determined up to unoriented equivalence by their complements.*

Proof. As in the proof of Corollary 3, it follows from Corollary 2 that if K is not determined by its complement then either $K(1)$ or $K(-1)$ is homeomorphic to S^3. But if $\alpha(K) \in \mathbf{Z}_2$ denotes the Arf invariant of K, then the Rohlin invariant of $K(1/n)$ is $n\alpha(K)$ (see [Gn]). □

Actually, a considerably stronger result follows from recent work of Casson's [C], [A-M]. He proves that any knot K whose Alexander polynomial $\Delta(t)$ satisfies $\Delta''(1) \neq 0$ has Property P. In particular, since the mod 2 reduction of $\frac{1}{2}\Delta''(1)$ is $\alpha(K)$, this implies that knots of Arf invariant 1 have Property P.

It is shown in [Wh] that Corollary 1 implies the following result.

COROLLARY 6. *Prime knots with isomorphic groups have homeomorphic complements.* □

Finally, combining our work with the fact, recently proved by Bleiler and Scharlemann [B-S], that strongly invertible knots have Property P, we can state the following result.

COROLLARY 7. *If K is a non-trivial knot which is invariant under a non-trivial periodic automorphism of S^3, then K has Property* P.

This will be proved in Section 2.8.

Before giving some indications of the method of proof of the Cyclic Surgery Theorem, we shall fix some conventions and terminology that will be used throughout the paper. We will work in the smooth category. All manifolds are understood to be orientable. If S is a surface in a 3-manifold, we define a *compressing disk* for S to be a disk $D \subset M$ such that $D \cap S = \partial D$ and such that ∂D is (homotopically) non-trivial in S. If S is properly embedded one can surger S using D to obtain a new properly embedded surface in M; this operation will be called a *compression*. A surface S in a 3-manifold M will be called *incompressible* if (i) no component of S is a sphere and (ii) there is no compressing disk for S in M. The incompressible surfaces that we discuss will all be either properly embedded (always the case in Chapter 1) or contained in the boundary of the ambient 3-manifold (sometimes the case in Chapter 2). In these cases, (ii) is equivalent to the condition that the fundamental group of each component of S maps injectively to that of M.

By an *essential surface* in a 3-manifold M, we shall mean a properly embedded surface which is incompressible and no component of which is parallel to a subsurface of ∂M. Now suppose that M is irreducible and that ∂M is a torus, and let S be any essential surface in M with ∂S non-empty. The boundary components of S are disjoint non-trivial simple closed curves in ∂M and hence all have the same slope, say r. We call r the *boundary slope* of S. A slope r on ∂M will be called a *boundary slope* if it is the boundary slope of some essential surface in M. We will call r a *strict boundary slope* if it is the boundary slope of some essential surface which is not a fiber in any fibration of M over the circle.

The Cyclic Surgery Theorem is proved in Chapter 1 for the case when M contains no essential torus and neither of the given slopes r, s is a strict boundary slope. Here the argument is based on the Thurston Geometrization Theorem, which asserts in this situation that the interior of M has a hyperbolic metric of finite volume. As in [C-S1], this is used to define a family of (characters of) representations of $\pi_1(M)$ in $SL_2(\mathbf{C})$, parametrized by the points of a complex affine algebraic curve X_0. The ideal points of a de-singularized projective completion $\tilde{X}_0$ of X_0 correspond to actions of $\pi_1(M)$ on trees, which can be used to define essential surfaces in M. One shows that for all but a very small set* of slopes r, one of the following alternatives holds. Either there is a point of X_0 corresponding to a representation of $\pi_1(M)$ in $SL_2(\mathbf{C})$ which induces a representation of the quotient group $\pi_1(M(r))$ onto a non-cyclic subgroup of $PSL_2(\mathbf{C})$; or there is some ideal point of $\tilde{X}_0$ which defines an essential surface S which is closed or has boundary slope r, and is not a fiber. Furthermore, if S is closed then it has positive genus and remains incompressible in $M(r)$. Thus in each of these cases, either $\pi_1(M(r))$ is non-cyclic or r is a strict boundary slope.

For a more precise outline of the proof in this case, see Sections 1.0 and 1.1.

Chapter 2 is devoted to the proof of the Cyclic Surgery Theorem in the case where either M contains an essential torus or one of the given slopes is a strict boundary slope. One shows that it is enough to prove the theorem for manifolds which are not cabled (in the sense of [Gr-L]). If M is not cabled and contains an essential torus T, it is shown that T remains incompressible in $M(r)$ for all but a very small set of slopes r. This is done by considering two slopes r and s such that T compresses in $M(r)$ and $M(s)$, and carrying out a graph-theoretic analysis of the intersection of the two planar surfaces in M corresponding to the compressing disks for T in $M(r)$ and $M(s)$. An analogous argument allows one to deal also with the case where the first Betti number of M is greater than 1. There, by finding restrictions on the slopes r such that $M(r)$ is reducible, it is shown that $\pi_1(M(r))$ is non-cyclic for all but a very small set of slopes r. For the case of boundary slopes, observe that if r is the boundary slope of an essential surface F in M, then F can be capped off to produce a closed surface $\hat{F}$ in $M(r)$. Assuming (as we now may) that the first Betti number of M is 1 and that r is a strict boundary slope, it is shown that if F is suitably chosen then either $\hat{F}$ is incompressible in $M(r)$, or $\hat{F}$ is a 2-sphere which decomposes $M(r)$ as a connected sum of two non-trivial lens spaces, or M contains a closed essential surface S with special properties. Finally, by a graph-theoretic analysis of the

*For the purpose of this informal discussion, a set of slopes is "very small" if $\Delta(r, s) \leq 1$ for all r and s in the set.

kind mentioned above, one shows that S remains incompressible in $M(s)$ whenever $\Delta(r, s) > 1$.

A more detailed summary of the contents of Chapter 2 is given in Section 2.0.

The proof of the Cyclic Surgery Theorem gives a rather stronger result. Let us define a closed 3-manifold L to be *small* if
(*) there exists no incompressible surface in L; and
(**) there exists no representation of $\pi_1(L)$ into $PSL_2(\mathbf{C})$ with non-cyclic image.
Then in both the statement and proof of the Cyclic Surgery Theorem, the hypothesis that $M(r)$ and $M(s)$ have cyclic fundamental groups may be replaced by the condition that they are small. (A connected sum of two non-trivial lens spaces violates (**) because a free product of two cyclic groups is Fuchsian and hence embeds in $PSL_2(\mathbf{R})$.)

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Chapter 1

In this chapter we will prove the following case of the Cyclic Surgery Theorem.

THEOREM 1.0.1. *Let M be a compact irreducible (orientable) 3-manifold with torus boundary. Suppose that M is not a Seifert fibered space and contains no essential torus. Let r and s be slopes such that $\pi_1(M(r))$ and $\pi_1(M(s))$ are cyclic groups. If neither r nor s is a strict boundary slope then $\Delta(r, s) \leq 1$.*

In Section 1.1 we will reduce the proof of this theorem to that of two other results, Propositions 1.1.2 and 1.1.3. Proposition 1.1.2 will be proved in Section 1.4, and Proposition 1.1.3 will be proved in Sections 1.5 and 1.6. In Sections 1.2 and 1.3 we reformulate and extend some of the ideas in [C-S1] to make them more directly applicable to the proofs of these propositions.

Throughout this chapter it is understood that M satisfies the hypotheses of Theorem 1.0.1. Thus it follows from the Thurston Geometrization Theorem [MB, Chapter V] that the interior of M admits a hyperbolic metric of finite volume.

1.0.2. We need to establish some conventions regarding the use of fundamental groups in order to avoid problems with base points. If we are given a connected polyhedron X and if we have fixed a universal cover $\tilde{X}$ of X, then we will denote by $\pi_1(X)$ the group of covering transformations of $\tilde{X}$. If x is a point of X then a lift $\tilde{x} \in \tilde{X}$ of x determines an identification of $\pi_1(X)$ with $\pi_1(X, x)$. An oriented closed curve in X determines, up to conjugacy, an element of $\pi_1(X)$. A map from a connected polyhedron Y to X determines, up to composition with inner automorphisms, a homomorphism from $\pi_1(Y)$ to $\pi_1(X)$. Thus many of the statements which are proved in this chapter apply to objects which are only defined up to conjugacy. Our convention is that this ambiguity will be ignored provided that the statements in question are invariant under conjugacy.

We fix, for the entire chapter, a universal cover $\tilde{M}$ of M.

It will be convenient to adopt the following notation. We will denote by L the group $H_1(\partial M; \mathbf{Z})$, which will be regarded as a lattice in the 2-dimensional real vector space $V = H_1(\partial M; \mathbf{R})$. There is a homomorphism $\pi_1(\partial M) \to \pi_1(M)$ which is defined up to composition with inner automorphisms of $\pi_1(M)$. We will identify $\pi_1(\partial M)$ with its image under this homomorphism. We will let $e\colon L \to \pi_1(\partial M)$ denote the inverse of the Hurewicz isomorphism. We will often view e as a homomorphism from L to $\pi_1(M)$ which is defined modulo inner automorphisms. Recall that a slope is an isotopy class of unoriented simple closed curves on ∂M. Thus each slope r corresponds to a pair $\{\pm\alpha\}$ of primitive elements of L; we shall write $M(\alpha) = M(r)$. Note that $\pi_1(M(\alpha))$ has presentation $|\pi_1(M)\colon e(\alpha) = 1|$. If r is a (strict) boundary slope then we will call α and $-\alpha$ *(strict) boundary classes*. If α and β are elements of L then we will write $\Delta(\alpha, \beta)$ for the absolute value of their intersection number.

1.1. The curve of characters

We will briefly review some notation and basic theorems from [C-S1] and [C-S2]. We denote by $R = R(\pi_1(M))$ the space of all representations of $\pi_1(M)$ in $\mathrm{SL}_2(\mathbf{C})$. This space has the structure of a complex affine algebraic set. The set $X = X(\pi_1(M))$ is the set of characters of representations in R, and $t\colon R \to X$ is the natural map which sends a representation ρ to its character χ_ρ. This set also has the structure of a complex affine algebraic set [C-S1, Corollary 1.4.5]. The map t is a regular map, and for each $\gamma \in \pi_1(M)$ there is a regular function $I_\gamma\colon X \to \mathbf{C}$ defined by $I_\gamma(\chi) = \chi(\gamma)$. The set of functions I_γ for $\gamma \in \pi_1(M)$ generates the coordinate ring of X. Note that since I_γ depends only upon the conjugacy class of $\gamma \in \pi_1(M)$, $I_{e(\alpha)}$ is a well-defined function on X for each $\alpha \in L$. Since M is hyperbolic there is a representation $\rho_0 \in R$ which is discrete and faithful [C-S1, Proposition 3.1.1]. We denote by R_0 an irreducible component of R containing ρ_0, and set $X_0 = t(R_0)$.

The following proposition is essentially due to Thurston.

PROPOSITION 1.1.1. *The set* $X_0 \subset X$ *is an irreducible affine variety of dimension* 1. *For each non-trivial element* α *of* L, *the function* $I_{e(\alpha)}$ *is non-constant on* X_0.

Proof. Proposition 1.4.4 of [C-S1] shows that X_0 is a variety. By Proposition 3.2.1 of [C-S1], dim $X_0 \geq 1$. The equality dim $X_0 = 1$ then follows from Proposition 2 of [C-S2], which also shows that $I_{e(\alpha)}$ is non-constant on X_0 for each non-trivial element α of L. □

The idea of the proof of Theorem 1.0.1 is to show, for most $\alpha \in L$, that $\pi_1(M(\alpha))$ is non-cyclic by producing a representation in $\mathrm{PSL}_2(\mathbf{C})$ with non-cyclic image. A representation $\rho \in R_0$ will induce such a representation if and only if $\rho(e(\alpha)) = \pm 1$. An obvious necessary condition for $\rho(e(\alpha)) = \pm 1$ is that the value of the function $I_{e(\alpha)}$ should be ± 2 at the point $t(\rho) \in X_0$. This suggests studying the zeroes of the functions f_α: $X_0 \to \mathbf{C}$ defined for all $\alpha \in L$ by

$$f_\alpha = I_{e(\alpha)}^2 - 4.$$

It is always easier to study the zeroes of a function defined on a smooth projective curve. Thus, as in [C-S1], we will denote by $\tilde{X}_0$ the smooth projective curve which is birationally equivalent to X_0. The birational equivalence from $\tilde{X}_0$ to X_0 is regular (i.e. defined) at all but a finite number of points of $\tilde{X}_0$. The points where it is defined will be called *ordinary points*, and the others will be called *ideal points*. We will identify the function fields of X_0 and $\tilde{X}_0$ under the isomorphism induced by the birational equivalence. Thus any rational function f on X_0 pulls back to a rational function on $\tilde{X}_0$ which will also be denoted f.

Recall that a rational function defined on a smooth curve is meromorphic and that its degree is equal to the number of poles, counted with multiplicities. For a non-zero rational function the degree is also equal to the number of zeroes. The following proposition describes the behavior of the degree of f_α: $X_0 \to \mathbf{C} \cup \{\infty\}$ as α varies over the group L.

PROPOSITION 1.1.2. *There exists a norm* $\|\cdot\|$ *on the real vector space* $V = H_1(\partial M; \mathbf{R})$ *with the following properties.*

(i) *For each* $\alpha \in L$, $\|\alpha\| =$ degree f_α.

(ii) *The unit ball is a finite-sided polygon whose vertices are rational multiples of strict boundary classes in* L.

This will be proved in Section 1.4 by examination of the poles of the functions f_α for $\alpha \in L$.

The next proposition concerns the orders of zeroes of the functions f_α for $\alpha \in L$. If x is a point of $\tilde{X}_0$ and f is a rational function on $\tilde{X}_0$ with $f(x) = 0$, then we will denote by $Z_x(f)$ the order of zero of f at x. If $f(x) \neq 0$, and in particular if f has a pole at x, we set $Z_x(f) = 0$.

PROPOSITION 1.1.3. *Suppose that α is a primitive element of L which is not a strict boundary class, and that $\pi_1(M(\alpha))$ is cyclic. Then, for each point $x \in \tilde{X}_0$, and for each non-zero $\delta \in L$, we have*

$$Z_x(f_\alpha) \leq Z_x(f_\delta).$$

This will be proved in Sections 1.5 and 1.6. It will be shown in Section 1.5 that if x is an ordinary point and α and δ are non-zero elements of L such that $Z_x(f_\alpha) > Z_x(f_\delta)$, then the image of x in X_0 is the character of a representation $\rho \in R_0$ which induces a homomorphism of $\pi_1(M(\alpha))$ onto a non-cyclic subgroup of $\mathrm{PSL}_2(\mathbf{C})$. In the case where x is an ideal point and α and δ are non-zero elements of L such that $Z_x(f_\alpha) > Z_x(f_\delta)$, it will be shown in Section 1.6 that either α is a strict boundary class or there exists a closed surface in M which is incompressible in $M(\alpha)$.

The rest of this section will be devoted to the proof that Propositions 1.1.2 and 1.1.3 imply Theorem 1.0.1.

Note that the following is a corollary to Propositions 1.1.2 and 1.1.3.

COROLLARY 1.1.4. *Suppose that α is a primitive element of L which is not a strict boundary class, and that $\pi_1(M(\alpha))$ is cyclic. Then for each non-zero $\delta \in L$ $\|\alpha\| \leq \|\delta\|$.* □

Define $m = \min_{0 \neq \delta \in L}\|\delta\|$, and consider the ball B of radius m in V. By Proposition 1.1.2 this is a compact, convex, finite-sided polygon which is balanced (i.e. $-B = B$). By construction there are no non-zero elements of L contained in the interior of B.

We will use the obvious area element defined on V in which any pair of generators of L spans a parallelogram of area 1. The above properties of B imply that the projection of V to the torus $V/2L$ is one-to-one on the interior of B. Thus the area of B is at most 4. (This observation is due to Minkowski.)

Corollary 1.1.4 implies that if $\alpha \in L$ is a not a strict boundary class and $\pi_1(M(\alpha))$ is cyclic then α must lie on the boundary of B. Now let α and β be two elements of L, neither of which is a strict boundary class. Suppose that $\pi_1(M(\alpha))$ and $\pi_1(M(\beta))$ are cyclic. Let P denote the parallelogram in V with vertices $\pm\alpha$, $\pm\beta$. We have

$$\Delta(\alpha, \beta) = \tfrac{1}{2}\text{Area } P \leq \tfrac{1}{2} \text{ Area } B \leq 2.$$

Furthermore, if $\Delta(\alpha, \beta) = 2$ then we must have $B = P$. Hence in this case, α and β are vertices of B. But by Proposition 1.1.2(ii), this implies that they are strict boundary classes, a contradiction. Thus we conclude that $\Delta(\alpha, \beta) \leq 1$.

This completes the proof that Propositions 1.1.2 and 1.1.3 imply Theorem 1.0.1. □

1.2. Ideal points and trees

In this section we redo the material from Sections 2 and 3 of [C-S1] in a form, suggested by the approach of [M-S1], which is better adapted for the applications in this paper. We begin by reviewing some of the ideas in [Se].

Let F be a field and v: $F^* \to \mathbf{Z}$ a (discrete) valuation. We denote by $\mathscr{O}$ the valuation ring $\{f \in F | f = 0 \text{ or } v(f) \geq 0\}$, and by $\mathscr{M}$ the maximal ideal in $\mathscr{O}$. The group $\mathrm{SL}_2(F)$ acts on a tree T_F which is constructed as follows. Define a *lattice* to be a finitely generated $\mathscr{O}$-submodule of the vector space F^2 which spans over F. Define an equivalence relation on lattices by $\Lambda \sim \Lambda'$ if $\Lambda = \alpha\Lambda'$ for some $\alpha \in F^*$. Given lattices Λ and Λ' there exists an element $\alpha \in F^*$ such that $\alpha\Lambda' \subset \Lambda$ and $\Lambda/\alpha\Lambda'$ is a cyclic $\mathscr{O}$-module. The annihilator of this cyclic module is determined by the equivalence classes $[\Lambda]$ and $[\Lambda']$, and is equal to $\mathscr{M}^d$ for some integer $d = d([\Lambda], [\Lambda'])$.

Let T denote the set of equivalence classes of lattices. The function d: $T \times T \to T$ is an integer-valued metric. Consider the 1-complex constructed by taking T as the vertex set and adding an edge joining each pair of vertices which are distance 1 apart. It is shown in [Se] that this is a tree; i.e. it is connected and simply-connected. For two vertices s and s', the number of edges in the shortest path joining them is given by $d(s, s')$. We shall denote this 1-complex by T as well.

1.2.1. The obvious action of $\mathrm{GL}_2(F)$ on the set of lattices in F^2 determines an action on T, and this action is transitive on vertices. The lattice $\mathscr{O}^2$ generated by the standard basis of F^2 is stabilized by the subgroup $\mathrm{GL}_2(\mathscr{O})$ of $\mathrm{GL}_2(F)$. This subgroup is, in fact, the entire stabilizer of the vertex $[\mathscr{O}^2]$. Restricting the action of $\mathrm{GL}_2(F)$ yields an action of $\mathrm{SL}_2(F)$ on T, with no inversions, such that the stabilizer of a vertex is a conjugate in $\mathrm{GL}_2(F)$ of the subgroup $\mathrm{SL}_2(\mathscr{O})$.

Next we consider the action of $\mathrm{SL}_2(\mathscr{O})$ on the link $\mathscr{L}$ of the standard vertex $[\mathscr{O}^2]$. (Of course, this action is conjugate to the action of the stabilizer of any vertex on the link of the vertex.)

1.2.2. The vertices of $\mathscr{L}$ are represented by lattices $\Lambda \subset \mathscr{O}^2$ such that the quotient $\mathscr{O}^2/\Lambda$ is isomorphic as an $\mathscr{O}$-module to the residue field $k = \mathscr{O}/\mathscr{M}$. The image of such a lattice in the 2-dimensional k-vector space $\mathscr{O}^2/\mathscr{M}^2 = k^2$ is a

1-dimensional subspace. This establishes a bijection between $\mathscr{L}$ and the projective line $P^1(k)$; we will use this to identify $\mathscr{L}$ with $\mathrm{P}^1(k)$. Then the action of $\mathrm{SL}_2(\mathscr{O})$ on $\mathscr{L}$ is the pull-back, under the obvious homomorphism $\mathrm{SL}_2(\mathscr{O}) \to \mathrm{PSL}_2(k)$, of the natural action of $\mathrm{PSL}_2(k)$ on $P^1(k)$. In particular, an element of $\mathrm{SL}_2(\mathscr{O})$ acts trivially on $\mathscr{L}$ if and only if it is congruent to ± 1 modulo $\mathscr{M}$.

In some of the arguments below we will use the following version of the extension theorem for valuations.

THEOREM 1.2.3. *Let F be a finitely generated extension of a field K. Let $w\colon K^* \to \mathbf{Z}$ be a valuation. Then there exists a valuation $v\colon F^* \to \mathbf{Z}$ such that $v|K^* = d \cdot w$ for some positive integer d.*

This is slightly more general than the usual form [Jb] of the extension theorem since F is allowed to be a transcendental extension of K. For a proof see, for example, [M-S1, Lemma II.4.4]

If v and w are as in the above theorem we will say that v is an *extension* of w.

We will now consider a 3-manifold M as in the introduction to the chapter, and the associated varieties R_0, X_0 and $\tilde{X}_0$ defined in Section 1.1. Let K be the function field of X_0, which we have identified with that of $\tilde{X}_0$, and let F be the function field of R_0. Since there is a regular map of R_0 onto X_0, we may regard F as an extension field of K.

1.2.4. Recall, from Section 1 of [C-S1], that there is a tautological representation $P\colon \pi_1(M) \to \mathrm{SL}_2(F)$ defined by

$$P(\gamma) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where the functions a, b, c, and d are defined by

$$\rho(\gamma) = \begin{bmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{bmatrix}$$

for all $\rho \in R_0$. Note that for any element γ of $\pi_1(M)$, the trace of $P(\gamma)$ is equal to $I_\gamma \in K \subset F$. Note that since ρ_0 is faithful, P is also faithful.

1.2.5. An ideal point x of $\tilde{X}_0$ determines a valuation $w\colon K^* \to \mathbf{Z}$, which can therefore be extended to a valuation $v\colon F^* \to \mathbf{Z}$. This valuation determines an action of $\mathrm{SL}_2(F)$ on a tree T which pulls back, under the homomorphism P, to an action of $\pi_1(M)$ on T. An action constructed in this way will be said to be *associated to* the ideal point x.

(The construction of this action depends on the choice of the extended valuation v. However, the action defines a translation length function which is shown in [M-S1] to be determined by the ideal point x. In most cases the translation length function essentially determines the action; see [C-M]. We will not use these facts here.)

The following proposition is proved, in a slightly different setting, in [C-S1]; we indicate a proof from the point of view of the present paper.

PROPOSITION 1.2.6. *Let $\pi_1(M) \times T \to T$ be an action associated to an ideal point x. An element γ of $\pi_1(M)$ has a fixed point in T if and only if $I_\gamma(x) \neq \infty$. No point of T is fixed by the entire group $\pi_1(M)$.*

Proof. Let w and v be as in 1.2.5, and set $\mathcal{O} = \mathcal{O}_v$. By 1.2.1, γ has a fixed point in T if and only if $P(\gamma)$ belongs to a conjugate of $\mathrm{PSL}_2(\mathcal{O})$ in $\mathrm{PSL}_2(F)$. Using the rational canonical form (cf. [C-S1, Theorem 2.2.1]) one sees that this is equivalent to the condition $I_\gamma = \text{trace } P(\gamma) \in \mathcal{O}$, which is in turn equivalent to $I_\gamma \in \mathcal{O}_w$, i.e. $I_\gamma(x) \neq \infty$. Since x is an ideal point, and since the functions I_γ for $\gamma \in \pi_1(M)$ generate the coordinate ring of X_0, $\mathcal{O}$ cannot contain all of the I_γ. Thus no vertex of T is fixed by the entire group $\pi_1(M)$. □

The final conclusion in the statement of Proposition 1.2.6 can be strengthened as follows.

PROPOSITION 1.2.7. *Let $\pi_1(M) \times T \to T$ be an action associated to an ideal point. Then no non-trivial normal subgroup of $\pi_1(M)$ fixes a point of T.*

The proof of Proposition 1.2.7 depends on the following lemma.

LEMMA 1.2.8. *Let F be a field with valuation v and let Γ be a subgroup of $\mathrm{SL}_2(F)$. Suppose that Γ contains a normal subgroup N, which is not contained in the center of $\mathrm{SL}_2(F)$, but which is contained in a conjugate of $\mathrm{SL}_2(\mathcal{O})$. Then either the trace of every element of Γ lies in $\mathcal{O}_v$ or else Γ contains an abelian subgroup of index at most 2.*

Proof. Assume that there exists $\gamma_0 \in \Gamma$ whose trace does not belong to $\mathcal{O}_v$. In particular, $\mathrm{tr}(\gamma_0) \neq \pm 2$. Thus, after possibly replacing F by a degree 2 extension and extending the valuation, we may assume that γ_0 is diagonalizable over F.

Consider the action of Γ on the tree $T = T_F$. The fixed set T^N of the subgroup N is non-empty since N is contained in a conjugate of $\mathrm{SL}_2(\mathcal{O})$. By [Se, p. 58], T^N is a subtree of T. Since N is normal, T^N is invariant under Γ.

Now let γ be any conjugate of γ_0. Since γ does not fix any vertex of T it follows from [Se, p. 63, Prop. 24] that γ has a unique *axis* A_γ, i.e. a subcomplex of T which is homeomorphic to $\mathbf{R}$ and invariant under γ. Moreover, any subtree of T which is invariant under γ must contain A_γ. In particular, $A_\gamma \subset T^N$.

Since γ is diagonalizable in $SL_2(F)$ the axis A_γ can be described explicitly as follows. Consider a basis $\{e, f\}$ of eigenvectors of γ. For each integer n let s_n be the vertex of T corresponding to the lattice with basis $\{\pi^n e, f\}$, where π is a generator of the maximal ideal in $\mathcal{O}$. One checks directly that s_n is joined to s_{n+1} by an edge e_n and that the union of the e_n and the s_n is a line in T which is invariant under the action of γ. Thus this line is the axis A_γ.

We claim that each element ν of N is diagonal in the basis $\{e, f\}$. In proving this we may assume that $\{e, f\}$ is the standard basis of F^2. We have seen that ν fixes all of the vertices of A_γ. Let Π denote the matrix $\begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(F)$, which takes s_0 to s_n. Thus if ν equals $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then for each $n \in \mathbf{Z}$, we have that $\Pi^{-n}\nu\Pi^n = \begin{bmatrix} a & \pi^{-n}b \\ \pi^n c & d \end{bmatrix}$ fixes s_0 and therefore belongs to $SL_2(\mathcal{O})$ by 1.2.1. Therefore $b = c = 0$ so that ν is diagonal, proving the claim.

Since N is not contained in the center of $SL_2(\mathcal{O})$ there is at most one basis of F^2 in which N is diagonal. Therefore all of the conjugates of γ_0 are simultaneously diagonalizable. Therefore there exist exactly two lines in F^2 which are invariant under every conjugate of γ_0. Every element of Γ either keeps these lines invariant or interchanges them. It follows that Γ has an abelian subgroup of index at most 2. □

Proof of Proposition 1.2.7. Let $\Gamma = P(\pi_1(M)) \subset SL_2(F)$, where P is the tautological representation and F is the function field of R_0. Let v and w be as in 1.2.4. Suppose that there is a non-trivial normal subgroup of $\pi_1(M)$ which fixes a vertex of T; let N denote its image under P. Since P is faithful and $\pi_1(M)$ is torsion-free, N cannot be contained in the center of $SL_2(F)$. By 1.2.1, N is contained in a conjugate of $SL_2(\mathcal{O})$.

By Lemma 1.2.7, either every element of $P(\pi_1(M))$ has trace in $\mathcal{O}_v$ or Γ has an abelian subgroup of index at most 2. The first possibility implies that $I_\gamma(x) \neq \infty$ for all γ in Γ. This is absurd since x is an ideal point. The second possibility implies, since P is faithful, that $\pi_1(M)$ has an abelian subgroup of index at most 2, contradicting the fact that the interior of M is a hyperbolic manifold of finite volume. □

1.3. Trees and incompressible surfaces

The results of this section are very similar to those of [C-S1, Section 2]. However, for the applications in Section 1.6 of this chapter it is necessary to state and prove them from a rather different point of view, suggested by [M-S1] and [M-S2].

We continue to work with a manifold M as in the introduction to this chapter. Let $\tilde{M}$ denote the universal cover of M, and let $p: \tilde{M} \to M$ be the covering projection. Throughout this section we will suppose that we are given an action without inversions of $\pi_1(M)$ on a (simplicial) tree T. Let E denote the set of midpoints of edges in T.

Definition 1.3.1. A properly embedded surface s in M is *associated to* the action if there exists a $\pi_1(M)$-equivariant map $\tilde{\phi}: \tilde{M} \to T$ which is transverse to E, and such that $\tilde{\phi}^{-1}(E) = p^{-1}(S)$.

PROPOSITION 1.3.2. *If S is a surface associated to the action then, for each component D of $M - S$, the image of $\pi_1(D)$ in $\pi_1(M)$ is contained in the stabilizer of a vertex. Thus if there is no fixed point for the action then any associated surface is non-empty.*

(Note that the above statement is consistent with the conventions of 1.0.2, since the set of vertex stabilizers in $\pi_1(M)$ is invariant under conjugation. The same comment applies to Proposition 1.3.4 below.)

Proof. The image of $\pi_1(D)$ in $\pi_1(M)$ is the stabilizer of a component of $p^{-1}(D)$. The equivariance implies that it fixes a vertex of T. In particular if $S = \varnothing$ then $\pi_1(M)$ fixes a vertex of T. □

In studying surfaces associated to the action of $\pi_1(M)$ on T it is convenient to work with an alternate characterization. This will use an explicit form of the construction in Section 2 of [C-S1].

1.3.3. As in [Sco-W] we consider the aspherical complex $\mathscr{K}$ which is the quotient of $\tilde{M} \times T$ under the diagonal action of $\pi_1(M)$. This action realizes $\tilde{M} \times T$ as the universal cover of $\mathscr{K}$. Thus there is a natural isomorphism between $\pi_1(M)$ and $\pi_1(\mathscr{K})$. The action of $\pi_1(M)$ on T pushes forward to an action of $\pi_1(\mathscr{K})$ on T. If x is an edge or vertex of T then the stabilizer of x in $\pi_1(\mathscr{K})$ will be denoted π_x. If e is an edge of T with midpoint m, we let $\mathscr{K}_e$ denote the quotient of $\tilde{M} \times m$ under π_e. If s is a vertex of T, Σ_s will denote the closed star of s in the barycentric subdivision of T. The quotient of $M \times \Sigma_s$ under π_s will be called $\mathscr{K}_s$. Notice that the $\mathscr{K}_e$ are the components of the bicollared subcomplex $\mathscr{E} = \tilde{M} \times E/\pi_1(\mathscr{K})$ of $\mathscr{K}$. The complexes $\mathscr{K}_s$ are the closures of the components of $\mathscr{K} - \mathscr{E}$.

In terms of the complex $\mathscr{K}$ we can characterize the surfaces associated to actions as follows.

PROPOSITION 1.3.4. *A surface S in M is associated to the action of* $\pi_1(M)$ *on T if and only if there exists a map* ϕ: $M \to \mathcal{K}$, *such that*

(i) ϕ *induces the natural isomorphism between* $\pi_1(M)$ *and* $\pi_1(\mathcal{K})$ *up to conjugacy;*

(ii) ϕ *is transverse to* $\mathcal{E}$; *and*

(iii) $S = \phi^{-1}(\mathcal{E})$.

Proof. Given $\tilde{\phi}$ as in 1.3.1, we may take ϕ to be induced by the equivariant map $\tilde{\phi} \times \mathrm{id}$ from $\tilde{M} \times T$ to T. Given ϕ as in the statement of the lemma, $\tilde{\phi}$ may be constructed by lifting ϕ to a map $\tilde{M} \to \tilde{M} \times T$ and composing with the projection to T. □

1.3.5. We record here some remarks about the complex $\mathcal{K}$ that will be needed in Section 1.6. Suppose that r is a point of $\mathcal{E}$, and that $\tilde{r}$ is a lift of r to the universal cover $\tilde{M} \times T$ of $\mathcal{K}$. Then $\tilde{r} \in \tilde{M} \times e$ for a unique edge e. The lift $\tilde{r}$ determines an identification of $\pi_1(\mathcal{K}, r)$ with $\pi_1(\mathcal{K})$, which is the group of covering transformations of $\tilde{M} \times T$. Under this identification we have $\pi_1(\mathcal{K}_e, r) = \pi_e$ and $\pi_1(\mathcal{K}_s, r) = \pi_s$, where s is either vertex adjacent to e.

Let S be a surface in M. By a *trivial region* for S we mean either a closed 3-ball bounded by a component of S or a closed region of parallelism between a component of S and a submanifold of ∂M.

PROPOSITION 1.3.6. *Let S be a surface associated to the action of* $\pi_1(M)$ *on T. Suppose that S′ is a surface obtained from S either by compressing S along a disk or by deleting the intersection of S with a trivial region for S. Then S′ is also associated to the action.*

Proof. Let S be defined by a map ϕ: $M \to \mathcal{K}$ as in Proposition 1.3.4. Suppose that S′ is obtained by compressing S along a disk. Then by performing surgery on the map ϕ (cf. [Hem, proof of Lemma 6.5]), one obtains a homotopic map ϕ' which is transverse to $\mathcal{E}$ and such that $(\phi')^{-1}(\mathcal{E}) = S'$. The conclusion then follows from Proposition 1.3.4.

Next suppose that S′ is obtained by deleting the intersection of S with a trivial region R for S. There is a map ϕ': $M \to \mathcal{K}$ which agrees with ϕ outside a small neighborhood V of R and maps V into $\mathcal{K} - \mathcal{E}$. Clearly S′ is the pre-image of $\mathcal{E}$ under ϕ', and ϕ' is homotopic to ϕ. The conclusion again follows from 1.3.4. □

COROLLARY 1.3.7. *Suppose that no point of T is fixed by* $\pi_1(M)$. *Let S be a surface associated to the action of* $\pi_1(M)$ *on T. Then there is a sequence*

$S = S_0, S_1, \ldots, S_n$ *of surfaces in M, each associated to the action, such that*

(i) S_{i+1} *is obtained from* S_i *by a compression or by deleting the intersection of* S_i *with a trivial region for* S_i; *and*

(ii) S_n *is essential.*

Proof. A well-known finiteness result (cf. [Hem, *loc. cit.*]) asserts that, given any properly embedded surface S in a compact, irreducible 3-manifold, there exists a finite sequence of operations of the type described in (i) that replaces S by a (possibly empty) surface whose components are all incompressible and non-peripheral. If S is associated to the action of $\pi_1(M)$ on T, Proposition 1.3.6 asserts that all the surfaces in the sequence are associated to the action. If the action has no fixed point then by 1.3.2 the final surface is non-empty; thus by definition it is essential. □

PROPOSITION 1.3.8. *Assume that no point of T is fixed by* $\pi_1(M)$. *Then there exists an essential surface in M associated to the action. Furthermore, if C is a connected subcomplex of* ∂M *such that the image of* $\pi_1(C)$ *in* $\pi_1(M)$ *is contained in a vertex stabilizer, then the surface may be taken to be disjoint from C.*

Proof. Let $\mathscr{K}$ be defined as in 1.3.3, and let $\phi\colon M \to \mathscr{K}$ be a map inducing the natural isomorphism of fundamental groups (up to conjugacy). If C is a connected subcomplex of ∂M such that the image of $\pi_1(C)$ in $\pi_1(M)$ stabilizes a vertex s of the tree, then it follows from 1.3.5 that ϕ may be taken to map C into $\mathscr{K}_s$. After a general-position homotopy we may assume that ϕ is transverse to $\mathscr{E}$. Then by Proposition 1.3.4, $S = \phi^{-1}(\mathscr{E})$ is a surface associated to the action. Clearly $S \cap C = \varnothing$. The conclusion now follows from Corollary 1.2.6 since the property of being disjoint from C is obviously preserved under compression and under deletion of components. □

PROPOSITION 1.3.9. *Let x be an ideal point of the curve* $\tilde{X}_0$ *of Section* 1.1. *Let* α *be a primitive element in L such that* $I_{e(\alpha)}(x) \neq \infty$. *Then either* $I_{e(\beta)}(x) \neq \infty$ *for all* $\beta \in L$ *or else* α *is a strict boundary class.*

Proof. Consider the action of $\pi_1(M)$ on the tree associated to the ideal point x. By 1.2.6 $e(\alpha)$ stabilizes a vertex. Thus, by the proposition, there exists an essential surface S associated to the action which is disjoint from a simple closed curve in ∂M representing α. It follows from Proposition 1.2.7 that S is not a fiber in any fibration of M over S^1. If ∂S is non-empty then α is a strict boundary class. Otherwise, by 1.3.2, $\pi_1(\partial M)$ stabilizes a vertex of T. Thus by 1.2.6 we have $I_e(\beta) \neq \infty$ for all β in L. □

1.4. The norm

In this section we prove Proposition 1.1.2, which asserts that the degree of f_α for $\alpha \in L$ is given by a norm on the vector space $V = H_1(\partial M; \mathbf{R})$, and that the unit ball in this norm is a finite-sided polygon whose vertices are rational multiples of strict boundary classes. The proof is based on a study of the poles of the functions f_α.

Let K denote the function field $\mathbf{C}(X_0) = \mathbf{C}(\tilde{X}_0)$ of $\tilde{X}_0$. If f is an element of K and x is a point of $\tilde{X}_0$ where f has a pole, then we denote the order of the pole by $\Pi_x(f)$. If f does not have a pole at x we set $\Pi_x(f)$ equal to 0. Note that

$$\text{degree } f = \sum_{x \in \tilde{X}_0} \Pi_x(f).$$

Note also that if $\alpha \in L$ then the poles of f_α all occur at ideal points of $\tilde{X}_0$.

LEMMA 1.4.1. *For each ideal point x of $\tilde{X}_0$ there is a homomorphism ϕ_x: $L \to \mathbf{Z}$ such that*

$$\Pi_x(f_\alpha) = |\phi_x(\alpha)|$$

for all $\alpha \in L$.

Proof. As in 1.2.4, we consider the tautological representation P: $\pi_1(M) \to \mathrm{SL}_2(F)$, where $F = \mathbf{C}(R_0)$. We regard F as an extension field of K. Note that $P(\pi_1(\partial M))$ is an abelian subgroup of $\mathrm{SL}_2(F)$, and that for $1 \neq \gamma \in \pi_1(\partial M)$, trace $P(\gamma) = I_\gamma \neq 2$ since I_γ is non-constant by 1.1.1. Hence there is a degree-two extension E of the field F such that P is equivalent (via an inner automorphism of $\mathrm{GL}_2(E)$) to a representation P': $\pi_1(M) \to \mathrm{SL}_2(E)$ which restricts to a diagonal representation of $\pi_1(\partial M)$. Let w: $K^* \to \mathbf{Z}$ be the valuation defined by the ideal point x of $\tilde{X}_0$. By 1.2.2, there is a valuation v: $E^* \to \mathbf{Z}$ such that $v|K^* = d \cdot w$ for some positive integer d.

Let us fix a basis α_1, α_2 for the lattice L. For $i = 1$ or 2, set

$$P(e(\alpha_i)) = \begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{bmatrix}$$

where the λ_i are elements of E^*. For any element $\alpha = m\alpha_1 + n\alpha_2$ of L, we have

$$I_{e(\alpha)} = \text{trace } P(e(\alpha)) = \lambda_1^m \lambda_2^n + \lambda_1^{-m} \lambda_2^{-n}.$$

Hence

$$f_\alpha = I_{e(\alpha)}^2 - 4 = (\lambda_1^m \lambda_2^n - \lambda_1^{-m} \lambda_2^{-n})^2.$$

But for any element g of E, g not equal to 0, 1 or -1, we have

$$-\min(0, v(g - g^{-1})) = |v(g)|.$$

Thus

$$\begin{aligned}\Pi_x(f_\alpha) &= \Pi_x\left((\lambda_1^m\lambda_2^n - \lambda_1^{-m}\lambda_1^{-n})^2\right)\\ &= -\min\left(0, w\left((\lambda_1^m\lambda_2^n - \lambda_1^{-m}\lambda_1^{-n})^2\right)\right)\\ &= \frac{-2}{d}\min(0, v(\lambda_1^m\lambda_2^n - \lambda_1^{-m}\lambda_1^{-n}))\\ &= \frac{2}{d}|v(\lambda_1^m\lambda_2^n)|\\ &= \frac{2}{d}|mv(\lambda_1) + nv(\lambda_2)|.\end{aligned}$$

The lemma follows upon setting $\phi_x(\alpha) = (2/d)(mv(\lambda_1) + nv(\lambda_2))$. □

Now for each ideal point x of $\tilde{X}_0$ we extend the homomorphism ϕ_x given by 1.4.1 to a linear function Φ_x: $V \to \mathbf{R}$. For $\alpha \in V$ we set $\|\alpha\| = \Sigma|\Phi_x(\alpha)|$, where the sum is taken over all ideal points x of $\tilde{X}_0$. We shall show that $\|\cdot\|$ is a norm on V and has the properties asserted in Proposition 1.1.2.

LEMMA 1.4.2. *For each* $\alpha \in L$, $\|\alpha\| =$ degree f_α.

Proof. We have

$$\text{degree } f_\alpha = \sum_{x \in \tilde{X}_0} \Pi_x(f_\alpha) = \sum_{x \text{ ideal}} \phi_x(\alpha) = \|\alpha\|. \qquad \square$$

LEMMA 1.4.3. $\|\cdot\|$ *is a norm.*

Proof. Since $\|\cdot\|$ is a sum of absolute values of linear functions, it is sub-additive and homogeneous. Thus we need only show that $\|\alpha\| \neq 0$ for $\alpha \neq 0$. Since the functions Φ_x restrict to homomorphisms ϕ_x: $L \to \mathbf{Z}$, it suffices to show that if α is a non-zero element of L then $\|\alpha\| \neq 0$. Suppose that α is a non-zero element of L with $\|\alpha\| = 0$. Then by Lemma 1.4.2, f_α would have degree zero, i.e. would be constant. Thus $I_{e(\alpha)}$ would be constant, contradicting 1.1.1. □

LEMMA 1.4.4. *The unit ball for* $\|\cdot\|$ *is a finite-sided polygon whose vertices are rational multiples of strict boundary classes in* L.

Proof. Let J denote the set of all ideal points $x \in \tilde{X}_0$ for which ϕ_x is not identically zero. Then for $\alpha \in V$ we have $\|\alpha\| = \Sigma_{x \in J}|\Phi_x(\alpha)|$. Hence each vertex of the ball is contained in the zero set of one of the linear functions

Φ_x, $x \in J$. But for $x \in J$, Φ_x vanishes exactly on a 1-dimensional subspace of V which is spanned by an element of L. Thus if ω is a vertex of the unit ball then $\omega = r\alpha$, where α is a primitive element of L such that $\Phi_x(\alpha) = 0$ for some $x \in J$. For such an element α we have $\Pi_x(f_\alpha) = 0$, which implies that $I_\alpha(x) \neq \infty$. But by the definition of J there exists $\beta \in L$ with $\Pi_x(f_\beta) = |\phi_x(\beta)| \neq 0$, and hence $I_\beta(x) = \infty$. By 1.3.9 this implies that α is a strict boundary class. Since $||\alpha||$ is an integer, r must be rational. □

Proposition 1.1.2 follows immediately from Lemmas 1.4.2, 1.4.3 and 1.4.4. □

1.5. Zeros at ordinary points

This section and the next are devoted to the proof of Proposition 1.1.3. Recall that we are given a primitive element α of L which is not a strict boundary class. In addition we are given a point x of $\tilde{X}_0$, and a non-zero element δ of L. We must show that if $\pi_1(M(\alpha))$ is cyclic then $Z_x(f_\alpha) \leq Z_x(f_\delta)$. In this section we will treat the case where x is an ordinary point, and in Section 6 we will treat the case where x is an ideal point.

1.5.1. We denote by X_0^ν the set of all ordinary points of $\tilde{X}_0$, and we let ν denote the restriction to X_0^ν of the birational equivalence $\tilde{X}_0 \to X_0$. Note that ν is a regular map, i.e. is well-defined at every point of X_0^ν. Thus we have a diagram:

$$\begin{array}{ccc} & & R_0 \\ & & \downarrow t \\ X_0^\nu & \xrightarrow{\nu} & X_0 \end{array} .$$

We shall prove:

PROPOSITION 1.5.2. *Let x be a point of X_0^ν, and assume that $Z_x(f_\alpha) > Z_x(f_\delta)$. Then there exists a representation $\rho \in R_0$ such that*

(i) $t(\rho) = \nu(x)$;

(ii) *the image in* $\mathrm{PSL}_2(\mathbf{C})$ *of the group* $\rho(\pi_1(M)) \subset \mathrm{SL}_2(\mathbf{C})$ *is non-cyclic; and*

(iii) $\rho(e(\alpha)) = \pm 1$.

Note that if there exists a ρ satisfying (ii) and (iii) then it induces a homomorphism of $\pi_1(M(\alpha)) = |\pi_1(M)\colon \alpha = 1|$ onto a non-cyclic subgroup of $\mathrm{PSL}_2(\mathbf{C})$, and thus $\pi_1(M(\alpha))$ cannot be cyclic. Thus Proposition 1.5.2 implies Proposition 1.1.3 for the case of an ordinary point.

The proof of Proposition 1.5.2 will be based on a study of the normalization of the variety R_0.

1.5.3. Recall, for example from [Shf, Chapter II, Section 5], that a complex affine variety is said to be *normal* if its coordinate ring is integrally closed in its function field. If V is any complex affine variety, there is a complex affine variety V^ν such that $\mathbf{C}[V^\nu]$ is isomorphic to the integral closure of $\mathbf{C}[V]$ in $\mathbf{C}(V)$. Thus V^ν is normal. Furthermore, there is a natural regular birational map $\nu\colon V^\nu \to V$ realizing the inclusion of $\mathbf{C}[V]$ in its integral closure; and ν is an isomorphism if and only if V is normal. The map ν is finite-to-one [Shf, p. 116, Theorem 6]; hence for any subvariety Z of V, the irreducible components of $\nu^{-1}(Z)$ each have the same dimension as Z.

These constructions are functorial in the sense that if $\phi\colon V \to W$ is a dominating regular map of affine varieties then there is a unique regular map $\phi^\nu\colon V^\nu \to W^\nu$ making the diagram

$$\begin{array}{ccc} V^\nu & \xrightarrow{\ \nu\ } & V \\ \phi^\nu \downarrow & & \downarrow \phi \\ W^\nu & \xrightarrow[\ \nu\ ]{} & W \end{array}$$

commute.

The singular set of a normal variety has codimension at least 2 [Shf, p. 111, Theorem 3]. In particular, a normal curve is smooth. Hence if V is an affine curve then V^ν is the curve obtained from V by resolving its singularities; this justifies the notation of Proposition 1.5.2.

Using the normalization of R_0 we can now complete the diagram of 1.5.1 to the commutative square below.

$$\begin{array}{ccc} R_0^\nu & \xrightarrow{\ \nu\ } & R_0 \\ t^\nu \downarrow & & \downarrow t \\ X_0^\nu & \xrightarrow[\ \nu\ ]{} & X_0 \end{array}$$

In order to prove Proposition 1.5.2 it clearly suffices to show that for every $x \in X_0^\nu$ satisfying $Z_x(f_\alpha) > Z_x(f_\delta)$, there is a point $\tilde\rho \in R_0^\nu$ such that $t^\nu(\tilde\rho) = x$ and such that $\rho = \nu(\tilde\rho)$ satisfies conditions (ii) and (iii) of 1.5.2. This assertion is implied by the next three results.

PROPOSITION 1.5.4. *Let α and δ be non-zero elements of L. Suppose that x is a point of X_0^ν such that $Z_x(f_\alpha) > Z_x(f_\delta)$. Then for every $\tilde\rho \in R_0^\nu$ with $t^\nu(\tilde\rho) = x$, the representation $\rho = \nu(\tilde\rho)$ satisfies $\rho(e(\alpha)) = \pm 1$.*

PROPOSITION 1.5.5. *For every point x of X_0^ν there is a dense subset U of $(t^\nu)^{-1}(x)$ such that for every representation $\rho \in \nu(U)$, the image of $\rho(\pi_1(M))$ in $\mathrm{PSL}_2(\mathbf{C})$ is noncyclic.*

PROPOSITION 1.5.6. *The map $t^\nu\colon R_0^\nu \to X_0^\nu$ is surjective.*

The remainder of this section is devoted to the proofs of Propositions 1.5.4, 1.5.5, and 1.5.6. The proof of Proposition 1.5.4 is based on the following algebraic lemma, which will also be used in Section 1.6.

LEMMA 1.5.7. *Let v be a discrete valuation of a field F. Let A and B be commuting elements of $\mathrm{SL}_2^-(\mathcal{O}_v)$ such that*

$$v\big((\text{trace } A)^2 - 4\big) > v\big((\text{trace } B)^2 - 4\big).$$

Then $A \equiv \pm 1 \pmod{\mathcal{M}_v}$.

Proof. Since A and B commute, there is an extension field E of degree at most 2 over F such that A and B have a common eigenvector ξ in E^2. By 1.2.3 we can extend v to a discrete valuation u of E. After multiplying ξ by an appropriate element of E^* we may assume that ξ is a primitive element of $\mathcal{O}_u^2$. Hence A and B are conjugate in $\mathrm{GL}_2(\mathcal{O}_u)$ to matrices

$$\begin{bmatrix} a & x \\ 0 & a^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b & y \\ 0 & b^{-1} \end{bmatrix}.$$

1.5.8. Since A and B commute we have $ay + b^{-1}x = bx + a^{-1}y$. Hence

$$(b - b^{-1})x = (a - a^{-1})y.$$

On the other hand, $(\text{trace } A)^2 - 4 = (a - a^{-1})^2$ and $(\text{trace } B)^2 - 4 = (b - b^{-1})^2$. Thus the hypothesis of the lemma implies that $u(a - a^{-1}) > u(b - b^{-1})$. With 5.8 this gives

$$u(x) \geq u(x) - u(y) = u(a - a^{-1}) - u(b - b^{-1}) > 0.$$

The inequality $u(x) > 0$ means that x is congruent to 0 modulo $\mathcal{M}_u$. We also have $u(a - a^{-1}) > 0$, which implies that $a \equiv \pm 1 \pmod{\mathcal{M}_u}$. It follows that A is congruent to ± 1 modulo $\mathcal{M}_u$ and therefore modulo $\mathcal{M}_v$. □

Proof of Proposition 1.5.4. Consider a component Q of the algebraic set $(t^\nu)^{-1}(x) \subset R_0^\nu$ which passes through the point $\tilde{\rho}$ of R_0^ν. Since t^ν is non-constant, Q has codimension 1 in R_0^ν. Since R_0 is normal, its singularities have codimension at least 2. In particular, a generic point of Q is a smooth point of R_0^ν. Thus, by [Shf, p. 128], Q determines a discrete rank 1 valuation v of $F = \mathbf{C}(R_0) = \mathbf{C}(R_0^\nu)$. A non-zero function in F is contained in $\mathcal{O}_v$ if and only if its divisor of poles does not contain Q; it is contained in $\mathscr{M}_v$ if and only if its divisor of zeros does contain Q. Let $K = \mathbf{C}(X_0) = \mathbf{C}(X_0^\nu)$ and let w denote the valuation of K determined by x. Since $t(Q) = x$, it is clear that $\mathscr{O}_v \cap K = \mathscr{O}_w$. Thus there is an integer d such that $v|_{K^*} = d \cdot w$.

1.5.9. Note that $\mathbf{C}[R_0] \subset \mathbf{C}[R_0^\nu] \subset \mathscr{O}_v$. Furthermore, for any $h \in \mathbf{C}[R_0]$, regarding h as a function on R_0, we have

$$h \in \mathscr{M}_v \Leftrightarrow h \circ \nu = 0 \text{ on } Q \Leftrightarrow h = 0 \text{ on } \nu(Q).$$

As in 1.2.3, we consider the tautological representation $P\colon \pi_1(M) \to \mathrm{SL}_2(F)$. Recall that

$$f_\alpha = I_{e(\alpha)}^2 - 4 = (\text{trace } P(e(\alpha)))^2 - 4;$$

similarly $f_\delta = (\text{trace } P(e(\delta)))^2 - 4$. Since x is an ordinary point we have

$$Z_x(f_\alpha) = w(f_\alpha) = \frac{1}{d} \cdot v\big((\text{trace } P(e(\alpha)))^2 - 4\big).$$

Likewise,

$$Z_x(f_\delta) = \frac{1}{d} \cdot v\big((\text{trace } P(e(\delta)))^2 - 4\big).$$

Thus the hypothesis of Proposition 1.5.4 implies

$$v\big((\text{trace } P(e(\alpha)))^2 - 4\big) > v\big((\text{trace } P(e(\delta)))^2 - 4\big).$$

By Lemma 1.5.7 we have $P(e(\alpha)) = \pm 1 \pmod{\mathscr{M}_v}$.

By the definition of the tautological representation P, the entries of the matrix $\rho(e(\alpha))$ are obtained by evaluating the entries of $P(e(\alpha))$ at the point $\rho \in R_0$. The entries of $P(e(\alpha))$ are elements of $\mathbf{C}[R_0] \subset \mathscr{O}_v$. Since $P(e(\alpha)) \equiv \pm 1 \pmod{\mathscr{M}_v}$, and since $\rho \in \nu(Q)$, observation 1.5.9 implies that $\rho(e(\alpha)) = \pm 1$. □

The proof of Proposition 1.5.5 is based on:

Lemma 1.5.10. *Let x be a point of X_0. Let Z denote the set of all representations $\rho \in t^{-1}(x)$ such that the image of $\rho(\pi_1(M))$ in $\mathrm{PSL}_2(\mathbf{C})$ is cyclic. Then Z is contained in a countable union of algebraic subsets of $t^{-1}(x)$, each of dimension at most 2.*

Proof. We consider the countable set $\mathscr{N} = \{\ker \rho\colon \rho \in Z\}$ of normal subgroups of $\pi_1(M)$. For each $N \in \mathscr{N}$, set $Y_N = \{\rho \in t^{-1}(x) \mid \rho(N) = \{1\}\}$. Clearly each Y_N is a closed algebraic subset of $t^{-1}(x)$, and $Z \subset \bigcup_{N \in \mathscr{N}} Y_N$. We shall prove the lemma by showing that each Y_N has dimension at most 2.

Let $N \in \mathscr{N}$ be given. Then $\pi_1(M)/N$ has a central subgroup of order at most 2 with a cyclic quotient D. Let γ denote an element of $\pi_1(M)$ which projects to a generator of D. Then a representation $\rho \in Y_N$ is uniquely determined by the matrix $\rho(\gamma)$ and by the image of the central subgroup. The image of the central subgroup is either $\{1\}$ or $\{\pm 1\}$. Furthermore, for any $\rho \in Y_N$, the trace of $\rho(\gamma)$ is $\tau_N = I_\gamma(x)$. Thus there is a finite-to-one regular map from Y_N to the set of matrices of trace τ_N in $SL_2(\mathbf{C})$. Since the latter set has dimension at most 2, so does Y_N. □

Proof of Proposition 1.5.5. First note that R_0 contains the discrete faithful representation ρ_0. Since int M has finite volume, ρ_0 is irreducible. Hence by [C-S1, Corollary 1.5.3] we have $\dim R_0 - \dim X_0 = 3$; i.e. $\dim R_0 = 4$. Since R_0^ν is birationally equivalent to R_0, it also has dimension 4. It follows that for any point $x \in X_0^\nu$, the components of $(t^\nu)^{-1}(x)$ are each of dimension at least 3. (Actually, since R_0^ν is irreducible and $t^\nu\colon R_0^\nu \to X_0^\nu$ is non-constant, these components are all of dimension exactly 3.)

By Lemma 1.5.10, there is a countable collection of algebraic subsets $A_1, A_2, \ldots$ of $t^{-1}(\nu(x))$, each of dimension at most 2, such that $\bigcup_{i \geq 1} A_i$ contains every representation ρ for which the image of $\rho(\pi_1(M))$ in $PSL_2(\mathbf{C})$ is cyclic. We set $U = (t^\nu)^{-1}(x) - \bigcup_{i \geq 1} \nu^{-1}(A_i)$. By 1.5.10, each of the sets $\nu^{-1}(A_i)$ has dimension at most 2. Hence U is dense in $(t^\nu)^{-1}(x)$. □

The proof of 1.5.6 is a refinement of the proof of Proposition 1.4.4 of [C-S1]. It is based on the following lemma, which is similar to a result deduced in [C-S1] from the "Burnside Lemma." Here we offer a more geometric proof which is extracted from [M-S1].

LEMMA 1.5.11. *Let F be a field and let $v\colon F^* \to \mathbf{Z}$ be a discrete valuation. Suppose that Γ is a finitely generated subgroup of $SL_2(F)$ such that for each $g \in \Gamma$ trace $g \in \mathcal{O}$. Then Γ is conjugate in $GL_2(F)$ to a subgroup of $SL_2(\mathcal{O})$.*

Proof. (cf. [M-S1, II.3.17]). By considering the rational canonical form, one sees that each element of Γ is conjugate in $GL_2(F)$ to an element of $SL_2(\mathcal{O})$. Thus, by 2.1, under the action of $SL_2(F)$ on the tree T determined by the valuation v, each element of Γ fixes a vertex. According to [Se, Cor. 3 to Prop. 26], if a finitely generated group acts on a tree so that each element fixes a vertex, then there is a vertex which is fixed by the entire group. Thus Γ stabilizes a vertex of the tree T, and hence by 1.2.1 is conjugate in $GL_2(F)$ to a subgroup of $SL_2(\mathcal{O})$. □

We will also need an elementary fact from algebraic geometry. A valuation v of the function field $\mathbf{C}(V)$ of an affine variety V will be said to be *supported in* V if $\mathbf{C}[V] \subset \mathcal{O}_v$.

1.5.12. Recall that for any valuation v of a field F, $\mathcal{O}_v$ is integrally closed in F. Hence a valuation of $\mathbf{C}(V) = \mathbf{C}(V^\nu)$ is supported in V if and only if it is supported in V^ν.

LEMMA 1.5.13. *Suppose that* $\phi\colon V \to C$ *is a non-constant regular map from a complex affine variety to a complex affine curve. Regard* $\mathbf{C}(V)$ *as an extension of* $\mathbf{C}(C)$. *Let* x *be a smooth point of* C *and let* w *be the valuation determined by* x. *Assume that* w *extends* (*in the sense of* Section 2) *to a discrete valuation of* $\mathbf{C}(V)$ *supported in* V. *Then* x *is in the image of* ϕ.

Proof. Note that $\mathcal{I} = \mathcal{M}_w \cap \mathbf{C}[C]$ is the ideal of functions in $\mathbf{C}[C]$ that vanish at x. Hence the algebraic subset $\phi^{-1}(x)$ of V is defined by the ideal $\mathcal{I} \cdot \mathbf{C}[V]$. Hence in order to prove that x is in the image of ϕ, it is enough to show that $\mathcal{I} \cdot \mathbf{C}[V]$ is a proper ideal in $\mathbf{C}[V]$.

By hypothesis, w extends to a valuation v of $\mathbf{C}(V)$ which is supported in V. We have $\mathbf{C}[V] \subset \mathcal{O}_v$ and $\mathcal{I} \subset \mathcal{M}_w \subset \mathcal{M}_v$. Hence $\mathcal{I} \cdot \mathbf{C}[V]$ is contained in $\mathcal{M}_v$ and is therefore proper. □

Proof of Proposition 1.5.6. Let x be a point of X_0^ν and let w denote the valuation of $\mathbf{C}(X_0) = \mathbf{C}(X_0^\nu)$ determined by x. By Theorem 1.2.3, w extends to some valuation v of $\mathbf{C}(R_0)$. As in 1.2.4, we consider the tautological representation $P\colon \pi_1(M) \to \mathrm{SL}_2(F)$. For each $\gamma \in \pi_1(M)$ we have

$$\operatorname{trace} P(\gamma) = I_\gamma \in \mathbf{C}[X_0] \subset \mathcal{O}_w \subset \mathcal{O}_v.$$

Hence by Lemma 1.5.11, there is an element A of $\mathrm{GL}_2(F)$ such that $A \cdot P(\pi_1(M)) \cdot A^{-1} \subset \mathrm{SL}_2(\mathcal{O}_\nu)$. Using A we define a rational map $\psi\colon R_0 \to R_0$ as follows. If α is an element of $\mathrm{GL}_2(\mathbf{C})$ then let i_α denote the automorphism of $\mathrm{SL}_2(\mathbf{C})$ given by $i_\alpha(\xi) = \alpha\xi\alpha^{-1}$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then for $\rho \in R_0$ let $A(\rho)$ denote the matrix $\begin{bmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{bmatrix} \in \mathrm{GL}_2(\mathbf{C})$. We then set $\psi(\rho) = i_{A(\rho)} \circ \rho$.

Since $\nu\colon R_0^\nu \to R_0$ is birational, ψ determines a rational map $\psi^\nu\colon R_0^\nu \to R_0^\nu$. Moreover, we have the following commutative diagrams of rational maps:

$$\begin{array}{ccc} R_0^\nu & \xrightarrow{\ \psi^\nu\ } & R_0^\nu \\ & {\scriptstyle t^\nu}\searrow \quad \swarrow{\scriptstyle t^\nu} & \\ & X_0^\nu & \end{array}$$

Let L denote the subvariety of R_0^ν which is the closure of the image under ψ^ν of the set of regular points of ψ^ν. Thus ψ^ν co-restricts to a dominating rational map $\Psi\colon R_0^\nu \to L$. We will show that $x \in t^\nu(L)$; this will complete the proof that t^ν is surjective.

Since Ψ is dominating, it induces a homomorphism $\mathbf{C}(L) \to \mathbf{C}(R_0)$ which may be used to pull the valuation v back to a valuation u on $\mathbf{C}(L)$. The commutative diagram

$$\begin{array}{ccc} R_0^{\nu} & \xrightarrow{\ \Psi\ } & L \\ & {}_{t^{\nu}}\searrow \quad \swarrow_{t^{\nu}|L} & \\ & X_0^{\nu} & \end{array}$$

gives rise to the commutative diagram

$$\begin{array}{ccc} F & \xleftarrow{\ \Psi^*\ } & \mathbf{C}(L) \\ & {}^{t^*}\nwarrow \quad \nearrow^{(t^{\nu}|L)^*} & \\ & K & \end{array}$$

of function fields. This shows that u is an extension of w. Thus by Lemma 1.5.13 we need only show that u is supported in L.

Note that the coordinate ring $\mathbf{C}[R_0]$ is generated by the entries of the matrices $P(\gamma)$ for $\gamma \in \pi_1(M)$. Since $i_A(P(\pi_1(M))) \subset \mathrm{SL}_2(\mathcal{O}_v)$, it follows that $\psi^*(\mathbf{C}[R_0]) \subset \mathcal{O}_v$. Since a discrete valuation ring is integrally closed we have $\psi^*(\mathbf{C}[R_0^{\nu}]) \subset \mathcal{O}_v$. In particular $\Psi^*(\mathbf{C}[L]) \subset \mathcal{O}_v$, or equivalently $\mathbf{C}[L] \subset \mathcal{O}_u$. □

1.6. Zeroes at ideal points

In this section we complete the proof of Proposition 1.1.3. In Section 1.5 we gave the proof in the case where the point $x \in \tilde{X}_0$ in the hypothesis of 1.1.3 is an ordinary point. In the case where x is an ideal point of $\tilde{X}_0$, the conclusion of 1.1.3 follows immediately from the following result:

Proposition 1.6.1. *Let x be an ideal point of $\tilde{X}_0$. Let α and δ be non-zero elements of L. Suppose that α is primitive and is not a strict boundary class, and that*

$$Z_x(f_\alpha) > Z_x(f_\delta).$$

Then there is a closed surface in M which is incompressible in $M(\alpha)$. In particular $\pi_1(M(\alpha))$ is not cyclic.

The proof of Proposition 1.6.1 will occupy this entire section. Throughout the section, we consider a fixed ideal point x and fixed elements α and δ of L which satisfy the hypotheses of 1.6.1.

1.6.2. In particular we have $Z_x(f_\alpha) > 0$, and hence $I_{e(\alpha)}(x)$ is finite. Since α is not a strict boundary class, it follows from Proposition 1.3.9 that $I_{e(\beta)}(x) \neq \infty$ for all $\beta \in L$.

Next we consider an action of $\pi_1(M)$ on a tree T associated to the ideal point x. Recall that by 1.2.6, no point of T is fixed by $\pi_1(M)$. Thus, it follows from 1.3.2 that any surface associated to the action is non-empty. Since for every $g \in \pi_1(\partial M)$ we have $I_g(x) \neq \infty$, each element of $\pi_1(\partial M)$ fixes some point of T. Hence by [Se, Cor. 3 to Prop. 26], there is one vertex of T which is fixed by $\pi_1(\partial M)$. It therefore follows from Proposition 1.3.8 that there is a closed surface associated to the action of $\pi_1(M)$ on T.

The method of proof of 1.6.1 is to show that in the set of all closed surfaces in M which are associated to the action of $\pi_1(M)$ on T, there is one which is incompressible in $M(\alpha)$.

If S is a closed surface then we define

$$\chi_-(S) = \sum -\chi(S_i)$$

where the sum is taken over all components S_i of S which are not 2-spheres. We denote by $\#(S)$ the number of components of S. The *complexity* of S is the pair $c(S) = (\chi_-(S), \#(S))$, which will be regarded as an element of the set $\mathbf{N} \times \mathbf{N}$ of pairs of non-negative integers with the lexicographic ordering.

1.6.3. Among all closed surfaces associated to the action of $\pi_1(M)$ on T we fix one, S, of minimal complexity. The rest of this section will be devoted to proving that S is incompressible in $M(\alpha)$. This will complete the proof of Proposition 1.6.1.

The first step is easy:

LEMMA 1.6.4. *The surface S is essential in M. In particular every component of S has genus greater than* 1.

Proof. We have observed that a surface associated to the action of $\pi_1(M)$ on T is non-empty. If some component of S is a 2-sphere or a boundary-parallel surface, then it follows from 1.3.6 that there is a surface which is associated to the action and which is obtained from S by deleting one or more components. This contradicts the minimality of $c(S)$.

Now suppose that there is a compressing disk D for S. By 1.3.6, the surface S′ obtained by compressing S along D is associated to the action. If the component S_0 of S containing ∂D has genus greater than 1, then $\chi_-(S') < \chi_-(S)$, and again the minimality of $c(S)$ is contradicted. If S_0 is a torus then $c(S') = c(S)$, and some component of S′ is a 2-sphere. By 1.3.6 we can then produce an associated surface by deleting one or more components of S. Again this contradicts the minimality of $c(S)$.

This proves that S is essential. Since M contains no essential tori, it follows that every component of S has genus greater than 1. □

The main step in the proof that S is incompressible in $M(\alpha)$ uses the following strong property of the action of $\pi_1(M)$ on T.

LEMMA 1.6.5. *Let s be any vertex of T which is fixed by $\pi_1(\partial M) \subset \pi_1(M)$. Then the element $e(\alpha)$ of $\pi_1(\partial M)$ acts trivially on the link of s in T.*

(Note that if the subgroup $\pi_1(\partial M)$ and the element α are conjugated by an element of $\pi_1(M)$, the truth of the above statement is unaffected. Thus the statement is consistent with the conventions of 1.0.2.)

Proof. Let w denote the valuation of $K = \mathbf{C}(X_0)$ defined by the ideal point x of $\tilde{X}_0$. Since by 1.6.2, $I_\alpha(x)$ and $I_\delta(x)$ are finite, we have $w(f_\alpha) = Z_x(f_\alpha)$ and $w(f_\delta) = Z_x(f_\delta)$. According to 1.2.5, the tree T is defined in terms of some extension v of w to $F = \mathbf{C}(R_0)$. There is a positive integer d such that $v|K^* = d \cdot w$.

The action of $\pi_1(M)$ on T is defined by the tautological representation P: $\pi_1(M) \to SL_2(F)$. The vertices of T are equivalence classes of $\mathcal{O}_v$-lattices in F^2. Since $GL_2(F)$ acts transitively on the vertices of T, we may assume after conjugation by an element of $GL_2(F)$ that the given vertex s is the class of the standard lattice $\mathcal{O}_v^2$. By 1.2.1 the stabilizer of this vertex in $SL_2(F)$ is $SL_2(\mathcal{O}_v)$. Thus the hypothesis of the lemma implies that $P(\pi_1(\partial M)) \subset SL_2(\mathcal{O}_v)$.

The hypothesis of Proposition 1.6.1 gives

$$\begin{aligned} v\Big((\text{trace } P(e(\alpha)))^2 - 4\Big) &= d \cdot w\big(I_\alpha^2 - 4\big) \\ &= d \cdot w(f_\alpha) \\ &= d \cdot Z_x(f_\alpha) \\ &> d \cdot Z_x(f_\delta) = v\Big((\text{trace } P(e(\delta)))^2 - 4\Big). \end{aligned}$$

Since $P(e(\alpha))$ and $P(e(\delta))$ commute, Lemma 1.5.7 implies $P(e(\alpha)) \equiv \pm 1 \pmod{\mathcal{M}_v}$. By 1.2.2 this is equivalent to saying that $e(\alpha)$ acts trivially on the link of s. □

1.6.6. *Proof of Proposition* 1.6.1. As observed earlier it suffices to show that S is incompressible in $M(\alpha)$. The proof will proceed by contradiction. Assume that S is compressible in $M(\alpha)$. Since S has no sphere components by Lemma 1.6.4, there must exist a compressing disk D for S in $M(\alpha)$.

Recall that $M(\alpha)$ is obtained from M by attaching a solid torus J along ∂M so that the boundary of a meridian disk in J represents the homology class α in ∂M. We may suppose D to be chosen so that each component of $D \cap J$ is a meridian disk. Thus $P = D \cap M$ is a planar surface whose boundary components are $\omega = \partial D$ and simple closed curves $\alpha_1, \dots, \alpha_k$ representing the homology class α in ∂M.

We shall denote by N the closure of the component of $M(\alpha) - S$ that contains ∂M. Let V be a regular neighborhood of D in $M(\alpha)$ which meets J in a regular neighborhood of $D \cap J$ in J. Consider the surface Q in $M(\alpha)$ obtained by compressing S along D, using the neighborhood V: Explicitly we have $Q = (S \cup \partial V) - \operatorname{int}(S \cap \partial V)$.

1.6.7. Since by Lemma 1.6.4, every component of S has negative Euler characteristic, and since Q is obtained from S by a compression, we have $\chi_-(Q) < \chi_-(S)$. Set $Q^- = Q \cap M$. Thus Q^- is obtained from S by replacing an annular neighborhood of ω by two parallel copies of P.

Claim 1.6.8. *The surface $Q^- \subset M$ is associated to the action of $\pi_1(M)$ on T.*

Proof of claim. We consider the complex $\mathscr{K}$ determined by the action as in 1.3.3. Recall that there is a natural isomorphism between the groups $\pi_1(M)$ and $\pi_1(\mathscr{K})$ of covering transformations. By Proposition 1.3.4 there is a map ϕ: $M \to \mathscr{K}$ which is transverse to $\mathscr{E}$ with $\phi^{-1}(\mathscr{E}) = S$, and which induces the natural isomorphism up to conjugation.

We now fix base points $q \in \omega \subset S \subset N \subset M$ and $r = \phi(q) \in \mathscr{K}$. Any choice of lifts $\tilde{q}$ and $\tilde{r}$ of q and r determines identifications of $\pi_1(M)$ with $\pi_1(M, q)$ and of $\pi_1(\mathscr{K})$ with $\pi_1(\mathscr{K}, r)$. We may choose $\tilde{q}$ and $\tilde{r}$ so that in terms of these identifications, ϕ_*: $\pi_1(M, q) \to \pi_1(\mathscr{K}, r)$ is the natural isomorphism between $\pi_1(M)$ and $\pi_1(\mathscr{K})$. By 1.3.5, there is an edge e of T with $\phi(S) \subset \mathscr{K}_e$, and $\pi_1(\mathscr{K}_e, r) = \pi_e$. There is a vertex s incident to e such that $\phi(N) \subset \mathscr{K}_s$; again by 1.3.5 we have $\pi_1(\mathscr{K}_s, r) = \pi_s$.

Also $\partial M \subset N$. Hence up to conjugacy in $\pi_1(N, q)$, there is a natural identification of $\pi_1(\partial M)$ with a subgroup of $\pi_1(N, q)$. Thus $\phi_*(\pi_1(\partial M))$ is a subgroup of $\pi_1(\mathscr{K}_s, r)$ defined up to conjugacy in $\pi_1(\mathscr{K}_s, r)$. Since $\pi_1(\mathscr{K}_s, r)$ fixes the vertex s of T, it follows from Lemma 1.6.5 that every conjugate of $\phi_*(e(\alpha))$ in $\pi_1(\mathscr{K}_s, r)$ acts trivially on the link of s.

All but one of the boundary components of the planar surface P are curves in $\partial M \subset N$ which represent the conjugacy class of $e(\alpha)$ in $\pi_1(N, q)$. Hence if θ: $P \to N$ denotes the restriction of ϕ, then $\theta_*(\pi_1(P, q))$ is contained in the normal closure of $\phi_*(e(\alpha))$ in $\pi_1(\mathscr{K}_s)$. Therefore $\theta_*(\pi_1(P, q))$ acts trivially on the link of s. In particular, $\theta_*(\pi_1(P, q)) \subset \pi_e = \pi_1(\mathscr{K}_e, r)$. Since $\mathscr{K}_s$ is aspherical, this implies that there exists a homotopy Θ: $P \times [0, 1] \to \mathscr{K}_s$, constant on ω, such that $\Theta_0 = \theta$ and $\Theta_1(P) \subset \mathscr{K}_e$. Such a homotopy Θ can clearly be extended to a homotopy Φ: $M \times [0, 1] \to \mathscr{K}$, constant outside a small neighborhood of V in N, such that $\Phi_0 = \phi$ and $\Phi_1^{-1}(\mathscr{K}_e) = S \cup \partial V$. Furthermore, Φ_1 can be perturbed to a map ψ: $M \to \mathscr{K}$, transverse to $\mathscr{E}$, such that $\psi^{-1}(\mathscr{E}) = Q$. Since ψ is homotopic to ϕ it then follows from Proposition 1.3.4 that Q is associated to the action. □

We now apply Corollary 1.3.7 to the surface Q^-. This gives a sequence of surfaces $Q^- = Q_0^-, Q_1^-, \ldots, Q_n^-$, each associated to the action of $\pi_1(M)$ on T. Each Q_{i+1}^- is obtained from Q_i^- by compressing along a disk or deleting certain components. Furthermore, Q_n^- is essential in M. For each i we clearly have $\partial Q_i^- \subset \partial Q^-$.

In particular, any boundary component of Q_n^- must represent the homology class α in L. Note also that, since Q_n^- is associated to the action of $\pi_1(M)$ on T, it follows from Proposition 1.2.7 that Q_n^- cannot be a fiber in a fibration of M over S^1. Since α is not a strict boundary class it follows that Q_n^- is closed.

Next we define a closed surface Q_i in $M(\alpha)$ by adding meridian disks of J to the boundary components of Q_i^-. We have $Q_0 = Q$ and $Q_n = Q_n^-$.

Note also that Q_{i+1} is obtained from Q_i either by surgery on a (possibly trivial) simple closed curve or by deleting certain components. Hence

$$\chi_-(Q_n) \leq \chi_-(Q_{n-1}) \leq \cdots \chi_-(Q_0) = \chi_-(Q)$$

which with 1.6.7 gives $\chi_-(Q_n) < \chi_-(S)$, and hence $c(Q_n) < c(S)$. But $Q_n = Q_n^-$ is a closed surface associated to the action. This contradicts the minimality of $c(S)$. Hence the assumption in 1.6.6 that S is compressible in $M(\alpha)$ is false, and the proof of Proposition 1.6.1 is complete. □

Chapter 2

2.0. Introduction

Let M be a compact, connected, irreducible 3-manifold such that ∂M is a torus. In light of Theorem 1.0.1, we shall be concerned in this chapter with the case of the Cyclic Surgery Theorem in which either M contains an essential torus or one of r, s is a strict boundary slope. To state the theorem that we shall use to deal with the first possibility, we need the following standard definition. Let V be a solid torus. A *cable space* is the complement of an open tubular neighborhood of a (p, q)-cable of the core of V, where p, q are coprime integers with $q \geq 2$.

THEOREM 2.0.1. *If M contains an essential torus S which compresses in $M(r)$ and $M(s)$, then either $\Delta(r, s) \leq 1$ or S and ∂M cobound a cable space in M.*

A closed, connected 3-manifold is a *Haken manifold* if it is irreducible and contains an incompressible surface. A *lens space* is a closed 3-manifold of Heegaard genus 1 whose fundamental group is finite (cyclic). (Thus S^3 and $S^1 \times S^2$ are excluded.)

The next two theorems, together with Addendum 2.0.4, establish the Cyclic Surgery Theorem in the case where one of the given slopes is a strict boundary slope.

THEOREM 2.0.2. *Suppose that* $\dim H_1(M; \mathbf{Q}) > 1$. *If* $M(r)$ *and* $M(s)$ *have cyclic first homology groups and are not Haken manifolds then* $\Delta(r, s) \leq 1$.

THEOREM 2.0.3. *Suppose that* $\dim H_1(M; \mathbf{Q}) = 1$. *If r is a boundary slope, then either*

(i) $M(r)$ *is a Haken manifold*; *or*

(ii) $M(r)$ *is a connected sum of two lens spaces*; *or*

(iii) M *contains a closed incompressible surface which remains incompressible in* $M(s)$ *whenever* $\Delta(r, s) > 1$; *or*

(iv) M *fibers over* S^1 *with fiber a planar surface having boundary slope* r.

We remark that possibility (iv) was overlooked in the statement of Proposition 2 of the announcement [CGLS]. Note that in that case $M(r)$ is homeomorphic to $S^1 \times S^2$, which has cyclic fundamental group. However, we have the following addendum to Theorem 2.0.3.

Addendum 2.0.4. *In case* (iv) *of Theorem* 2.0.3, *if r is a strict boundary slope then conclusion* (iii) *holds also*.

The chapter is organized as follows. In Sections 2.1, 2.2 and 2.3 we take the first steps towards proving Theorem 2.0.3. We start with an essential surface F in M having boundary slope r, and consider the corresponding closed surface $\hat{F}$ in $M(r)$. We show that if F is suitably chosen, then either $\hat{F}$ is an incompressible surface of positive genus and conclusion (i) holds; or $\hat{F}$ is a separating 2-sphere which decomposes M as a connected sum of two lens spaces, giving conclusion (ii); or $\hat{F}$ is a non-separating 2-sphere and conclusion (iv) holds; or M contains a closed essential surface S, disjoint from F, with certain additional properties. The main tool used is a mild extension of a result of Jaco [J] giving conditions under which the addition of a 2-handle to a 3-manifold yields a manifold with incompressible boundary. This work is begun in Section 2.1, and completed in Section 2.2, in the case that F has positive genus, and in Section 2.3, in the case that F is planar.

To complete the proof of Theorem 2.0.3, we need to show that the closed surface S remains incompressible in $M(s)$ whenever $\Delta(r, s) > 1$. More generally, we consider the question: For which slopes r does a given closed essential surface S in M compress in $M(r)$? To study this, we assume that there exist compressing disks for S in $M(r)$ and $M(s)$, say, and carry out a graph-theoretic analysis of the intersection of the two corresponding planar surfaces in M. In the case that S is a torus, this leads to Theorem 2.0.1. (An earlier result along these lines was obtained by Litherland [L], who used the same general approach.) A very similar argument proves Theorem 2.0.2. Here, if $M(r)$ and $M(s)$ are not Haken manifolds, then, since they have positive first Betti number, they each contain a 2-sphere which does not bound a 3-ball, and we apply the same analysis to the intersection of the two planar surfaces in M corresponding to these 2-spheres. Again, an earlier result along these lines was proved in [Gr-L]. By refining the graph-theoretic analysis used to prove Theorems 2.0.1 and 2.0.2, we obtain information on the general question of when a closed essential surface in M compresses in $M(r)$. In particular, making use of the special properties of the surface S that arises in the proof of Theorem 2.0.3, we show that it satisfies conclusion (iii).

Section 2.4 contains the statements of the results that we prove on the question of when a closed essential surface compresses under Dehn surgery. We also show in that section how Theorems 2.0.1 and 2.0.2 follow from these results. In Section 2.5 we set up the graph-theoretic machinery that we will use in the proofs, and establish some basic connections between the graph theory and the topology. We also state the main graph-theoretic propositions that we require, and show how they imply the theorems stated in Section 2.4 and enable us to complete the proof of Theorem 2.0.3, as well as prove Addendum 2.0.4. The main graph-theoretic analysis is carried out in Section 2.6, in which we prove the propositions stated in Section 2.5. After Theorems 1.0.1, 2.0.1, 2.0.2 and 2.0.3, and Addendum 2.0.4, the proof of the Cyclic Surgery Theorem reduces to a consideration of manifolds which contain cable spaces. This is discussed in Section 2.7. Finally, in Section 2.8, we give a proof of Corollary 7.

2.1. Dehn surgery along a boundary slope

In this and the next two sections M will be a compact, connected, irreducible 3-manifold, such that ∂M is a torus and $\dim H_1(M; \mathbf{Q}) = 1$.

We wish to analyze $M(r)$ in the case that r is a boundary slope. To do this we consider an essential surface F in M with boundary slope r which is minimal in an appropriate sense, and study the handle decompositions of certain associated submanifolds of M and $M(r)$. We begin this analysis in the present section, and refine it in Sections 2.2 and 2.3, which treat the cases that F is non-planar and planar respectively. Specifically, we will show that either F is non-planar and conclusion (i) of Theorem 2.0.3 holds, or F is planar and either conclusion (ii) or (iv) of Theorem 2.0.3 holds, or M contains a closed incompressible surface S with certain properties. Sections 2.4, 2.5, and 2.6 are devoted largely to showing that this surface S remains incompressible in $M(s)$ whenever $\Delta(r, s) > 1$. This is conclusion (iii) of Theorem 2.0.3.

It will be convenient to introduce the following notation. If S is a surface, and $C_1, \dots, C_n$ are disjoint simple loops in S, then $\sigma(S; \bigcup_{i=1}^n C_i)$ will denote the surface resulting from surgery on S along $C_1, \dots, C_n$.

If Q is a 3-manifold, and $C_1, \dots, C_n$ are disjoint simple loops in ∂Q, then $\tau(Q; \bigcup_{i=1}^n C_i)$ will denote the 3-manifold obtained by attaching 2-handles to Q along disjoint regular neighborhoods of $C_1, \dots, C_n$.

Note that if $C_1, \dots, C_n \subset S \subset \partial Q$, then $\sigma(S; \bigcup_{i=1}^n C_i) \subset \partial \tau(Q; \bigcup_{i=1}^n C_i)$.

The following *handle addition lemma* will play a key role in our analysis of $M(r)$.

LEMMA 2.1.1. *Let Q be an irreducible 3-manifold, S a surface in ∂Q which is compressible in Q, and C a simple loop in S such that $S - C$ is incompressible in Q. Suppose that $\sigma(S; C)$ is not a 2-sphere. Then $\tau(Q; C)$ is irreducible and $\sigma(S; C)$ is incompressible in $\tau(Q; C)$.*

In the form stated, this is proved in [C-G]. See also [J], [Jo], [P], [Sch1], and [D-S].

By a *compression body* we shall mean a cobordism W (rel ∂) between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times I \cup$ 2-handles $\cup$ 3-handles and $\partial_- W$ has no 2-sphere components. It follows that W is irreducible and $\partial_- W$ is incompressible in W. If Q is an irreducible 3-manifold and F is a surface in ∂Q, then there is a maximal compression body $W \subset Q$ with $\partial_+ W = F$, which is unique up to isotopy. (See [Bo, §2].) The *inner boundary* of W is $F^- = \partial_- W \cup \partial F \times I$. Thus $\partial F^- = \partial F$, and F^- is incompressible in Q.

There is a disjoint union of disks $(D^*, \partial D^*) \subset (Q, F)$ such that, for a regular neighborhood $N(D^*)$ of D^* compatible with a collar $\partial Q \times I$ of ∂Q in Q, we have $W = F \times I \cup N(D^*) \cup$ 3-handles. Note that $F^- \cong \sigma(F; \partial D^*)$ with all 2-sphere components removed.

Our assumption that $\dim H_1(M; \mathbf{Q}) = 1$ implies that $H_2(M) = 0$. It follows that if $(G, \partial G) \subset (M, \partial M)$ is a non-separating surface then $[\partial G] \neq 0$ in $H_1(\partial M)$, and hence, by successively tubing adjacent oppositely oriented components of ∂G and then compressing, we obtain an essential non-separating surface $(G, \partial G) \subset (M, \partial M)$ such that $\partial G \neq \emptyset$ and all components of ∂G, when given the orientations induced by some orientation of G, are homologous on ∂M.

Now let r be a boundary slope on ∂M, and let $(F, \partial F) \subset (M, \partial M)$ be an essential separating surface with boundary slope r, such that each component of F has non-empty boundary, and such that the number of components of ∂F is minimal subject to these conditions. Thus F either is connected or has exactly two components, each of which is non-separating. In the latter case we shall assume (as we may without loss of generality) that F consists of two parallel copies of a non-separating surface G, which (by the argument given in the previous paragraph) necessarily has all its boundary components oriented coherently on ∂M. Note that $[F, \partial F] = 0$ in $H_2(M, \partial M)$.

We shall refer to the two cases described above as the *connected* and *disconnected* cases, respectively. They will for the most part be treated in parallel; the main divergence occurs when F is planar.

Note that the disconnected case can only occur when r is the unique slope that corresponds to an element in the kernel of the map $H_1(\partial M; \mathbf{Q}) \to H_1(M; \mathbf{Q})$.

The manifold obtained by cutting M along F has two components, X and X', say. (In the disconnected case, X' (say) is homeomorphic to $G \times I$.) The number of components of ∂F is even, say $2n$ ($n \geq 1$), and these components cut ∂M into parallel annuli $A_1, \dots, A_n, A'_1, \dots, A'_n$ such that $\partial X = F \cup \bigcup_{i=1}^n A_i$, $\partial X' = F \cup \bigcup_{i=1}^n A'_i$. Note that each of ∂X and $\partial X'$ has, in the connected case, genus $f + n$, where f is the genus of F, and in the disconnected case, genus $2g + (n-1)$, where g is the genus of G. Let C_i, C'_i be a core of A_i, A'_i respectively. Regarding the solid torus J as the union of $2n$ 2-handles with attaching regions $A_1, \dots, A_n, A'_1, \dots, A'_n$, we see that $M(r) = M \cup J \cong \tau(X; \bigcup_{i=1}^n C_i) \cup_{\hat{F}} \tau(X'; \bigcup_{i=1}^n C'_i) = \hat{X} \cup_{\hat{F}} \hat{X}'$, say, where $\hat{F} = \partial \hat{X} = \partial \hat{X}'$ is the closed surface obtained from F by capping off ∂F with meridian disks of J.

For $1 \leq i \leq n$, let $F_i = F \cup A_i \subset \partial X$; so $\partial F_i = \bigcup_{j \neq i} \partial A_j$. Let V_i be a maximal compression body for F_i in X. The inner boundary of V_i, F_i^-, is incompressible in X. Since $M = X \cup_F X'$ and F is incompressible in M, it follows that F_i^- is also incompressible in M. Since the number of boundary components of F_i^- is $2(n-1)$, our assumption on F implies that every separating component of F_i^- with non-empty boundary is a boundary-parallel annulus. Furthermore, since $[F_i^-, \partial F_i^-] = [F_i, \partial F_i] = [F, \partial F] = 0$ in $H_2(M, \partial M)$ (and $H_2(M, \partial M) \cong \mathbf{Z}$), the number of non-separating components of F_i^- is even. It follows that if F_i^- has a non-separating component then it has one with at most $(n-1)$ boundary components. Since, as noted above, any non-separating surface in M has non-empty boundary, taking two parallel copies of such a component would contradict our minimality assumption on F. We conclude that F_i^- consists of $(n-1)$ annuli, each parallel into ∂M, together with some closed surface (possibly empty) in int X. Let B be one of the annuli. Since B separates M, and int $B \cap F_i = \varnothing$, and F_i is connected, it follows that $\partial B = \partial A_j$ for some $j \neq i$. We may therefore number the annuli $B_i^{(j)}$, $j \neq i$, so that $\partial B_i^{(j)} = \partial A_j$ $(j \neq i)$.

LEMMA 2.1.2. *For $1 \leq i \leq n$, there exist disjoint disks $E_i^{(j)}$ in X, $j \neq i$, such that $\partial E_i^{(j)}$ meets C_j transversely in a single point and is disjoint from C_k if $k \neq i$ or j.*

Proof. Fix i, $1 \leq i \leq n$, and recall the annuli $B_i^{(j)}$, $j \neq i$. Each $B_i^{(j)}$ is parallel into ∂M. Let U_j be the solid torus realizing this parallelism. Then $U_j \subset X$

(otherwise $F \subset U_j$), $\partial U_j = A_j \cup B_i^{(j)}$, and the U_j are disjoint. Choose a meridian disk of U_j with boundary $\alpha_j \cup \beta_j$, where α_j, β_j are spanning arcs of the annuli A_j, $B_i^{(j)}$ respectively. Now $V_i \cong F_i \times I \cup$ 2-handles $\cup$ 3-handles; dually, $V_i \cong \partial_- V_i \times I \cup$ 0-handles $\cup$ 1-handles. An isotopy of β_j(rel ∂) in $B_i^{(j)}$ will move it off the disks that constitute the attaching regions of the 1-handles, and then a further isotopy (rel ∂) in V_i (using the product structure of $\partial_- V_i \times I$) will take it to an arc β_j' in $F_i \subset \partial X$. A corresponding isotopy and extension of the chosen meridian disk of U_j gives a disk $E_i^{(j)}$ in X with $\partial E_i^{(j)} = \alpha_j \cup \beta_j'$. These disks $E_i^{(j)}$, $j \neq i$, satisfy the conditions stated. □

Let W_i be the (possibly punctured) compression body in X with $\partial_+ W_i = \partial X$ defined by $W_i = \partial X \times I \cup N(\bigcup_{j \neq i} E_i^{(j)})$, where $N(\bigcup_{j \neq i} E_i^{(j)})$ is a suitable regular neighborhood of $\bigcup_{j \neq i} E_i^{(j)}$. Thus ∂W_i is the disjoint union of ∂X and $\partial_- W_i$, where $\partial_- W_i$ is a (closed) connected surface which, in the connected case, has genus $f + 1$, and, in the disconnected case, genus $2g$. Hence W_i is a compression body unless F is planar and disconnected, in which case $X = W_i \cup$ 3-ball is a handlebody of genus $n - 1$.

LEMMA 2.1.3. *For* $1 \leq i \leq n$, $\tau(W_i, \bigcup_{j \neq i} C_j) \cong \partial_- W_i \times I$.

Proof. This follows by canceling the 2-handle corresponding to C_j with $E_i^{(j)}$, $j \neq i$. □

For $0 \leq k \leq n$, let $X_k = \tau(X; \bigcup_{i=1}^{k} C_i)$. Thus $X_0 = X$, and $X_n = \hat{X}$. We shall apply the following lemma with $k = n - 1$ in Section 2.2, and with $k = n - 2$ in Section 2.3.

LEMMA 2.1.4. *If F is either non-planar or connected then, for* $0 \leq k \leq n - 1$, *X_k is irreducible and* $\partial X_k - \bigcup_{i=k+1}^{n} C_i$ *is incompressible in* X_k.

Proof. We prove this by induction on k, using Lemma 2.1.1. The assertion holds for $k = 0$, since X is irreducible and F is incompressible in X. So suppose $1 \leq k \leq n - 1$, and that the assertion holds for $k - 1$. Thus X_{k-1} is irreducible and $\partial X_{k-1} - \bigcup_{i=k}^{n} C_i = (\partial X_{k-1} - \bigcup_{i=k+1}^{n} C_i) - C_k$ is incompressible in X_{k-1}. Write $S_k = \partial X_{k-1} - \bigcup_{i=k+1}^{n} C_i$. We shall apply Lemma 2.1.1 with $Q = X_{k-1}$, $S = S_k$, and $C = C_k$. Since $X_k = \tau(X_{k-1}; C_k)$, this will establish the conclusion of the lemma for k, provided we show that S_k is compressible in X_{k-1}.

To do this, let $(D_k^*, \partial D_k^*) \subset (X, F_k)$ be a disjoint union of disks such that the maximal compression body V_k for F_k in X can be expressed as $F_k \times I \cup N(D_k^*)$ for a suitable regular neighborhood $N(D_k^*)$ of D_k^*. Since $\partial D_k^* \cap C_i = \varnothing$, $i \neq k$, we have $(D_k^*, \partial D_k^*) \subset (X_{k-1}, S_k)$. We claim that some component of ∂D_k^* is essential in ∂X_{k-1}, and a fortiori in S_k. For if not, then $\sigma(\partial X_{k-1}; \partial D_k^*)$ would be homeomorphic to the disjoint union of ∂X_{k-1} with some 2-spheres. However, ∂X_{k-1} is a connected surface which has, in the connected case, genus $f + n - (k - 1) \geq f + 2 \geq 2$, and, in the disconnected case, genus $2g + (n - 1) - (k - 1) \geq 2g + 1 \geq 3$ (since $g \geq 1$ here by hypothesis). On the other hand, $\sigma(\partial X; \partial D_k^*)$ contains $n - 1$ tori $T_k^{(j)}$, corresponding to the tori $A_j \cup B_k^{(j)}$, $j \neq k$, and therefore $\sigma(\partial X_{k-1}; \partial D_k^*) = \sigma(\sigma(\partial X; \partial D_k^*); \bigcup_{j=1}^{k-1} C_j)$ contains $\bigcup_{j=k+1}^{n} T_k^{(j)}$, a disjoint union of tori which is non-empty since $k \leq n - 1$. This contradiction completes the proof of the lemma. □

We note for completeness that if F is disconnected and planar then X_{n-1} is a 3-ball.

2.2. The non-planar case

As a first approximation to the conclusion of Theorem 2.0.3 in the case that F is non-planar, we have the following proposition.

PROPOSITION 2.2.1. *Suppose that F is non-planar. Then either*
(i) *$M(r)$ is irreducible and $\hat{F}$ is incompressible in $M(r)$; or*
(ii) *M contains a closed incompressible surface S which is disjoint from F.*

We shall ultimately show that the surface S remains incompressible in $M(s)$ whenever $\Delta(r, s) > 1$. Our proof of this will require some additional properties of S. These are contained in the following addendum to Proposition 2.2.1.

Addendum 2.2.2. *In conclusion* (ii) *of Proposition* 2.2.1, *we may assume that the surface S cobounds with ∂M a manifold N in M such that either*
(i) *N contains an annulus with one boundary component in S and the other having slope r in ∂M; or*
(ii) (a) *$N(r)$ is irreducible; and*
(b) *there is a compressing disk for S in $N(r)$ which misses $\hat{F}$; and*
(c) *no compressing disk as in* (b) *meets only a single component of $\partial M - \partial F$.*

Proof of Proposition 2.2.1. Recall the notation established in Section 2.1, and suppose that $\partial_- W_n$ is compressible in X. Since $\tau(W_n; \bigcup_{i=1}^{n-1} C_i) \cong \partial_- W_n \times I$ by Lemma 2.1.3, it follows that ∂X_{n-1} is compressible in X_{n-1}. By Lemma 2.1.4 (with $k = n - 1$), X_{n-1} is irreducible and $\partial X_{n-1} - C_n$ is incompressible in X_{n-1}. Hence, by Lemma 2.1.1, $\hat{X} = \tau(X_{n-1}; C_n)$ is irreducible and $\hat{F}$ is incompressible in $\hat{X}$. Similarly, if $\partial_- W_n'$ is compressible in X', then $\hat{X}'$ is irreducible and $\hat{F}$ is incompressible in $\hat{X}'$. Therefore $M(r) = \hat{X} \cup_{\hat{F}} \hat{X}'$ is irreducible and $\hat{F}$ is incompressible in $M(r)$, which is conclusion (i).

Otherwise, $\partial_- W_n$ (say) is incompressible in X, and hence in $M = X \cup_F X'$, giving conclusion (ii). □

The argument above in the case that $n = 1$ and F is disconnected is the same as that given in [J]. Also, the case in which $n = 1$, F is connected and X, X' are handlebodies is treated in [P].

Let $\hat{W}_n = \tau(W_n; \bigcup_{i=1}^{n} C_i)$. The observations contained in the following lemma will be needed in the proof of Addendum 2.2.2.

LEMMA 2.2.4. *If F is non-planar then*

(a) *$\hat{W}_n$ is irreducible;*
(b) *$\hat{F}$ is incompressible in $\hat{W}_n$; and*
(c) *$\partial_- W_n$ is compressible in $\hat{W}_n$.*

Proof. By Lemma 2.1.3, $\tau(W_n; \bigcup_{i=1}^{n-1} C_i) \cong \partial_- W_n \times I$, and hence $\hat{W}_n \cong \tau(\partial_- W_n \times I; C_n)$. Also, C_n is essential in $\partial X_{n-1} = \partial_- W_n \times \{0\}$, since each component of ∂X cut along $\bigcup_{i=1}^{n} C_i$ (which is homeomorphic to F) is non-planar. Hence $\hat{W}_n$ is a compression body with a single 2-handle whose attaching circle is essential. Conclusions (a), (b) and (c) follow. □

We remark that if F is planar and connected, then $\partial_- W_n$ is a torus and C_n is essential in $\partial_- W_n \times \{0\}$, since $C_1, \dots, C_n$ are homologically independent in ∂X. Hence $\hat{W}_n$ is a punctured solid torus, so that conclusion (c) still holds. We shall use this in Section 2.3.

Proof of Addendum 2.2.2. Recall that conclusion (ii) of Proposition 2.2.1 arises when (without loss of generality) $\partial_- W_n$ is incompressible in X. We distinguish two subcases.

(2.A) *$\partial_- W_n'$ is incompressible in X';*

(2.B) *$\partial_- W_n'$ is compressible in X'.*

(In the disconnected case, (2.B) always holds, since X' is a handlebody.)

First note that since $\partial_- W_n$ is incompressible in X, W_n is the unique (up to isotopy) maximal compression body for ∂X in X. Since $\partial_- W_i$ is homeomorphic to $\partial_- W_n$ for all i, $1 \le i \le n$, it follows that all the W_i are isotopic. Let us write $W = W_i$, $S_- = \partial_- W_i$. Similarly, in case (2.A) above, let $W' = W_i'$, $S_-' = \partial_- W_i'$.

In case (2.A), let $S = S_- \cup S_-'$, and $N = W \cup_F W'$. In case (2.B), let $S = S_-$ and $N = W \cup_F X'$.

In case (2.A), $N(r) = \hat{W} \cup_{\hat{F}} \hat{W}'$, which is irreducible by Lemma 2.2.3. In case (2.B), $N(r) = \hat{W} \cup_{\hat{F}} \hat{X}'$, which is irreducible by Lemma 2.2.3 and the fact that $\hat{X}'$ is irreducible and $\hat{F}$ is incompressible in $\hat{X}$ (see the proof of Proposition 2.2.1). Thus (ii)(a) of the addendum is satisfied.

By Lemma 2.2.4, S_- is compressible in $\hat{W}$. This shows, in both cases (2.A) and (2.B) that (ii)(b) holds.

Finally, consider (ii)(c). First suppose that we are in case (2.B). Any compressing disk for $S = S_-$ in $N(r)$ which misses $\hat{F}$ lies in $\hat{W}$. If there were such a disk which met only a single component of $\partial M - \partial F$, then there would be a planar surface in W with one boundary component essential in S_- and all other boundary components in a single annulus A_k, say. Hence S_- would compress in $\tau(W; C_k) = \tau(W_i; C_k)$, $1 \le i \le n$. If $n > 1$, we could choose $i \ne k$, and thereby obtain a contradiction to Lemma 2.1.3. Therefore $n = 1$. But in that case, W is just a collar $\partial X \times I$ of ∂X in X, and condition (i) of the addendum is satisfied by taking the annulus $C_1 \times I \subset \partial X \times I \subset N$. In case (2.A), we additionally apply the same argument to S'_- in W', showing that again either $S = S_- \cup S'_-$ satisfies condition (ii)(c) or $n = 1$ and S satisfies condition (i). □

Remark. It is actually easier in case (2.A) to show that S remains incompressible in $M(s)$ if $\Delta(r, s) > 1$, than in case (2.B). Specifically, this follows from Theorem 2.4.5, which in turn is a direct consequence of the graph-theoretic Proposition 2.5.6, and the latter is easier to prove than the corresponding graph-theoretic proposition (Proposition 2.5.9) that is needed to handle case (2.B). However, since the argument in case (2.B) also applies to case (2.A), for ease of organization we have chosen not to separate the two cases.

2.3. The planar case

The purpose of this section is to prove the following proposition. Recall that if F is disconnected it consists of two copies of a non-separating surface G.

PROPOSITION 2.3.1. *Suppose that F is planar. In the connected case, either*

(i) *$\hat{F}$ decomposes $M(r)$ as a connected sum of two lens spaces; or*

(ii) *M contains an incompressible torus which is disjoint from F and which compresses in $M(r)$.*

In the disconnected case,

(iii) *M fibers over S^1 with fiber G.*

For the connected case we shall need the following lemma.

LEMMA 2.3.2. *Let C_1, C_2 be disjoint simple loops in the boundary of a handlebody X of genus 2, with the property that there exist disks E_1, E_2 in X such that C_i intersects ∂E_i transversely in a single point, $i = 1, 2$. Then either*

(i) *$\tau(X; C_1 \cup C_2)$ is a punctured lens space; or*

(ii) *$\partial X - C_1 \cup C_2$ is compressible in X.*

To prove Lemma 2.3.2, we first prove a purely algebraic lemma, which is an easy consequence of Nielsen's "sign condition" for a primitive element of a free group of rank 2 [N].

[ ] will denote the conjugacy class of an element in a group. A set of elements $\{x_1, \ldots, x_n\}$ of a free group F is a *basis-up-to-conjugacy* for F if there exists a basis $\{a_1, \ldots, a_n\}$ for F such that $[x_i] = [a_i]$, $1 \leq i \leq n$. Recall that an element of a free group F is *primitive* if it belongs to some basis for F. Finally, $\langle\ \rangle$ denotes normal closure.

LEMMA 2.3.3. *Let x and y be primitive elements of a free group F of rank 2. Then either*

(i) $H_1(F/\langle x, y\rangle)$ *is cyclic of order k, $1 < k < \infty$; or*
(ii) $\{x, y\}$ *is a basis-up-to-conjugacy; or*
(iii) $[x] = [y^{\pm 1}]$.

Proof. Since x is primitive, F has a basis of the form $\{x, z\}$. Let w be a cyclically reduced word in x, z such that $[y] = [w]$. By [N], since y is primitive, all the exponents of z in w have the same sign. Hence either conclusion (i) holds or z occurs at most once in w, with exponent ± 1. Therefore, we may take w to be either x^n or $x^n z^{\pm 1}$, for some integer n. In the first case we must have $n = \pm 1$ (since y is primitive), and we obtain conclusion (iii). In the second case, since $\{x, x^n z^{\pm 1}\}$ is a basis for F, we obtain conclusion (ii). □

Proof of Lemma 2.3.2. First note that since C_1 and ∂E_1 (say) are dual, $\tau(X; C_1 \cup C_2)$ is obtained by attaching a 2-handle to a solid torus.

The loops C_1 and C_2 (when oriented) determine conjugacy classes $[C_1]$, $[C_2]$ in $\pi_1(X)$. The existence of the disk E_i implies that $[C_i]$ is the conjugacy class of a primitive element of $\pi_1(X)$, $i = 1, 2$. Therefore Lemma 2.3.3 applies.

If case (i) of Lemma 2.3.3 holds, then we obtain conclusion (i) of Lemma 2.3.2.

In cases (ii) and (iii) of Lemma 2.3.3, we use the result of Zieschang [Z] that the minimal geometric length of a system of disjoint simple loops in the boundary of a handlebody is equal to its minimal algebraic length. (The geometric (resp. algebraic) length is computed from the cyclic words (resp. cyclic reduced words) that record the intersections of the loops with the boundaries of a complete system of meridian disks for the handlebody. For further discussion and a short proof of Zieschang's theorem, see [Wa2]. For an even shorter proof, see [K].) In the present situation, this implies that there exist disjoint disks D_1, D_2 in X, such that X cut along $D_1 \cup D_2$ is a 3-ball, with the property that, in case (ii) of Lemma 2.3.3, C_i meets ∂D_i transversely in a single point and is disjoint from ∂D_j, $i = 1, 2$, $j \neq i$, and in case (iii), each of C_1 and C_2 meets ∂D_1 transversely in a single point and is disjoint from ∂D_2. In both cases it follows that $\partial X - C_1 \cup C_2$ is compressible in X, giving conclusion (ii). □

Proof of Proposition 2.3.1. (a) *The connected case.* Recall the surfaces $\partial_- W_n \subset X$, $\partial_- W_n' \subset X'$ defined in Section 2.1. Since F is planar these are tori.

First suppose that $\partial_- W_n$ (say) is incompressible in X. Then M contains an incompressible torus which is disjoint from F. Also, $\partial_- W_n$ compresses in $\hat{W}_n = \tau(W_n; \cup_{i=1}^n C_i)$ (see the remark after the proof of Lemma 2.2.3) and hence in $M(r)$. We thus obtain conclusion (ii).

So we suppose that $\partial_- W_n$ compresses in X (and that $\partial_- W_n'$ compresses in X'). Since X is irreducible, $\partial_- W_n$ bounds a solid torus in X, implying that X is a handlebody of genus n. It is convenient to distinguish two cases:

(1) $n = 1$. Here F is an annulus and X is a solid torus. Since F is essential in $M = X \cup_F X'$, the image of $H_1(F)$ in $H_1(X)$, which coincides with the image of $H_1(C_1)$ in $H_1(X)$, has index k, say, where $1 < k < \infty$. Therefore $\hat{X} = \tau(X; C_1)$ is a punctured lens space whose fundamental group has order k.

(2) $n > 1$. Recall Lemma 2.1.2. By canceling the 2-handle corresponding to C_j with the disk $E_n^{(j)}$, $j = 1, \ldots, n-2$, we see that X_{n-2} is a handlebody of genus 2. Consider the disjoint simple loops $C_{n-1}, C_n \subset \partial X_{n-2}$. Since the boundaries of the disks $E_n^{(n-1)}$ and $E_{n-1}^{(n)}$ are disjoint from $C_1, \ldots, C_{n-2}$, we have $E_n^{(n-1)}, E_{n-1}^{(n)} \subset X_{n-2}$. Also, $\partial E_n^{(n-1)}$ (resp. $\partial E_{n-1}^{(n)}$) intersects C_{n-1} (resp. C_n) transversely in a single point. Application of Lemma 2.1.4 with $k = n-2$ shows that $\partial X_{n-2} - C_{n-1} \cup C_n$ is incompressible in X_{n-2}. Lemma 2.3.2 now implies that $\hat{X}$ is a punctured lens space.

Similarly (in both cases (1) and (2)), since $\partial_- W_n'$ is compressible in X', $\hat{X}'$ is also a punctured lens space, and we obtain conclusion (i) of the proposition.

(b) *The disconnected case.*

(1) $n = 1$. Here G is a disk and M is a solid torus, so that we have conclusion (iii).

(2) $n > 1$. Recall from Section 2.1 that here X is a handlebody of genus $n-1$. Also, the disks $\{E_i^{(j)}: j \neq i\}$, for $1 \leq i \leq n$, given by Lemma 2.1.2, show that the set of loops $\{C_j: j \neq i\}$ is primitive (in the sense of [Gr2]) for each i, $1 \leq i \leq n$. Therefore, by [Gr2, Proposition 2.1], $(X, G) \cong (G \times I, G \times \{0\})$. Recalling that $(X', G) \cong (G \times I, G \times \{0\})$ also, we see that M fibers over S^1 with fiber G. □

We remark that as far as the Cyclic Surgery Theorem is concerned, the cases $n = 1$ of both (a) and (b) are actually excluded by the assumption that M is not Seifert fibered. However, as there is no need to exclude them in the statement of Proposition 2.3.1 or Theorem 2.0.3, we do not do so.

2.4. Reduction and boundary reduction under Dehn surgery

Throughout this section M will be an irreducible 3-manifold (not necessarily compact) with a torus boundary component T. If r is a slope on T, $M(r)$ will denote the manifold obtained by attaching a solid torus to M along T so that the boundary of a meridian disk of the solid torus has slope r. Let S be a surface in

$\partial M - T$ which is incompressible in M. We are interested in the question of when S compresses in $M(r)$. Note that the case in which S is a closed incompressible surface in int M can be reduced to the situation just described by cutting M along S.

We make the following conjecture.

Conjecture 2.4.1. *If S compresses in $M(r_1)$ and $M(r_2)$ then $\Delta(r_1, r_2) \leq 1$, unless M contains an incompressible annulus with one boundary component in S and the other in T.*

Examples showing the need to rule out annuli are provided by the cable spaces (see the remarks immediately following the statement of Theorem 2.4.4 below). Examples (with S a torus) where S compresses in $M(r_i)$ for two slopes r_1, r_2 (with $\Delta(r_1, r_2) = 1$) have been given by Berge (unpublished) and Gabai [Ga]. In fact an example is given in [Ga] where S compresses in $M(r_i)$ for three slopes r_1, r_2, r_3 (with $\Delta(r_i, r_j) = 1$, $i \neq j$).

In the direction of Conjecture 2.4.1, we are able to prove the following.

THEOREM 2.4.2. *If S compresses in $M(r_1)$ and $M(r_2)$, then either*

(i) $\Delta(r_1, r_2) \leq 2$; *or*

(ii) *M contains an annulus with one boundary component in S and the other having slope r_0 in T, where $\Delta(r_0, r_i) = 1$, $i = 1, 2$; or*

(iii) *M is homeomorphic to $T \times I$.*

To complete the proof of Theorem 2.0.3, after Propositions 2.2.1 and 2.3.1, it would suffice to prove Conjecture 2.4.1. This can be achieved when S is a torus (see Theorem 2.4.4 below). In particular, this completes the proof of Theorem 2.0.3 when F is planar. In the non-planar case, the completion of the proof of Theorem 2.0.3 may be described as a modification of the proof of Theorem 2.4.2 which makes use of the special properties of the surface S that are described in Addendum 2.2.2. (Thus Theorem 2.4.2 itself is not used in the proof of the Cyclic Surgery Theorem, but we include it for its interest as a general result on the problem of when surfaces compress under Dehn surgery.)

If conclusion (ii) of Theorem 2.4.2 holds, then it is possible to obtain more explicit information about the compressibility of S in $M(r)$. This is described in the following theorem.

THEOREM 2.4.3. *Suppose that M contains an annulus with one boundary component in S and the other having slope r_0 in T. Then*

(a) *S compresses in $M(r_0)$.*

(b) *S is incompressible in $M(r)$ if $\Delta(r, r_0) > 1$, unless M is homeomorphic to $T \times I$.*

(c) *If* $\Delta(r_i, r_0) = 1$, $i = 1, 2$, *then* $M(r_1) \cong M(r_2)$. *In particular, S compresses in* $M(r)$ *for some r such that* $\Delta(r, r_0) = 1$ *if and only if it compresses in* $M(r)$ *for all such r.*

To state the improvement that can be made to Theorem 2.4.2 when S is a torus, recall the definition of a *cable space* given in Section 2.0.

THEOREM 2.4.4. *If S is a torus which compresses in* $M(r_1)$ *and* $M(r_2)$, *then either*

(i) $\Delta(r_1, r_2) \leq 1$; *or*
(ii) *M is a cable space*; *or*
(iii) *M is homeomorphic to* $T \times I$.

The examples mentioned above shows that this result is best possible.

If M is a cable space, with boundary components S and T (since there is an automorphism of M interchanging S and T, the labeling is irrelevant), then in fact S compresses in $M(r)$ for infinitely many slopes r. This is accounted for by the existence of an incompressible annulus in M connecting the two boundary components, as described in Theorem 2.4.3(c) (see [Gr-L, Remark on p. 125]). The cable spaces thus provide simple examples to illustrate Theorem 2.4.3 and to show the need to exclude annuli in the formulation of Conjecture 2.4.1.

Another situation in which Theorem 2.4.2 can be improved is the following.

THEOREM 2.4.5. *If* S_1 *and* S_2 *are disjoint surfaces in* $\partial M - T$, *each incompressible in M, such that* S_i *compresses in* $M(r_i)$, $i = 1, 2$, *then* $\Delta(r_1, r_2) \leq 1$.

The above theorems can be regarded as restricting the boundary slopes in T of essential punctured disks in M. The next result deals with the analogous question for punctured spheres.

THEOREM 2.4.6. *If* $(P_i, \partial P_i) \subset (M, T)$ *is an essential planar surface with boundary slope* r_i, $i = 1, 2$, *then either*

(i) $\Delta(r_1, r_2) \leq 1$; *or*
(ii) $M(r_1)$ *or* $M(r_2)$ *contains a lens space as a connected summand.*

An immediate corollary is the following.

COROLLARY 2.4.7. *If* $M(r_1)$ *and* $M(r_2)$ *are reducible, then either*

(i) $\Delta(r_1, r_2) \leq 1$; *or*
(ii) $M(r_1)$ *or* $M(r_2)$ *contains a lens space as a connected summand.*

Corollary 2.4.7 will be used to prove Theorem 2.0.2.

Theorem 2.4.6 and Corollary 2.4.7 may be compared with [Gr-L, Theorem 1.1], where it is shown that the alternative conclusions (i) and (ii) can be replaced by the single conclusion that $\Delta(r_1, r_2) \leq 4$. Probably none of these results is best possible.

In Section 2.5 we shall show how the theorems stated above follow from certain graph-theoretic propositions; these in turn will be proved in Section 2.6.

The remainder of this section is devoted to showing how Theorems 2.0.1 and 2.0.2 follow quickly from Theorems 2.4.4 and 2.4.5, and Corollary 2.4.7, respectively.

Proof of Theorem 2.0.1. Let S be an essential torus in int M which compresses in $M(r)$ and $M(s)$. Let M' be the component of the manifold obtained by cutting M along S that contains ∂M. There are two cases.

(1) *S does not separate M*. Then $\partial M' = S_1 \cup S_2 \cup \partial M$, where S_1 and S_2 are copies of S. If some S_i compresses in both $M'(r)$ and $M'(s)$, then Theorem 2.4.4 implies that $\Delta(r, s) \leq 1$. On the other hand, if S_1 (say) compresses in $M'(r)$ while S_2 compresses in $M'(s)$, then Theorem 2.4.5 implies that $\Delta(r, s) \leq 1$.

(2) *S separates M*. Then $\partial M' = S \cup \partial M$. If S compresses in $M'(r)$ and $M'(s)$, then by Theorem 2.4.4 either $\Delta(r, s) \leq 1$ or M' is a cable space. (Conclusion (iii) of Theorem 2.4.4 is impossible since S is essential in M.) □

Proof of Theorem 2.0.2. Since $\dim H_1(M; \mathbf{Q}) > 1$, we have that $\dim H_1(M(r); \mathbf{Q}) \geq 1$ for all slopes r. Hence $M(r)$ is a Haken manifold unless it is reducible. By Corollary 2.4.7, $M(r)$ and $M(s)$ reducible implies that either $\Delta(r, s) \leq 1$ or $M(r)$ (say) contains a lens space as a connected summand. But the latter possibility is ruled out by the assumption that $H_1(M(r))$ is cyclic. □

2.5. The basic graph-theoretic analysis of reduction and boundary reduction

In this section we describe the basic machinery with which the theorems stated in Section 2.4 will be proved. In particular, we show how these theorems, and the remaining part of Theorem 2.0.3, follow from certain graph-theoretic propositions.

Throughout, we shall use the indices α and β to denote 1 or 2, with the convention that, when they are used together, $\{\alpha, \beta\} = \{1, 2\}$.

Let M and S be as described at the beginning of Section 2.4, and let D_α be a compressing disk for S in $M(r_\alpha) = M \cup J_\alpha$. We may assume that D_α meets the solid torus J_α in a disjoint union of meridian disks of J_α. (Note that $D_\alpha \cap J_\alpha$ is necessarily non-empty since S is incompressible in M.) Then $P_\alpha = D_\alpha \cap M$ is a planar surface, with one *outer boundary component* $\partial_0 P_\alpha = \partial D_\alpha$, lying in S, and n_α, say, *inner boundary components* $\partial_x P_\alpha$, $x = 1, \dots, n_\alpha$, each having slope r_α in T. We shall always assume that P_α is incompressible in M. This is guaranteed if D_α is chosen so that n_α is minimal (for example, over all compressing disks for S in $M(r_\alpha)$), and we shall usually make this choice. On the one occasion when this may not be possible (this occurs for P_1 in the final step of the proof of Theorem 2.0.3), it will be easy to see that we may still assume that P_1 is incompressible in M.

By an isotopy of P_2, say, we may assume that P_1 and P_2 meet in general position. Note that $\partial_0 P_\alpha$ meets no inner boundary component of P_β. Then $P_1 \cap P_2 = \mathscr{A} \amalg \mathscr{B} \amalg \mathscr{C}$, where $\mathscr{A}$ is a disjoint union of arcs with at least one endpoint in an inner boundary component of P_α, $\mathscr{B}$ is a disjoint union of arcs with both endpoints in $\partial_0 P_\alpha$, and $\mathscr{C}$ is a disjoint union of simple closed curves. By a standard innermost disk argument, a further isotopy of P_2 will ensure that no component of $\mathscr{C}$ bounds a disk in P_α, since P_β is incompressible and M is irreducible. Finally, again by an isotopy of P_2, say, we may assume that each inner boundary component of P_α meets each inner boundary of P_β in $\Delta = \Delta(r_1, r_2)$ points. In fact, if the inner boundary components of P_β are numbered $\partial_1 P_\beta, \dots, \partial_{n_\beta} P_\beta$ so that they are consecutive on T, then going round each inner boundary component of P_α (in some direction) the indices of the inner boundary components of P_β that we encounter are, in order, $1, \dots, n_\beta, \dots, 1, \dots, n_\beta$ (repeated Δ times).

A *graph in a disk* D will consist of a finite number of vertices in int D, together with a finite number of edges. Each edge either connects a vertex to a (possibly non-distinct) vertex, or connects a vertex to ∂D. The former are *interior edges*, the latter, *boundary edges*. We assume that the endpoints on ∂D of all boundary edges are distinct.

Let Γ_α be the graph in the disk D_α obtained by taking the union of $\mathscr{A}$ with the cone (in the corresponding meridian disk of J_α) on $\mathscr{A} \cap \partial_x P_\alpha$ for each inner boundary component $\partial_x P_\alpha$ of P_α. Thus the vertices x of Γ_α are in one-to-one correspondence with the inner boundary components $\partial_x P_\alpha$ of P_α, and the edges of Γ_α are in one-to-one correspondence with the components of $\mathscr{A}$. Note that each vertex has valency Δn_β. Let e be an edge of Γ_α, with one of its endpoints the vertex x, say. Then e corresponds to an arc in $\mathscr{A}$ with one endpoint on $\partial_x P_\alpha$ which is a point of intersection of $\partial_x P_\alpha$ with some inner boundary component $\partial_y P_\beta$, say, of P_β. We say that *e has label y at x*. In this way, each incidence of an edge of Γ_α at a vertex of Γ_α is labeled with a vertex of Γ_β.

Two vertices of Γ_α are *parallel* if the corresponding inner boundary components of P_α, when given the orientation induced by some orientation of P_α, are homologous in T. Otherwise, they are *antiparallel*.

Since all our manifolds are orientable, each arc in $P_1 \cap P_2$ must join points of intersection of ∂P_1 with ∂P_2 of opposite sign. Thus we have the following *parity rule*:

If e is an interior edge of Γ_α connecting parallel (resp. antiparallel) vertices of Γ_α, then the labels at the endpoints of e represent antiparallel (resp. parallel) vertices of Γ_β.

Although in general the edges of our graphs are unoriented, it will sometimes be convenient to temporarily orient an edge e. Then $\partial_+ e$ and $\partial_- e$ will denote the head and tail endpoints of e respectively.

A *cycle* in Γ_α is a subgraph homeomorphic to a circle. Equivalently, a cycle is a subgraph consisting of interior edges $e_0, \dots, e_{k-1}$, $k \geq 1$, which may be oriented so that $\partial_+ e_i = \partial_- e_{i+1}$, $i \in \mathbf{Z}_k$, and so that the vertices $\partial_+ e_i$, $i \in \mathbf{Z}_k$, are distinct. We call k the *length* of the cycle.

Let x be a vertex of Γ_α. An *x-cycle* in Γ_β is a cycle σ such that

(a) all the vertices of σ are parallel, and

(b) for some consistent orientation of the edges in σ, each edge e has label x at $\partial_- e$.

A *Scharlemann cycle* in Γ_β is an x-cycle σ (for some vertex x of Γ_α) such that the interior of the disk in D_β bounded by σ is disjoint from Γ_β. (The terminology is explained by the remark preceding Lemma 2.5.2 below.)

It follows easily from the definition that if the edges of a Scharlemann cycle in Γ_β are oriented consistently, then each edge e has label x at $\partial_- e$ and label $\bar{x}$ at $\partial_+ e$, where x and $\bar{x}$ are antiparallel and represent inner boundary components of P_α which are adjacent on T. (See Figure 2.1.)

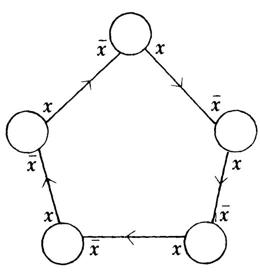


FIGURE 2.1

A Scharlemann cycle of length 1 will be called a *trivial loop*.

The next three lemmas are concerned with the topological interpretation of certain properties of the graphs Γ_1 and Γ_2.

LEMMA 2.5.1. *Γ_α contains no trivial loop.*

Proof. A trivial loop in Γ_α gives rise to a boundary-compression of P_β towards T. It is easy to show that this implies that P_β is compressible. □

Parts (a) and (b) of the next lemma are essentially contained in [Sch2, proof of Proposition 4.7] and [Sch3, proof of Proposition 5.6] respectively. We include the proofs for the convenience of the reader.

LEMMA 2.5.2. *Suppose that Γ_β contains a Scharlemann cycle. Then:*

(a) *There exists a compressing disk D'_α for S in $M(r_\alpha)$ with $\partial D'_\alpha = \partial D_\alpha$ such that the corresponding planar surface P'_α in M has two fewer inner boundary components than P_α; and*

(b) *$M(r_\alpha)$ has a lens space as a connected summand.*

Proof. Let σ be a Scharlemann cycle in Γ_β. Then the length of σ, say k, is greater than 1 by Lemma 2.5.1.

(a) σ gives rise to a disk $E \subset P_\beta$ whose boundary is the union of consecutive arcs $a_0, b_0, \ldots, a_{k-1}, b_{k-1}$, where the a_i are in $P_1 \cap P_2$ and correspond to the edges of σ, and the b_i are in the inner boundary components of P_β corresponding to the vertices of σ. (See Figure 2.2.) Note that int $E \cap P_\alpha = \varnothing$. The labels at the endpoints of the edges of σ represent inner boundary components of P_α which are adjacent on T. These boundary components cobound an annulus A in T containing all the b_i. Let Q_α be the punctured torus $P_\alpha \cup A$, pushed slightly into int M, and let P'_α be Q_α surgered along E. Since ∂E has algebraic (and geometric) intersection number k with a core of A, P'_α is a planar surface with $\partial_0 P'_\alpha = \partial_0 P_\alpha$ and with two fewer inner boundary components than P_α.

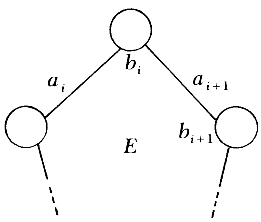


FIGURE 2.2

(b) Consider the disk $D_\alpha \subset M(r_\alpha) = M \cup J_\alpha$. The inner boundary components of P_α represented by the labels at the endpoints of the edges of σ bound meridian disks D, $\overline{D}$, say, in J_α. Since these boundary components are antiparallel in T, the union of the 3-cell in J_α bounded by $D \cup A \cup \overline{D}$ with a suitable regular neighborhood of D_α (pushed slightly off S) will be a solid torus V in int $M(r_\alpha)$. Then $\partial E \subset \partial V$, and there is a regular neighborhood $N(E)$ of E such that $V \cup N(E)$ is a punctured lens space in $M(r_\alpha)$ whose fundamental group has order k. $\square$

Remark. Usually we may assume without loss of generality that the disk D_α is chosen so that the number n_α of inner boundary components of P_α is minimal. Then by Lemma 2.5.2(a), the existence of a Scharlemann cycle in Γ_β simply leads to a contradiction. One exception, however, occurs in the final step of the proof of Theorem 2.0.3. There the set-up is more complicated, and in particular it is not clear that we may assume that the number of inner boundary components of P_1 is minimal while still maintaining the other hypotheses. In that case we use Lemma 2.5.2(b) instead.

Two edges e, e' of Γ_α are *parallel* if the corresponding arcs a, a' in $P_1 \cap P_2$ are parallel in P_α, that is, there exists a disk E in P_α such that $\partial E = a \cup b \cup a' \cup b'$, where b, b' are arcs in ∂P_α. (Note that $E \cap P_\beta$ may contain arcs other than a and a'.)

The next lemma is essentially contained in [Sho]. Recall that $\Delta = \Delta(r_1, r_2)$.

LEMMA 2.5.3. *Suppose that P_α is an annulus. Then either $\Delta \leq 1$ or M is homeomorphic to $T \times I$.*

Proof. Since P_α is an annulus, all the edges of Γ_α, and hence of Γ_β, are boundary edges (by Lemma 2.5.1). Suppose that $\Delta \geq 2$. Then by a simple outermost arc argument, there is a vertex y of Γ_β such that two adjacent (boundary) edges of Γ_β containing y are parallel. Let the corresponding arcs in $P_1 \cap P_2$ be a and a'. Then, for $i = 1, 2$, there are arcs $d_i \subset \partial D_i$, $b_i \subset \partial P_i - \partial D_i$, and disks $E_i \subset P_i$ such that $\partial E_i = a \cup b_i \cup a' \cup d_i$. Note that $E_1 \cap E_2 - a \cup a'$ consists of arcs which are parallel into both d_1 and d_2. These are easily removed, for example by an isotopy of E_2 (using the incompressibility of S in M and the irreducibility of M). Then $A = E_1 \cup E_2$ is an annulus in M with one boundary component in S and the other, $b_1 \cup b_2 = \partial_0 A$, say, in T. Since int $b_\beta \cap \partial P_\alpha = \varnothing$, we see that $\partial_0 A$ has slope r_0, say, where $\Delta(r_0, r_\alpha) = 1$ (see Figure 2.3). Hence, after an isotopy of A, $A \cap P_\alpha$ will consist of a single spanning arc.Thus $A \cup P_\alpha$ is homeomorphic to $X \times I$, where $X = X \times \{0\}$ is

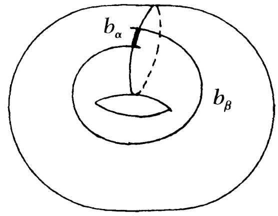


FIGURE 2.3

the union of two simple loops in T which intersect transversely in a single point, and has a regular neighborhood homeomorphic to $N \times I$ where $N = N \times \{0\}$ is a regular neighborhood of X in T. Since $\partial N = \partial N \times \{0\}$ bounds a disk B_0 in T and S is incompressible in M, $\partial N \times \{1\}$ bounds a disk B_1 in S. Then $B_0 \cup \partial N \times I \cup B_1$ is a 2-sphere, which bounds a 3-ball B in M since M is irreducible. Then $N \times I \cup B \cong T \times I$, showing that $M \cong T \times I$. □

Lemma 2.5.4. *Suppose that Γ_α contains boundary edges which are parallel in both Γ_1 and Γ_2. Then either M contains an annulus with one boundary component in S and the other having slope r_0 in T, where $\Delta(r_0, r_i) = 1$, $i = 1, 2$, or M is homeomorphic to $T \times I$.*

Proof. Let the arcs in $P_1 \cap P_2$ corresponding to the edges that are parallel in Γ_1 and Γ_2 be a and a'. Then, for $i = 1, 2$, there are arcs $d_i \subset \partial D_i$, $b_i \subset \partial P_i - \partial D_i$, and disks $E_i \subset P_i$, such that $\partial E_i = a \cup b_i \cup a' \cup d_i$. By possibly rechoosing a' and performing an isotopy of E_2 (say), we may assume that $\operatorname{int} E_1 \cap \operatorname{int} E_2 = \emptyset$. Then $A = E_1 \cup E_2$ is an annulus in M with one boundary component in S and the other, $b_1 \cup b_2 = \partial_0 A$, say, in T. By moving $\partial_0 A$ slightly into general position with respect to the component of ∂P_i containing b_i, we see that $\partial_0 A$ has slope r_0 in T where $\Delta(r_0, r_i) > 0$, $i = 1, 2$. Now apply Lemma 2.5.3 with A replacing P_α in the statement of that lemma. We conclude that either $M \cong T \times I$ or $\Delta(r_0, r_i) = 1$, $i = 1, 2$. □

We shall usually use Lemma 2.5.4 in conjunction with the following observation.

Lemma 2.5.5. *Suppose that Γ_α contains a family of $2n_\beta$ mutually parallel boundary edges. Then Γ_α contains boundary edges which are parallel in both Γ_1 and Γ_2.*

Proof. The edges in the parallel family may be numbered $e(1), \ldots, e(n_\beta), e'(1), \ldots, e'(n_\beta)$ so that $e(y)$ and $e'(y)$ have label y (at their vertex endpoint). Thus $e(y)$ and $e'(y)$ correspond to boundary edges in Γ_β with vertex endpoint y; let the union of these boundary edges be $f(y)$. Then $f(y)$ is a properly embedded arc in D_β containing the vertex y. Let y_0 be a vertex such that $f(y_0)$ is an outermost such arc. Then the edges $e(y_0)$ and $e'(y_0)$ are parallel in both Γ_1 and Γ_2. □

We now state the main graph-theoretic propositions on which the proof of the theorems in Section 2.4 and the proof of conclusion (iii) of Theorem 2.0.3 will be based. These propositions will be proved in Section 2.6.

PROPOSITION 2.5.6. *Suppose that* $\Delta \geq 2$. *Then either*
(i) Γ_1 *or* Γ_2 *contains a Scharlemann cycle; or*
(ii) *every vertex of* Γ_α *belongs to a boundary edge of* Γ_α.

Addendum 2.5.7. *In conclusion* (ii) *of Proposition* 2.5.6, *every vertex* x *of* Γ_α *belongs to a boundary edge* $e(x)$ *of* Γ_α *such that the vertices of* Γ_β *in the set* $\{$*label of* $e(x)$ *at* x: x *a vertex of* $\Gamma_\alpha\}$ *are all parallel.*

PROPOSITION 2.5.8. *Suppose that* $\Delta \geq 3$. *Then either*
(i) Γ_1 *or* Γ_2 *contains a Scharlemann cycle; or*
(ii) *there exists a pair of boundary edges which are parallel in both* Γ_1 *and* Γ_2.

To complete the proof of Theorem 2.0.3 we need to make use of some additional structure. Thus, for the next proposition we assume that there is an essential surface $(F, \partial F) \subset (M, T)$ with $\partial F \neq \varnothing$ and having boundary slope r_1, and that the compressing disk D_1 for S in $M(r_1)$ is disjoint from F. We may assume that P_1 is incompressible in $M - F$, for example, by choosing D_1 so that n_1 is minimal among all compressing disks for S in $M - F$. Since F is incompressible in M, this implies that P_1 is also incompressible in M.

PROPOSITION 2.5.9. *Suppose that* $\Delta \geq 2$. *Then either*
(i) Γ_1 *contains a Scharlemann cycle; or*
(ii) Γ_2 *contains a Scharlemann cycle; or*
(iii) *all the inner boundary components of* P_1 *lie in a single component of* $T - \partial F$.

This proposition will be used to establish conclusion (iii) of Theorem 2.0.3. Its proof is a modification of the proof of Proposition 2.5.8.

In the remainder of this section we show how Theorems 2.4.2–2.4.6, and the rest of Theorem 2.0.3 (together with Addendum 2.0.4), follow from the above graph-theoretic propositions.

Proof of Theorem 2.4.2. We may assume that n_α is minimal over all compressing disks for S in $M(r_\alpha)$. The theorem then follows immediately from Proposition 2.5.8, Lemma 2.5.2(a), and Lemma 2.5.4. □

Proof of Theorem 2.4.3. By hypothesis, M contains an annulus A_0 with one boundary component in S and the other having slope r_0 in T.
(a) A_0 gives rise to a compressing disk for S in $M(r_0)$.
(b) This follows immediately from Lemma 2.5.3.
(c) Let $t\colon M \to M$ be the homeomorphism defined by Dehn twisting (in some direction) along A_0. If $\Delta(r_i, r_0) = 1$, $i = 1, 2$, then, for some integer k,

t^k takes the slope r_1 to r_2. It therefore extends to a homeomorphism from $M(r_1)$ to $M(r_2)$. □

Proof of Theorem 2.4.4. Suppose that $\Delta \geq 2$, and that D_α is chosen so that n_α is minimal. By Proposition 2.5.6, Addendum 2.5.7, and Lemma 2.5.2(a), each vertex x of Γ_α is an endpoint of a boundary edge $e(x)$ of Γ_α such that the labels of the edges $e(x)$ at the corresponding vertices x represent parallel vertices of Γ_β. Since S is a torus, we may also assume that all the intersections of ∂D_1 with ∂D_2 have the same sign. By the same reasoning as that which led to the parity rule, these together imply that the vertices of Γ_α are all parallel; similarly, the vertices of Γ_β are all parallel. By the parity rule, this implies that Γ_α contains no interior edges. By a straightforward outermost arc argument, there exists a vertex x of Γ_α such that all the (boundary) edges of Γ_α incident to x are parallel. Since $\Delta \geq 2$, Lemmas 2.5.4 and 2.5.5 imply that either M is homeomorphic to $T \times I$ or there is an annulus A in M with one boundary component, say $\partial_1 A$, in S, and the other, say $\partial_0 A$, having slope r_0 in T, where $\Delta(r_0, r_i) = 1$, $i = 1, 2$.

Let K be a core of the solid torus J, where $M(r_1) = M \cup J$. Since $\Delta(r_1, r_0) = 1$, a meridian disk of J meets $\partial_0 A$ in one point. Hence A can be extended to an annulus B such that $\partial B = \partial_1 A \cup K$, which can be used to isotope K into S. Now let Σ be a 2-sphere in $M(r_1)$. By the previous sentence, an isotopy of Σ in $M(r_1)$ will move it off K, and hence into M, where it bounds a 3-ball. This shows that $M(r_1)$ is irreducible. Therefore, since S compresses in $M(r_1)$ by hypothesis, $M(r_1)$ is a solid torus with boundary S. Thus M is the complement of an open regular neighborhood of a curve, K, in a solid torus which can be isotoped to an essential curve in the boundary of the solid torus. It follows that M is either a cable space or homeomorphic to $T \times I$. □

Proof of Theorem 2.4.5. Let D_α be a compressing disk for S_α in $M(r_\alpha)$ with n_α minimal. Since $S_1 \cap S_2 = \varnothing$, the graph Γ_α contains no boundary edges. The result now follows from Proposition 2.5.6 and Lemma 2.5.2(a). □

Proof of Theorem 2.4.6. Here the arcs in $P_1 \cap P_2$ give rise to graphs Γ_1, Γ_2 in 2-spheres Σ_1, Σ_2. All the remarks made at the beginning of this section apply, *mutatis mutandis*, to Γ_1 and Γ_2. Hence, formally removing an open disk from $\Sigma_i - \Gamma_i$, $i = 1, 2$, we obtain two graphs in disks with no boundary edges, to which we may apply Proposition 2.5.6. We conclude that either $\Delta(r_1, r_2) \leq 1$ or there is a Scharlemann cycle in Γ_1 or Γ_2. It is easy to see that Lemma 2.5.1 still applies, since P_1 and P_2 are essential in M, so that any Scharlemann cycle in Γ_1 or Γ_2 has length greater than 1. But if Γ_β contains such a Scharlemann cycle, then an argument entirely analogous to the proof of Lemma 2.5.2(b) shows that $M(r_\alpha)$ contains a lens space as a connected summand. □

Remark. Although a Scharlemann cycle in Γ_β can also be used to construct a new planar surface P'_α with two fewer boundary components, as in Lemma 2.5.2(a), there is now no guarantee that P'_α is essential.

Proof of Theorem 2.0.3. There are two cases.

(1) *F is non-planar*. By Proposition 2.2.1, either conclusion (i) of Theorem 2.0.3 holds or M contains a closed incompressible surface S which is disjoint from F. Moreover, S has the additional properties described in Addendum 2.2.2.

If condition (i) of Addendum 2.2.2 holds, then S remains incompressible in $M(s)$ whenever $\Delta(r, s) > 1$ by Theorem 2.4.3(b). (Note that S is not parallel to ∂M, for example since it is disjoint from F.) This is conclusion (iii) of Theorem 2.0.3.

So suppose that S satisfies condition (ii) of Addendum 2.2.2. Write $r_1 = r$ and let D_1 be a compressing disk for S in $N(r_1)$ which misses $\hat{F}$, as guaranteed by (ii)(b) of Addendum 2.2.2. By the remark immediately preceding the statement of Proposition 2.5.9, we may assume that the corresponding punctured disk P_1 is incompressible in M. Suppose that S also compresses in $M(r_2)$, and therefore in $N(r_2)$, for some slope r_2. Let D_2 be a compressing disk for S in $N(r_2)$, such that the number of inner boundary components of the corresponding planar surface P_2 is minimal. Let Γ_1, Γ_2 be the graphs in D_1, D_2 obtained in the usual way. Now apply Proposition 2.5.9. Conclusion (i) of that proposition contradicts the minimality of the number of inner boundary components of P_2, by Lemma 2.5.2(a). Conclusion (ii) contradicts condition (ii)(a) of Addendum 2.2.2, in view of Lemma 2.5.2(b). Finally, conclusion (iii) contradicts condition (ii)(c) of Addendum 2.2.2. We must therefore have $\Delta(r_1, r_2) \leq 1$, and thus obtain conclusion (iii) of Theorem 2.0.3.

We point out that in the above proof we are not able to use the existence of a Scharlemann cycle in Γ_2 to reduce the number of inner boundary components of P_1, as this might destroy the condition $P_1 \cap F = \varnothing$ which is crucial for the proof of Proposition 2.5.9.

(2) *F is planar*. First suppose that F is connected. Then by Proposition 2.3.1, either conclusion (ii) of Theorem 2.0.3 holds or M contains an incompressible torus S which is disjoint from F and compresses in $M(r)$. Therefore, by Theorem 2.0.1, either S remains incompressible in $M(s)$ if $\Delta(r, s) > 1$ (and we have conclusion (iii) of Theorem 2.0.3), or S and ∂M cobound a cable space. If the latter holds, then in particular there is an annulus A in M with one boundary component in S and the other having slope r_0, say, in T. Now consider $A \cap F$. Since $F \cap S = \varnothing$ and F is essential in M, a standard outermost arc argument shows that we must have $r_0 = r$. Then again we obtain conclusion (iii) of Theorem 2.0.3, by Theorem 2.4.3(b).

If F is disconnected then, by Proposition 2.3.1(iii), M fibers over S^1 with fiber G, and we have conclusion (iv) of Theorem 2.0.3. □

Before giving the proof of Addendum 2.0.4, we need the following lemma.

LEMMA 2.5.10. *Let M be a 3-manifold which fibers over S^1 with fiber a connected surface, and let F be an essential connected surface in M. If either*

(a) *F is disjoint from some fiber; or*

(b) *F is planar and ∂F is disjoint from some fiber,*

then F is isotopic to a fiber.

Proof. (a) Let H be a fiber such that $F \cap H = \varnothing$. Then F is a connected incompressible surface in $H \times I$ (that is, M cut along H) such that $F \cap H \times \partial I = \varnothing$. But it follows easily from [Wal, Proposition 3.1] that any such surface is either a boundary-parallel annulus or parallel to $H \times \{0\}$.

(b) Move F into general position with respect to some fiber H, keeping $\partial F \cap H = \varnothing$. Thus $F \cap H$ consists of disjoint simple loops; let C be one that is innermost on F (which is planar by hypothesis), and let F_0 be the corresponding innermost component of F cut along C. Then, when we cut M along H, F_0 is an incompressible surface in $H \times I$ with ∂F_0 contained in (say) $H \times \{0\} \cup \partial H \times I$ and $\partial F_0 \cap H \times \{0\} \neq \varnothing$. Again by [Wal, Proposition 3.1] any such surface is parallel to a subsurface of $H \times \{0\}$, and so we may perform an isotopy of F to reduce the number of components of $F \cap H$. Continuing in this way, we eventually move F off H, at which point we apply part (a). □

Proof of Addendum 2.0.4. Let P be a fiber. We work with a surface satisfying the following slight modification of the definition given in Section 2.1. Let $(F, \partial F) \subset (M, \partial M)$ be an essential separating surface with boundary slope r, such that each component of F has non-empty boundary and is not isotopic to P, and such that the number of components of ∂F is minimal subject to these conditions. Since r is a strict boundary slope by hypothesis, there exists such a surface F. In addition, we assume (as we may) that if F is disconnected then it consists of two parallel copies of some connected non-separating surface. Note that by Lemma 2.5.10(b), F is not planar.

Recall the definition of F_i^- from Section 2.1. Our minimality assumption on F here implies that each component of F_i^- is either closed, or a boundary-parallel annulus, or non-separating, and that if there are any non-separating components, then at least one of them is isotopic to P. On the other hand, if F_i^- did have a component isotopic to P, then, since by a small isotopy we can make $F \cap F_i^- = \varnothing$, F would consist of copies of P by Lemma 2.5.10(a). We conclude that F_i^- consists of boundary-parallel annuli together with a closed surface.

The proofs of Proposition 2.2.1 and Addendum 2.2.2 then apply verbatim to this surface F. Since $M(r)$ is homeomorphic to $S^1 \times S^2$, conclusion (i) of

Proposition 2.2.1 does not hold; so we have conclusion (ii), together with Addendum 2.2.2. The proof of the relevant part of Theorem 2.0.3 (case (1) above) now applies to give conclusion (iii) of that theorem. □

2.6. The combinatorics of graphs in disks

This section is graph-theoretic in nature. In it we assume that we are given graphs Γ_1, Γ_2 in disks D_1, D_2 as described in Section 2.5, and our goal is to prove Propositions 2.5.6 (including Addendum 2.5.7), 2.5.8, and 2.5.9.

The search for Scharlemann cycles is facilitated by consideration of the following more general type of cycle (see Lemma 2.6.2 below).

A *great cycle* (specifically, a *great x-cycle*) in Γ_β is an x-cycle σ such that all the vertices of Γ_β that lie in the (closed) disk in D_β bounded by σ are parallel.

Let Λ be a subgraph of Γ_β, and let x be a vertex of Γ_α. We say that Λ satisfies condition $P(x)$ if:

For each vertex y of Λ there exists an edge of Λ incident to y with label x, connecting y to a parallel vertex of Λ.

LEMMA 2.6.1. *Suppose that Λ satisfies condition $P(x)$. Then each component of Λ contains an x-cycle.*

Proof. For each vertex y of Λ, choose an edge $e(y)$ of Λ incident to y with label x, connecting y to a parallel vertex of Λ.

Let Λ_0 be a component of Λ, and let y_1 be any vertex of Λ_0. Consider the edge $e(y_1)$, connecting y_1 to y_2, say. Since y_1 and y_2 are parallel, the label of $e(y_1)$ at y_2 is not x, by the parity rule. Hence if $y_2 \neq y_1$, the edge $e(y_2)$ is distinct from $e(y_1)$. Continue in this way, obtaining edges $e(y_i)$ connecting vertices y_i and y_{i+1}, with label x at y_i, until a vertex is repeated for the first time, say $y_m = y_n$, $m < n$, but $y_i \neq y_j$ for $1 \leq i < j < n$. Then the edges $e(y_m), \dots, e(y_{n-1})$ form an x-cycle in Λ_0. □

LEMMA 2.6.2. *If Γ_β contains a great cycle, then it contains a Scharlemann cycle.*

Proof. Let σ be a great x-cycle in Γ_β, for some vertex x of Γ_α, and let E be the disk in D_β bounded by σ. Let $\varepsilon(\sigma)$ be the number of edges e in Γ_β such that int $e \subset$ int E. We prove the result by induction on $\varepsilon(\sigma)$.

If $\varepsilon(\sigma) = 0$, then σ is a Scharlemann cycle by definition. So suppose that $\varepsilon(\sigma) > 0$. We distinguish two cases:

(1) *Any edge in $\Gamma_\beta \cap E$ incident to a vertex of σ lies in σ.*

Since all the vertices of $\Gamma_\beta \cap E$ are parallel, the graph $\Lambda = \Gamma_\beta \cap E - \sigma$ satisfies $P(x')$ for any vertex x' of Γ_α, and hence contains an x'-cycle σ' by

Lemma 2.6.1. Clearly σ' is a great x'-cycle in Γ_β and $\varepsilon(\sigma') < \varepsilon(\sigma)$. Hence the result follows by induction.

(2) *There exists an edge of* $\Gamma_\beta \cap E$ *incident to a vertex of* σ *but not contained in* σ.

Since all the vertices of σ are parallel, there is a label x' such that x and x' represent inner boundary components of P_α which are adjacent in T, with the property that for every vertex y of σ there is an edge of $\Gamma_\beta \cap E$ incident to y with label x'. This is also (trivially) true for any vertex y in int E. Hence $\Gamma_\beta \cap E$ satisfies condition $P(x')$. An application of Lemma 2.6.1 would now give an x'-cycle σ' in $\Gamma_\beta \cap E$; however, we want to avoid the possibility that $\sigma' = \sigma$. To do this, note that since by hypothesis there exists an edge of $\Gamma_\beta \cap E$ incident to a vertex of σ but not contained in σ, there exists such an edge incident to a vertex y, say, in σ with label x'. Let e be the edge of σ which, when oriented so that it has label x at $\partial_- e$, has $\partial_+ e = y$. Then the graph $\Lambda = \Gamma_\beta \cap E - \operatorname{int} e$ also satisfies condition $P(x')$. By Lemma 2.6.1 we obtain an x'-cycle σ' in Λ, which is clearly a great x'-cycle in Γ_β with $\varepsilon(\sigma') < \varepsilon(\sigma)$. Then Γ_β contains a Scharlemann cycle by induction. □

Let Λ be a graph in a disk D, and choose a point $\infty \in \partial D - \Lambda$. We may then define a partial ordering on the set of components of Λ by declaring that $\Lambda_1 < \Lambda_2$ if and only if every path in D from Λ_1 to ∞ meets Λ_2. Call a component of Λ *extremal* if it is minimal with respect to this partial ordering for some choice of ∞.

To prove Proposition 2.5.6 we focus on the following condition:

$(*)_\alpha$. *There exists a vertex x of Γ_α such that for each vertex y of Γ_β there is an edge of Γ_α incident to x with label y, connecting x to an antiparallel vertex of Γ_α.*

LEMMA 2.6.3. *Suppose that condition $(*)_\alpha$ holds. Then Γ_β contains a great x-cycle.*

Proof. By the parity rule, the hypothesis $(*)_\alpha$ is equivalent to the condition that there exists a vertex x of Γ_α such that for each vertex y of Γ_β there is an edge $e(y)$ of Γ_β incident to y with label x, connecting y to a parallel vertex of Γ_β. For each vertex y of Γ_β, choose such an edge $e(y)$, and let Λ be the subgraph of Γ_β consisting of all these edges $e(y)$. Then Λ satisfies condition $P(x)$. Note also that each component of Λ has all its vertices parallel.

Let Λ_0 be an extremal component of Λ. By Lemma 2.6.1, Λ_0 contains an x-cycle σ. Since Λ_0 is extremal, all vertices of Γ_β in the disk bounded by σ belong to Λ_0; in particular, they are parallel. Hence σ is a great x-cycle. □

The negation of condition $(*)_\alpha$ is:

$(**)_\alpha$. *For each vertex x of Γ_α there exists a vertex $y(x)$ of Γ_β such that each edge of Γ_α incident to x with label $y(x)$ connects x either to a parallel vertex of Γ_α or to ∂D_α.*

LEMMA 2.6.4. *Suppose that $\Delta \geq 2$ and that condition $(**)_\alpha$ holds. Then either*

(i) *Γ_α contains a great cycle; or*

(ii) *every vertex of Γ_β belongs to a boundary edge of Γ_β.*

Proof. For each vertex x of Γ_α choose a vertex $y(x)$ of Γ_β as in condition $(**)_\alpha$, and define the subgraph Λ of Γ_α to be the union over all vertices x of Γ_α of {edges of Γ_α incident to x with label $y(x)$}. Note that each vertex of Λ has valency $\geq \Delta \geq 2$. Also, each component of Λ has all its vertices parallel.

Let Λ_0 be an extremal component of Λ with respect to some point $\infty \in \partial D_\alpha - \Lambda$. Let R be the component of $D_\alpha - \Lambda_0$ containing ∞. The frontier FrR of R can be expressed as the union of a sequence of oriented edges $e_1, \ldots, e_n$ of Λ_0, with $\partial_+ e_i = \partial_- e_{i+1}$, $1 \leq i \leq n-1$, such that as we traverse these edges in order, the component of $D_\alpha - \Lambda_0$ immediately on (say) our left is always R. (As oriented edges, $e_1, \ldots, e_n$ are distinct, although as unoriented edges there may be repetitions. Note however that, since each vertex of Λ_0 has valency at least 2, consecutive edges in the sequence are distinct as unoriented edges.)

Recall from Section 2.5 the definition of a cycle. We distinguish two cases:

(1) *The sequence $e_1, \ldots, e_n$ contains a cycle.*

Note that this is necessarily the case if $\Lambda_0 \subset \operatorname{int} D_\alpha$.

Let the cycle be σ, with edges $f_0 = e_m, \ldots, f_{k-1} = e_{m+k-1}$, say; so $\partial_+ f_i = \partial_- f_{i+1}$, $i \in \mathbf{Z}_k$. Let x_i be the vertex $\partial_- f_i$, $i \in \mathbf{Z}_k$. Let E be the disk in D_α bounded by σ. (See Figure 2.4.)

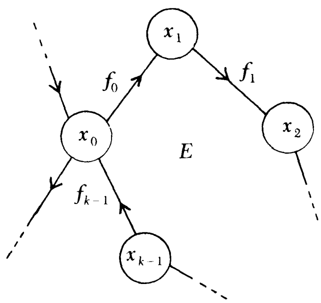


FIGURE 2.4

Observe that, except possibly for $i = 0$, as we encircle the vertex x_i in the clockwise direction, the edges of Γ_α incident to x_i between f_{i-1} and f_i have their interiors in R, while the remaining edges of Γ_α incident to x_i lie in E. (See Figure 2.5.) Furthermore, since by the definition of Λ there are edges of Λ_0 incident to x_i at least at each of the Δ occurrences of the label $y(x_i)$, there are at most $n_\beta - 1$ incidences of edges in the interior of R between f_{i-1} and f_i. It follows that if $i \neq 0$, then for each vertex y of Γ_β there is an edge of $\Gamma_\alpha \cap E$ incident to x_i with label y. This is also (trivially) true for any vertex in int E.

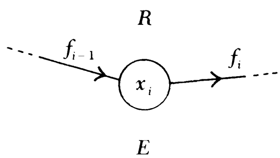


FIGURE 2.5

Since Λ_0 is extremal, all vertices of $\Gamma_\alpha \cap E$ are in Λ_0, and hence are parallel. Let y_0 be the label at x_0 of any edge in $\Gamma_\alpha \cap E$ incident to x_0 (for example, f_0). Then $\Gamma_\alpha \cap E$ satisfies condition $P(y_0)$. By Lemma 2.6.1, $\Gamma_\alpha \cap E$ contains a y_0-cycle. Since all vertices of $\Gamma_\alpha \cap E$ are parallel, this is necessarily a great y_0-cycle, giving conclusion (i).

(2) *The sequence $e_1, \ldots, e_n$ does not contain a cycle.*

Then FrR is an arc in Λ_0 with its endpoints in ∂D_α. Let E be the disk $D_\alpha - R$. (See Figure 2.6.) As in case (1) above, for every vertex x in FrR and every vertex y of Γ_β there is an edge of $\Gamma_\alpha \cap E$ incident to x with label y.

Suppose that conclusion (ii) does not hold, so that there is a vertex y_0 of Γ_β which does not lie in any boundary edge of Γ_β. Then any edge of Γ_α incident to

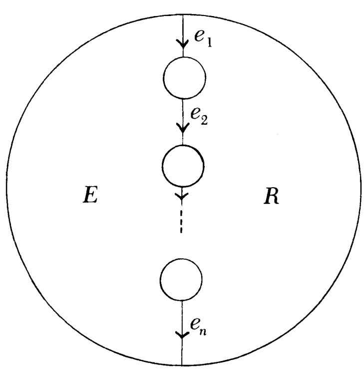


FIGURE 2.6

any vertex with label y_0 is an interior edge. Since Λ_0 is extremal, all vertices of $\Gamma_\alpha \cap E$ are in Λ_0, and hence are parallel. Therefore $\Gamma_\alpha \cap E$ satisfies condition $P(y_0)$. By Lemma 2.6.1, $\Gamma_\alpha \cap E$ contains a y_0-cycle, which is necessarily a great y_0-cycle since all vertices of $\Gamma_\alpha \cap E$ are parallel. Thus we have conclusion (i). □

Remark. In the last paragraph of the above proof, we only used the fact that there was a vertex y_0 of Γ_β such that any edge of $\Gamma_\alpha \cap E$ (as opposed to any edge of Γ_α) incident to any vertex with label y_0 was an interior edge. Since all the vertices of $\Gamma_\alpha \cap E$ are parallel, conclusion (ii) of Lemma 2.6.4 may therefore be strengthened to say that every vertex y of Γ_β is an endpoint of a boundary edge $e(y)$ of Γ_β such that the vertices of Γ_α in the set $\{$label of $e(y)$ at y: y a vertex of $\Gamma_\beta\}$ are all parallel.

We can now prove Proposition 2.5.6 and Addendum 2.5.7.

Proof of Proposition 2.5.6. This follows immediately from Lemmas 2.6.2, 2.6.3 and 2.6.4. □

Proof of Addendum 2.5.7. This follows from the remark after the proof of Lemma 2.6.4. □

The following lemma will be needed in the proofs of Propositions 2.5.8 and 2.5.9.

LEMMA 2.6.5. *Let* Γ *be a graph in a disk* D *with no trivial loops or parallel edges, such that every vertex of* Γ *belongs to a boundary edge. Then* Γ *has a vertex of valency at most* 3 *which belongs to a single boundary edge.*

Thus Γ has a vertex of one of the types illustrated in Figure 2.7 (which shows all the edges of Γ incident to the vertex).

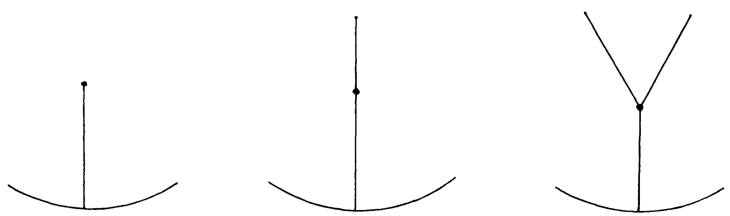


FIGURE 2.7

Proof. For the purposes of this proof only, let us call vertices x, x' of Γ *adjacent* if there exist edges d, d' of Γ connecting x, x' to ∂D such that $d \cap \partial D$ and $d' \cap \partial D$ are adjacent on ∂D among all points of $\Gamma \cap \partial D$.

Let the number of vertices of Γ be n. The lemma is clearly true if $n \leq 3$. For $n \geq 4$ we will prove the following stronger assertion by induction on n:

(*) *There exist two non-adjacent vertices of valency at most* 3, *each belonging to a single boundary edge.*

So suppose $n \geq 4$ and assume that (*) is true for graphs with m vertices, $4 \leq m < n$. We consider three cases:

(1) *There is an edge of* Γ *connecting non-adjacent vertices.*
Let e be such an edge connecting x and x', say. Let d, d' be edges connecting x, x' to ∂D. (See Figure 2.8.) Let D_1, D_2 be the closures of the components of $D - d \cup e \cup d'$, and consider the graphs $\Gamma_i = \Gamma \cap D_i$ in D, $i = 1, 2$. Suppose Γ_i has n_i vertices. Since x, x' are not adjacent in Γ, we must have $n_i \geq 3$, $i = 1, 2$, or equivalently $n_i < n$, $i = 1, 2$. Note that x, x' are adjacent in Γ_i. We claim that there exists a vertex y_i of Γ_i of valency at most 3 which belongs to a single boundary edge of Γ_i, with $y_i \neq x$ or x'. If $n_i \geq 4$, this follows from the inductive hypothesis applied to Γ_i. If $n_i = 3$, then take y_i to be the vertex of Γ_i not equal to x or x'. Now y_1, y_2 are non-adjacent vertices of Γ of valency at most 3, each belonging to a single boundary edge.

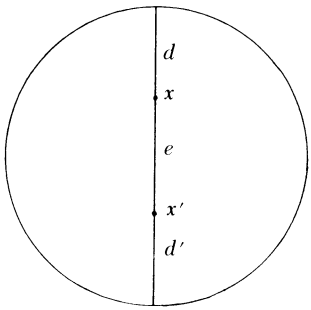


FIGURE 2.8

(2) *There is a vertex of* Γ *which belongs to two distinct boundary edges.*

Let x be such a vertex, contained in boundary edges d, d', say. (See Figure 2.9.) Let D_1, D_2 be the closures of the components of $D - d \cup d'$, and consider

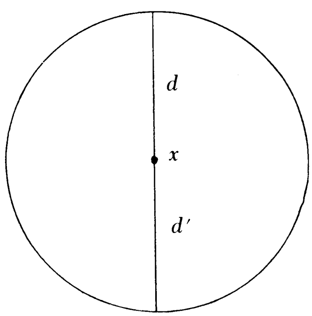


FIGURE 2.9

the graphs $\Gamma_i = \Gamma \cap D_i$ in D, $i = 1, 2$. Suppose that Γ_i has n_i vertices. Since d and d' are not parallel in Γ, we must have $n_i \geq 2$, $i = 1, 2$, or equivalently $n_i < n$, $i = 1, 2$. We claim that there exists a vertex y_i of Γ_i of valency at most 3, which belongs to a single boundary edge of Γ_i, with $y_i \neq x$. If $n_i \geq 4$, this follows from the inductive hypothesis applied to the graph $\overline{\Gamma}_i$ obtained from Γ_i by amalgamating the parallel edges d, d'. If $n_i = 2$ or 3, then the claim is easily verified by inspection. Now y_1, y_2 satisfy $(*)$ for Γ.

(3) *Neither* (1) *nor* (2) *holds.*

Then every vertex belongs to a single boundary edge, and is connected to at most two vertices (since they must be adjacent to it). Thus any two non-adjacent vertices satisfy $(*)$. □

LEMMA 2.6.6. *If Γ_α contains a parallel family of edges connecting parallel vertices, then either the sets of labels at the two ends of the family are disjoint, or Γ_α contains a Scharlemann cycle.*

The following corollary is immediate.

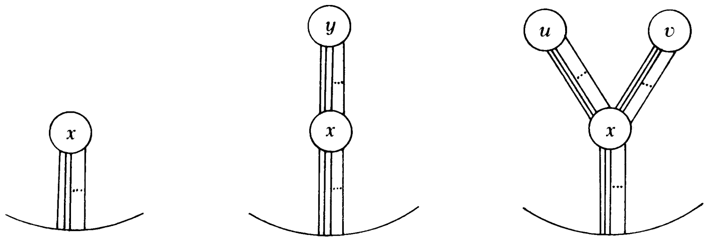
COROLLARY 2.6.7. *If Γ_α contains a parallel family of more than $n_\beta/2$ edges connecting parallel vertices, then Γ_α contains a Scharlemann cycle.*

Proof of Lemma 2.6.6. Assume that the sets of labels at the two ends of the parallel family are not disjoint, and let y be a vertex of Γ_β which appears in both sets. Since no edge in the family can have label y at both ends by the parity rule, we see that the family contains a y-cycle σ of length 2. Since there are no vertices in the interior of the disk that σ bounds, σ is a great y-cycle. Now apply Lemma 2.6.2 to conclude that Γ_α contains a Scharlemann cycle.

LEMMA 2.6.8. *If Γ_α contains a parallel family of n_β edges connecting antiparallel vertices, then Γ_β contains a Scharlemann cycle.*

Proof. This follows immediately from Lemmas 2.6.3 and 2.6.2. □

Proof of Proposition 2.5.8. By Proposition 2.5.6 we may assume that each vertex of Γ_α belongs to a boundary edge. Let $\overline{\Gamma}_\alpha$ be the reduced graph corresponding to Γ_α, obtained by amalgamating all mutually parallel edges in the obvious way. Applying Lemma 2.6.5 to $\overline{\Gamma}_\alpha$, we conclude that Γ_α has a vertex x of one of the forms illustrated in Figure 2.10.



Type I Type II Type III

FIGURE 2.10

We treat these separately.

Type I. In this case Γ_α contains $\Delta n_\beta \geq 3n_\beta$ parallel boundary edges, giving a pair of boundary edges which are parallel in both Γ_1 and Γ_2 by Lemma 2.5.5.

Type II. Since $\Delta \geq 3$, there must be either at least n_β parallel edges connecting x to y, or at least $2n_\beta$ parallel edges connecting x to ∂D_α. In the first case, we obtain a Scharlemann cycle in Γ_1 or Γ_2 by Corollary 2.6.7 and Lemma 2.6.8, and in the second case we obtain a pair of boundary edges which are parallel in both Γ_1 and Γ_2 by Lemma 2.5.5.

Type III. Let U, V, W be the sets of labels around x corresponding to the edges connecting x to u, v and ∂D_α respectively. (See Figure 2.11.)We denote by $|U|$ the number of labels (counted with multiplicity) in U, etc. Then $|U| + |V| + |W| = \Delta n_\beta \geq 3n_\beta$. There are three sub-cases:

(1) *u, v and x parallel*. By Corollary 2.6.7, either Γ_α contains a Scharlemann cycle or $|U| \leq (1/2)n_\beta$, $|V| \leq (1/2)n_\beta$. But then $|W| \geq 2n_\beta$, giving a pair of boundary edges which are parallel in both Γ_1 and Γ_2.

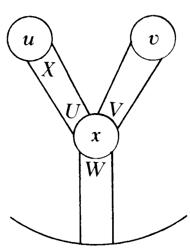


FIGURE 2.11

(2) *u, v parallel, antiparallel to x.* If $|W| \geq 2n_\beta$, we obtain a pair of boundary edges which are parallel in both Γ_1 and Γ_2. If not, then $|U| + |V| \geq n_\beta$, which implies that every vertex of Γ_β occurs as the label at x of some edge of Γ_α connecting x to either u or v. Then Γ_β contains a great x-cycle by Lemma 2.6.3, and consequently a Scharlemann cycle by Lemma 2.6.2.

(3) *u, v antiparallel.* We may assume without loss of generality that u is parallel to x.

Let $A = \{$vertices y of Γ_β: y appears at least twice as a label in $W\}$, and $B = \{$vertices y of Γ_β: y appears as a label in both U and $V\}$. Since each vertex of Γ_β appears $\Delta \geq 3$ times in $U \cup V \cup W$, and since by Corollary 2.6.7 and Lemma 2.6.8 we may assume that no vertex appears more than once in either U or V, it follows that every vertex of Γ_β belongs to either A or B.

If $y \in A$, then there exist two distinct edges $e_1(y), e_2(y)$ in Γ_β connecting y to ∂D_β (with label x at y), such that $e_1(y)$ and $e_2(y)$ are parallel in Γ_α.

If $y \in B$, then there exist edges $e_u(y)$ (resp. $e_v(y)$) in Γ_β, incident to y with label x, connecting y to an antiparallel (resp. parallel) vertex of Γ_β.

Let Λ be the subgraph of Γ_β consisting of $\{e_1(y), e_2(y)\colon y \in A\}$. Then Λ is a disjoint union of properly embedded arcs in Γ_β, each containing a single vertex. Consider an extremal component Λ_0 of Λ, containing the vertex y_0, say. Then (at least) one of the two components of $D_\beta - \Lambda_0$, call it R, contains no vertices of Λ. Thus all vertices of Γ_β in R belong to B.

We now distinguish two cases.

(3.A) *R contains no vertices of Γ_β.* Then the edges $e_1(y)$ and $e_2(y)$ are parallel in Γ_β. Since they are also parallel in Γ_α, we are done.

(3.B) *R contains a vertex of Γ_β.* Every vertex of R belongs to B, and hence in particular is connected by edges of Γ_β to both a parallel and an antiparallel vertex of Γ_β. It follows that the vertices of Γ_β in R cannot all be parallel to y_0. Let Π be the subgraph of Γ_β defined by $\Pi = \{e_v(y)\colon y \in R$ and y is antiparallel to $y_0\}$. Then Π satisfies condition $P(x)$, and hence contains an x-cycle σ by Lemma 2.6.1. By Proposition 2.5.6, we may assume that every vertex of Γ_β belongs to a boundary edge, which implies that there are no vertices of Γ_β in the interior of the disk bounded by σ. Hence σ is a great x-cycle, and so Γ_β contains a Scharlemann cycle by Lemma 2.6.2. □

Proof of Proposition 2.5.9. Recall that, in addition to the usual set-up, there is an essential surface $(F, \partial F) \subset (M, T)$ with $\partial F \neq \varnothing$ and boundary slope r_1, such that $P_1 \cap F = \varnothing$. Consider $P_2 \cap F$. The arcs in $P_2 \cap F$ define a graph Γ_0, say, in D_2, such that $\Gamma_0 \cap \Gamma_2 = \{$vertices of $\Gamma_0\} = \{$vertices of $\Gamma_2\}$. Since F is essential in M, it is easy to see that the proof of Lemma 2.5.1 allows us to assume that Γ_0 contains no trivial loops. Also, since $F \cap S = \emptyset$ and $\partial D_2 \subset S$, Γ_0 contains no boundary edges.

Claim 2.6.9. *Γ_2 contains no parallel family of more than n_1 boundary edges.*

Proof. As we go around an inner boundary component of P_2, between successive intersections with any given inner boundary component of P_1, we must encounter all the boundary components of F. Since Γ_0 has no boundary edges and no trivial loops, the claim follows. □

Continuing with the proof of Proposition 2.5.9, suppose that $\Delta \geq 2$. Exactly as in the first paragraph of the proof of Proposition 2.5.8, we conclude that if neither Γ_1 nor Γ_2 contains a Scharlemann cycle, then Γ_2 has a vertex x of one of the types illustrated in Figure 2.10.

Type I. Then Γ_2 contains a parallel family of $\Delta n_1 \geq 2n_1$ boundary edges, contradicting Claim 2.6.9.

Type II. Since there are at most n_1 parallel edges connecting x to ∂D_2 by Claim 2.6.9, there are at least n_1 parallel edges connecting x to y. Therefore either Γ_1 or Γ_2 contains a Scharlemann cycle, by Corollary 2.6.7 and Lemma 2.6.8.

Type III. We shall use the same notation as in the corresponding case of the proof of Proposition 2.5.8. There are again three sub-cases.

(1) *u, v, and x parallel.* By Corollary 2.6.7, either Γ_2 contains a Scharlemann cycle or $|U| \leq n_1/2$, $|V| \leq n_1/2$. Then $|W| \geq n_1$. Therefore, by Claim 2.6.9, $|W| = n_1$. Since Γ_0 has no boundary edges and no trivial loops, it follows that on the torus T, all the inner boundary components of P_1 lie between some pair of boundary components of F. This is conclusion (iii) of the proposition.

(2) *u, v parallel, antiparallel to x.* By Claim 2.6.9, $|W| \leq n_1$. Hence $|U| + |V| \geq n_1$, which implies that every vertex of Γ_1 occurs as the label at x of some edge of Γ_2 connecting x to either u or v. Then Γ_1 contains a great x-cycle by Lemma 2.6.3, and consequently a Scharlemann cycle by Lemma 2.6.2.

(3) *u, v antiparallel.* We may assume without loss of generality that u is parallel to x. Let X be the set of labels around u of the edges of Γ_2 connecting u to x. (See Figure 2.11.) By Lemma 2.6.8, we may assume that $|V| < n_1$, and hence that $|U| + |W| > n_1$. By Lemma 2.6.6, we may also assume that $X \cap U = \emptyset$. Therefore $X \subset W$. By the argument in case (1) above, we may assume that $|W| < n_1$. Then, since Γ_0 has no boundary edges and no trivial loops, no boundary components of F occur between any two adjacent inner boundary components of P_1 corresponding to the labels in W. Since $X \subset W$, the

same is true for X, and hence (since Γ_0 has no trivial loops) for U also. Since $|U| + |W| > n_1$, it follows that all the boundary components of F occur between some pair of inner boundary components of P_1, namely those corresponding to the two adjacent extremal labels of U and W. This is equivalent to conclusion (iii). □

2.7. Dehn surgery on cabled manifolds

In this section we complete the proof of the Cyclic Surgery Theorem. Let M be a compact, connected, irreducible 3-manifold with torus boundary, which is not Seifert-fibered. The following is an immediate consequence of Theorems 1.0.1, 2.0.1, 2.0.2, and 2.0.3, and Addendum 2.0.4.

COROLLARY 2.7.1. *If $\pi_1(M(r))$ and $\pi_1(M(s))$ are cyclic groups, then either $\Delta(r, s) \leq 1$ or M contains an essential torus which cobounds with ∂M a cable space in M.*

Suppose, then, that $M = M' \cup C_{p,q}$, where $C_{p,q}$ is a cable space of type (p, q) and $\partial M'$ is incompressible in M'. Suppose that $\pi_1(M(r_i))$ is cyclic, $i = 1, 2$. We must show that $\Delta(r_1, r_2) \leq 1$. Since $\pi_1(M(r_i))$ is cyclic, $\partial M'$ compresses in $C_{p,q}(r_i)$, $i = 1, 2$. By [Gr1, Lemma 7.2] (or [Gr-L, Lemma 3.1]), this implies that there are co-ordinates on ∂M such that either $r_i = pq$ or $r_i = (1 + k_i pq)/k_i$. Since $\Delta(pq, (1 + kpq)/k) = 1$ for all k, it follows that either $\Delta(r_1, r_2) \leq 1$ or $r_i = (1 + k_i pq)/k_i$ for some integer k_i, $i = 1, 2$. Note that $\Delta(r_1, r_2) = |k_1 - k_2|$. Then $C_{p,q}(r_i)$ is a solid torus ([Gr1, Lemma 7.2]), and hence $M(r_i) \cong M'(r_i')$ for some slopes r_1', r_2' on $\partial M'$. Moreover, there are co-ordinates on $\partial M'$ such that $r_i' = (1 + k_i pq)/k_i q^2$, $i = 1, 2$ (see [Gr1, Corollary 7.3] or [Gr-L, Lemma 3.1]). Then $\Delta(r_1', r_2') = |k_1 - k_2| q^2 = \Delta(r_1, r_2) q^2$ which is greater than 1 if $r_1 \neq r_2$. Therefore applying Corollary 2.7.1 to M' we infer that $M' = M'' \cup C_{p',q'}$, say, where $\partial M''$ is incompressible in M''. By the same argument that we originally applied to M, there are co-ordinates on $\partial M'$ such that $r_i' = (1 + k_i' p' q')/k_i'$, $i = 1, 2$. But the argument in [Gr-L, p. 137] shows that this is incompatible with the previous co-ordinate expressions of the r_i', unless $q = 2$ and $|k_1 - k_2| = \Delta(r_1, r_2) = 1$.

This completes the proof of the Cyclic Surgery Theorem.

2.8. Property P for symmetric knots

The purpose of this section is to give a proof of Corollary 7, which was stated in the introduction to the paper.

Let K be a non-trivial knot in S^3 such that there exists a periodic automorphism h of S^3, not equal to the identity, satisfying $h(K) = K$. Our goal is to show that K has Property P. Replacing h by a suitable power, we may assume that h has prime order, say p. Let $\mathrm{Fix}(h)$ denote the fixed-point set of h.

We distinguish three cases.

(1) *h orientation-reversing*. In this case K is amphicheiral and the result follows from Corollary 4.

(2) *h orientation-preserving*, $\mathrm{Fix}(h) \cap K \neq \varnothing$. By the Smith Conjecture [M-B], $\mathrm{Fix}(h) \neq K$, so that $\mathrm{Fix}(h) \cap K \cong S^0$ and h is an involution whose restriction to K is orientation-reversing. Thus K is strongly invertible, and hence has Property P by [B-S].

(3) *h orientation-preserving*, $\mathrm{Fix}(h) \cap K = \varnothing$. Let M be the complement of an h-invariant open tubular neighborhood of K, and let M^* be the quotient of M by $h|M$. Note that, here, the quotient map $\partial M \to \partial M^*$ is a p-fold (cyclic) covering. For any slope r on ∂M, $h|M$ extends to a periodic automorphism of $M(r)$ (which may fix pointwise the core of the attached solid torus J). Let h_n denote the extension to $M(1/n)$, where we are using the standard parametrization of slopes on ∂M. Note that if r^*, s^* are slopes on ∂M^* which lift to slopes r, s on ∂M, then $\Delta(r^*, s^*) = p\Delta(r, s)$ (consider the intersection between the pre-images of r^* and s^*, which consist of p copies of r and s respectively).

There are two sub-cases.

(A) $\mathrm{Fix}(h) \neq \varnothing$. In this case $\mathrm{Fix}(h_n) = \mathrm{Fix}(h)$. Let N be the complement in M of an h-invariant open tubular neighborhood of $\mathrm{Fix}(h)$, and let N^* be the quotient of N by $h|N$. Since the linking number of K and $\mathrm{Fix}(h)$ is non-zero, N is irreducible, and hence so is N^*. With the obvious notation, h_n induces a free $\mathbf{Z}_p$-action on $N(1/n)$ with quotient $N^*(r_n^*)$ for some slope r_n^* on ∂M^*. (In fact, using the usual meridian-longitude co-ordinates for ∂M^*, we have $r_n^* = 1/pn$.) Now suppose that $M(1/n)$ is simply-connected. Then, by the Generalized Smith Conjecture [M-B], $\mathrm{Fix}(h_n)$ is unknotted, so that $N(1/n)$ is a homotopy solid torus. Hence $N^*(r_n^*)$ is also a homotopy solid torus. Since $M(1/0) \cong S^3$ is simply-connected, and since $\Delta(r_n^*, r_0^*) = p\Delta(1/n, 1/0) > 1$ if $n \neq 0$, it follows from Theorem 2.4.4 applied to N^* that, if $n \neq 0$, then N^* is either a cable space or homeomorphic to $\partial M^* \times I$. Therefore N is either a cable space or homeomorphic to $\partial M \times I$, and hence (since $\mathrm{Fix}(h)$ is unknotted) K is a torus knot. Since non-trivial torus knots have Property P [Ms], we are done.

(B) $\mathrm{Fix}(h) = \varnothing$. Suppose that $M(1/n)$ is simply-connected.

If h_n fixes the core of J, then by the Generalized Smith Conjecture, M is a solid torus, and hence K is trivial.

If $\mathrm{Fix}(h_n) = \varnothing$, then h_n generates a free $\mathbf{Z}_p$-action on $M(1/n)$, with quotient $M^*(r_n^*)$ for some slope r_n^* on ∂M^*. Hence $\pi_1(M^*(r_n^*))$ is cyclic. Since $M(1/0) \cong S^3$, the Cyclic Surgery Theorem implies that either

$\Delta(r_n^*, r_0^*) \leq 1$ or M^* is Seifert-fibered. Since $\Delta(r_n^*, r_0^*) = p\Delta(1/n, 1/0)$, the first conclusion is impossible if $n \neq 0$. If the second conclusion holds, then M is also Seifert-fibered. Hence K is a torus knot, and therefore has Property P.

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