Characteristic subsurfaces, character varieties and Dehn fillings

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Let M be a one-cusped hyperbolic 3–manifold. A slope on the boundary of the compact core of M is called exceptional if the corresponding Dehn filling produces a non-hyperbolic manifold. We give new upper bounds for the distance between two exceptional slopes α and β in several situations. These include cases where $M(\beta)$ is reducible and where $M(\alpha)$ has finite π_1, or $M(\alpha)$ is very small, or $M(\alpha)$ admits a π_1–injective immersed torus.

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1 Introduction

Throughout this paper, M will denote a compact, connected, orientable 3–manifold whose boundary is a torus. We also assume that M is *simple*. In other words, it is irreducible, ∂–irreducible, acylindrical, and atoroidal. Thus M is homeomorphic to the compact core of a finite-volume hyperbolic 3–manifold with one cusp. For convenience, we will call such a manifold M hyperbolic. A slope α on ∂M (defined in Section 2) is said to be *exceptional* if the Dehn filling $M(\alpha)$ does not admit a hyperbolic structure. By the *distance* between two slopes α and β, we will mean their geometric intersection number $\Delta(\alpha, \beta)$.

Cameron Gordon has conjectured in [18] that the distance between any two exceptional slopes for M is at most 8, and also that there are exactly four specific manifolds M which have a pair of exceptional slopes with distance greater than 5. The results in this paper give upper bounds for the distance between two exceptional slopes in several special cases. We assume for most of these results that $M(\beta)$ is reducible, and that $M(\alpha)$ is a non-hyperbolic manifold of one of several types. Here, and throughout the paper, we will write L_p to denote a lens space whose fundamental group has order $p \geq 2$.

Our first result applies in the case that $M(\alpha)$ has finite fundamental group.

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Theorem 1.1 *If $M(\beta)$ is reducible and if $\pi_1(M(\alpha))$ is finite, then $\Delta(\alpha,\beta) \le 2$. Moreover, if $\Delta(\alpha,\beta)=2$, then $H_1(M)\cong\mathbb{Z}\oplus\mathbb{Z}/2$, $M(\beta)=L_2\#L_3$ and $\pi_1(M(\alpha))\cong O^*_{24}\times\mathbb{Z}/j$, where O^*_{24} denotes the binary octahedral group.*

Although we expect that the case $\Delta(\alpha,\beta) = 2$ does not arise, this theorem is a considerable improvement on the previously known bounds (see Boyer–Zhang [6]).

Recall that a closed 3–manifold N is said to be *very small* if $\pi_1(N)$ has no non-Abelian free subgroup. The next result deals with the situation where $M(\beta)$ is reducible and $M(\alpha)$ is very small. The proof is based on an analysis of the $PSL_2(\mathbb{C})$ character variety of a free product of cyclic groups. (See Section 2 for the definition of a strict boundary slope.)

Theorem 1.2 *Suppose that $M(\beta)$ is a reducible manifold and β is a strict boundary slope. If $M(\alpha)$ is very small, then $\Delta(\alpha,\beta)\le 3$.*

A closed orientable 3–manifold N is said to admit a *geometric decomposition* if the pieces of its prime and torus decompositions either admit geometric structures or are I–bundles over the torus. According to Thurston's Geometrization Conjecture, which has been claimed by Perelman, any closed orientable 3–manifold admits a geometric decomposition. If we strengthen the hypotheses of Theorem 1.2 by assuming that $M(\alpha)$ admits a geometric decomposition, we obtain the following stronger result.

Theorem 1.3 *Suppose that $M(\beta)$ is a reducible manifold and $M(\alpha)$ is a very small manifold that admits a geometric decomposition, then $\Delta(\alpha,\beta)\le 2$.*

This result is sharp. Indeed, if M is the hyperbolic manifold obtained by doing a Dehn filling of slope 6 on one boundary component of the (right-hand) Whitehead link exterior, then $M(1)\cong L_2\#L_3$ is reducible, while $M(3)$ is Seifert with base orbifold of the form $S^2(3,3,3)$, and so is very small.

The next result applies in the case where $M(\alpha)$ contains an immersed π_1–injective torus. Note that in this case, $M(\alpha)$ is either reducible, toroidal, or a Seifert fibred space with base orbifold of the form $S^2(r,s,t)$ (see Scott [24, Torus Theorem] and Gabai [15, Corollary 8.3]). The bound $\Delta(\alpha,\beta)\le 3$ holds in the first two cases by Gordon–Luecke [19], Oh [23] and Wu [28]. Thus the new information contained in this theorem concerns the case where $M(\alpha)$ is Seifert fibred and geometrically atoroidal.

Theorem 1.4 *Suppose that β is a strict boundary slope for M. If $M(\beta)$ is a reducible manifold and if $M(\alpha)$ admits a π_1–injective immersed torus, then $\Delta(\alpha,\beta)\le 4$. Moreover, if $\Delta(\alpha,\beta)=4$, then $M(\alpha)$ is a Seifert-fibred manifold with base orbifold $S^2(r,s,t)$, where (r,s,t) is a hyperbolic triple and at least one of r, s or t is divisible by 4.*

The inequalities we obtain in the last two results are significantly sharper than those obtained under comparable hypotheses in Boyer–Culler–Shalen–Zhang [3]. For Theorem 1.4, this is due to the fact that in [3] it is only assumed that β is the boundary slope of an essential, planar surface in M. Here we are using additional information about the topological structure of the connected sum decomposition of $M(\beta)$.

Since a Seifert fibred manifold is either very small or contains a π_1–injective immersed torus, the results above immediately yield the following corollary.

Corollary 1.5 *If $M(\beta)$ is a reducible manifold, β is a strict boundary slope, and $M(\alpha)$ is Seifert fibred, then $\Delta(\alpha,\beta) \leq 4$. Further, if $\Delta(\alpha,\beta) = 4$, then the base orbifold $\mathcal{B}$ of $M(\alpha)$ is $S^2(r,s,t)$, where (r,s,t) is a hyperbolic triple and 4 divides at least one of r,s,t.*

We also obtain the following result in the case where $M(\beta)$ is only assumed to be non-Haken, rather than reducible.

Theorem 1.6 *If β is a strict boundary slope and $M(\beta)$ is not a Haken manifold, then*

(1) $\Delta(\alpha,\beta) \leq 2$ *if $M(\alpha)$ has finite fundamental group;*

(2) $\Delta(\alpha,\beta) \leq 3$ *if $M(\alpha)$ is very small;*

(3) $\Delta(\alpha,\beta) \leq 4$ *if $M(\alpha)$ admits a π_1–injective immersed torus.*

We will show that our results imply the following restricted version of Gordon's conjecture.

Theorem 1.7 *If $M(\beta)$ is a reducible manifold and β is a strict boundary slope, then $M(\alpha)$ is a hyperbolic manifold for any slope α such that $\Delta(\alpha,\beta) > 5$. If we assume that the geometrization conjecture holds, then $M(\alpha)$ is a hyperbolic manifold for any slope α such that $\Delta(\alpha,\beta) > 4$.*

We remark that we expect the following to hold in this subcase of Gordon's Conjecture.

Conjecture 1.8 *If $M(\beta)$ is a reducible manifold, then $M(\alpha)$ is a hyperbolic manifold for any slope α such that $\Delta(\alpha,\beta) > 3$.*

The bound in the conjecture cannot be lowered. For instance, if M is the hyperbolic manifold obtained by doing a Dehn filling of slope 6 on one boundary component of the Whitehead link exterior, then $M(1) \cong L_2 \# L_3$ is reducible while $M(4)$ is toroidal.

The paper is organized as follows. Basic definition and notational conventions are given in Section 2. We review the notion of a singular slope for a closed, essential surface in Section 3 and prove Proposition 3.5, which characterizes the situations in which a boundary slope can fail to be a singular slope. At the end of Section 3 we prove Theorem 1.6 and Theorem 1.7, assuming Theorem 1.1, Theorem 1.3 and Theorem 1.4. Section 4 contains the proof of a technical result (Proposition 3.3) about singular slopes in $L(p,1)\#L(q,1)$ which is stated and applied earlier, in Section 3. In Section 5 we reduce the proofs of Theorems 1.1 – 1.4 to more specific propositions, which are proved in Sections 8, 9, 10 and 12 respectively. Section 6 is a review of $PSL_2(\mathbb{C})$–character variety theory and Section 7 contains results about the representation varieties of fundamental groups of very small 3–manifolds. Section 11 is based on the characteristic submanifold methods used in [3], and extends some of those results under the additional topological assumptions that are available in the setting of this paper. These results are applied in Section 12.

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2 Notation and definitions

We will use the notation $|X|$ to denote the number of components of a topological space X. The first Betti number of X will be denoted $b_1(X)$.

By a *lens space* we mean a closed orientable 3–manifold with a genus 1 Heegaard splitting. A lens space will be called non-trivial if it is not homeomorphic to $S^2 \times S^1$ or S^3.

By an *essential surface* in a compact, orientable 3–manifold, we mean a properly embedded, incompressible, orientable surface such that no component of the surface is boundary-parallel and no 2–sphere component of the surface bounds a 3–ball.

A *slope* α on ∂M is a pair $\{\pm a\}$ where a is a primitive class in $H_1(\partial M)$. The manifold $M(\alpha)$ is the Dehn filling of M obtained by attaching a solid torus to the boundary of M so that the meridian is glued to an unoriented curve representing the classes in α.

Definition 2.1 A slope β on ∂M is called a *boundary slope* if there is an essential surface F in M such that ∂F is a non-empty set of parallel, simple closed curves in

∂M of slope β. In this case we say that F has slope β. If $M(\beta)$ is reducible, then of course β is a boundary slope.

Next consider a connected surface F properly embedded in a 3–manifold W with bicollar $N(F) = F \times [-1, 1]$ in W. Denote by W_F the manifold $W \setminus F \times (-\frac{1}{2}, \frac{1}{2})$ and set $F_+ = F \times \{\frac{1}{2}\}, F_- = F \times \{-\frac{1}{2}\} \subset \partial W_F$. We say that W *fibres over* S^1 *with fibre* F if W_F is connected and $(W_F, F_+ \cup F_-)$ is a product $(I, \partial I)$–bundle pair. We say that W *semi-fibres over* I *with semi-fibre* F if W_F is not connected and $(W_F, F_+ \cup F_-)$ is a twisted $(I, \partial I)$–bundle pair.

Definition 2.2 A slope β on ∂M is called a *strict boundary slope* if there is an essential surface F in M of slope β which is neither a fibre nor a semi-fibre.

Definition 2.3 Given a closed, essential surface S in M, we let $\mathcal{C}(S)$ denote the set of slopes δ on ∂M such that S compresses in $M(\delta)$. A slope η on ∂M is called a *singular slope* for S if $\eta \in \mathcal{C}(S)$ and $\Delta(\delta, \eta) \leq 1$ for each $\delta \in \mathcal{C}(S)$.

3 Reducible Dehn fillings and singular slopes

A fundamental result of Wu [27] states that if $\mathcal{C}(S) \neq \varnothing$, then there is at least one singular slope for S.

The following result, which links singular slopes to exceptional surgeries, is due to Boyer, Gordon and Zhang.

Proposition 3.1 (Boyer–Gordon–Zhang [4, Theorem 1.5]) *If η is a singular slope for some closed essential surface S in M, then for an arbitrary slope α we have*

$$\Delta(\alpha, \eta) \leq \begin{cases} 1 & \text{if } M(\alpha) \text{ is either small or reducible} \\ 1 & \text{if } M(\alpha) \text{ is Seifert fibred and } S \text{ does not separate} \\ 2 & \text{if } M(\alpha) \text{ is toroidal and } \mathcal{C}(S) \text{ is infinite} \\ 3 & \text{if } M(\alpha) \text{ is toroidal and } \mathcal{C}(S) \text{ is finite.} \end{cases}$$

Consequently if $M(\alpha)$ is not hyperbolic, then $\Delta(\alpha, \eta) \leq 3$.

If $b_1(M) \geq 2$ and $M(\beta)$ is reducible, then work of Gabai [14] (see Corollary, page 462) implies that β is a singular slope for some closed, essential surface. This is also true generically when $b_1(M) = 1$, as the following result indicates.

Theorem 3.2 (Culler–Gordon–Luecke–Shalen [11, Theorem 2.0.3]) *Suppose that $b_1(M) = 1$ and that η is a boundary slope on ∂M. Then one of the following possibilities holds.*

(1) *$M(\eta)$ is a Haken manifold.*

(2) *$M(\eta)$ is a connected sum of two non-trivial lens spaces.*

(3) *η is a singular slope for some closed essential surface in M.*

(4) *$M(\eta) \cong S^1 \times S^2$ and η is not a strict boundary slope.*

Thus when $M(\beta)$ is reducible, either β is a singular slope for some closed, essential surface in M, or $M(\beta)$ is $S^1 \times S^2$ and β is not a strict boundary slope, or $M(\beta)$ is a connected sum of two lens spaces. In particular, the inequalities of Proposition 3.1 hold unless, perhaps, $M(\beta)$ is a very special sort of reducible manifold.

In order to prove our main results we must narrow the profile of a reducible filling slope which is not a singular slope.

The following result will be proved in the next section of the paper.

Proposition 3.3 *Suppose that $M(\beta) = L(p,1)\#L(q,1)$ and there are at least two isotopy classes of essential surfaces in M of slope β. Then β is a singular slope for some closed essential surface in M.*

Corollary 3.4 *Suppose that $M(\beta) = P^3\#P^3$ and β is a strict boundary slope. Then β is a singular slope for some closed essential surface in M.*

The proposition below, which follows immediately from Theorem 3.2 and Corollary 3.4, summarizes the situation.

Proposition 3.5 *Suppose that $b_1(M) = 1$ and $M(\beta)$ is a reducible manifold. Then one of the following three possibilities occurs:*

(1) *β is a singular slope for some closed essential surface in M; or*

(2) *$M(\beta)$ is homeomorphic to $L_p\#L_q$, where $q > 2$; or*

(3) *$M(\beta)$ is homeomorphic to $S^2 \times S^1$ or $P^3\#P^3$, and β is not a strict boundary slope.*

We end this section by giving the proofs of Theorem 1.6 and Theorem 1.7, assuming Theorem 1.1, Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.6 Since we have assumed that β is a strict boundary slope, if $M(\beta)$ is reducible, then Theorem 1.1, Theorem 1.3 and Theorem 1.4 imply that the corollary holds. On the other hand, if $M(\beta)$ is irreducible, then $b_1(M) = 1$ as $M(\beta)$ is non-Haken. Since β is a boundary slope, Theorem 3.2 implies that β is a singular slope for a closed essential surface in M. Proposition 3.1 now shows that the conclusion holds. □

Proof of Theorem 1.7 First suppose that $M(\beta)$ is either $S^1 \times S^2$ or $P^3 \# P^3$. Since β is a strict boundary slope, it follows from Proposition 3.5 that it must be a singular slope for some closed, essential surface in M. Thus Proposition 3.1 shows that the desired conclusion holds.

Next suppose that $M(\beta) \neq S^1 \times S^2, P^3 \# P^3$. Boyer–Zhang [8, Theorem 0.6] implies that if $\Delta(\alpha, \beta) > 5$, then $M(\alpha)$ is virtually Haken. In particular, $M(\alpha)$ admits a geometric decomposition (Casson–Jungreis [10], Gabai [16; 15], Gabai–Meyerhoff–Thurston [17]). According to Gordon–Luecke [19] and either Wu [28] or Oh [23], $M(\alpha)$ is irreducible and geometrically atoroidal as long as $\Delta(\alpha, \beta) > 3$. Further, Theorem 1.4 shows that $M(\alpha)$ is not Seifert fibred as long as $\Delta(\alpha, \beta) > 4$. Thus $M(\alpha)$ is hyperbolic if $\Delta(\alpha, \beta) > 5$. This proves the first claim of the theorem. The second follows similarly since $M(\alpha)$ admits a geometric decomposition for any slope α if the geometrization conjecture holds. □

4 Singular slopes when $M(\beta)$ is $L(p, 1) \# L(q, 1)$

This section contains the proofs of Proposition 3.3 and Corollary 3.4.

Let $\mathcal{S}(M)$ denote the set of essential surfaces in M. For each slope β on ∂M, set

$$\mathcal{S}_\beta(M) = \{F \in \mathcal{S}(M) : \partial F \neq \varnothing \text{ and } \beta \text{ is the boundary slope of } F\}.$$

For each surface $F \in \mathcal{S}_\beta(M)$, we use $\widehat{F}$ to denote the closed surface in $M(\beta)$ obtained by attaching meridian disks to F.

We begin with two propositions that give conditions on $\mathcal{S}_\beta(M)$ which guarantee that β is a singular slope for some closed essential surface in M. The first is a consequence of the proof of Theorem 3.2 (cf [11, chapter 2]).

Proposition 4.1 (Culler–Gordon–Luecke–Shalen [11]) *Suppose $M(\beta) \cong L_p \# L_q$ and that $F \in \mathcal{S}_\beta(M)$ satisfies $|\partial F| \leq |\partial F'|$ for each $F' \in \mathcal{S}_\beta(M)$. If $\widehat{F}$ is not an essential 2–sphere in $M(\beta)$, then β is a singular slope for a closed, essential surface in M.*

Proposition 4.2 *Suppose that $M(\beta) \cong L_p \# L_q$ and let $F \in \mathcal{S}_\beta(M)$. If there exists a closed, essential surface S in M which is disjoint from F, then β is a singular slope for S.*

Proof Since S is closed, essential, and disjoint from F, F is not a semi-fibre in M. On the other hand, S compresses in $M(\beta) \cong L_p \# L_q$, so $\beta \in \mathcal{C}(S)$ (see Boyer–Culler–Shalen–Zhang [3, Corollary 6.2.3]) then shows that S is incompressible in

$M(\gamma)$ for each slope γ on ∂M such that $\Delta(\gamma,\beta) \gg 0$. Wu's theorem [27] states that either $\Delta(\gamma,\gamma') \le 1$ for each $\gamma,\gamma' \in \mathcal{C}(S)$, or there is a slope $\gamma_0 \in \mathcal{C}(S)$ such that $\mathcal{C}(S) = \{\gamma : \Delta(\gamma,\gamma_0) \le 1\}$. In the first case, it is immediate that β is a singular slope for S. In the second case, observe that we must have $\gamma_0 = \beta$, since otherwise there would exist slopes $\gamma \in \mathcal{C}(S)$ with $\Delta(\gamma,\beta)$ arbitrarily large. Thus β is a singular slope for S in either case. □

We now proceed with the proof of Proposition 3.3, which depends on the two lemmas below. First we introduce some notational conventions that will be used in the lemmas.

Conventions 4.3 Suppose that $M(\beta) \cong L_p \# L_q$ and that β is not a singular slope for a closed essential surface. It is evident that $b_1(M) = 1$ and, since β is not the slope of the rational longitude of M, that each surface $F \in \mathcal{S}_\beta(M)$ is separating. Fix a surface $P \in \mathcal{S}_\beta(M)$ such that

$$|\partial P| \le |\partial F| \text{ for each } F \in \mathcal{S}_\beta(M).$$

Since P is connected and separating, we have that $n = |\partial P|$ is even. It follows from Proposition 4.1 that $\widehat{P}$ is an essential 2–sphere which bounds two punctured lens spaces $\widehat{X}$ and $\widehat{X}'$ in $M(\beta)$. We shall make the convention that $\widehat{X}$ is a punctured L_p and $\widehat{X}'$ is a punctured L_q. We let X and X' denote the submanifolds bounded by P in M, where $X \subset \widehat{X}$ and $X' \subset \widehat{X}'$.

In Conventions 4.3, we shall say that (X,P) is *unknotted* if there is a solid torus $V \subset X$ and an n–punctured disk D_n with outer boundary $\partial_o D_n$ such that

$$X = V \cup_A (D_n \times I),$$

where $A = (\partial_o D_n) \times I$ is identified with an essential annulus in ∂V.

Note that if (X,P) is unknotted and $p = 2$, then (V,A) is a twisted I–bundle pair over a Möbius band and the induced I–fibring of A coincides with that from $D_n \times I$. Thus (X,P) is a twisted I–bundle.

Lemma 4.4 *Assume that $M(\beta) \cong L_p \# L_q$ and that β is not a singular slope for a closed essential surface. Let $P \in \mathcal{S}_\beta(M)$ be chosen to have the minimal number of boundary components. Suppose that (X,P) is unknotted. If $F \in \mathcal{S}_\beta(M)$ is contained in X, then F is isotopic to P.*

Proof Write $X = V \cup_A (D_n \times I)$ as above and isotope F so as to minimize $|A \cap F|$. Then F intersects V and $D_n \times I$ in incompressible surfaces. If $A \cap F = \varnothing$, then $F \subset D_n \times I$, and therefore Waldhausen [26, Proposition 3.1] implies that F is parallel into $D_n \times \{0\} \subset P$. But then, $|\partial F| \leq n = \frac{1}{2}|\partial P|$, which contradicts our choice of P. Thus $F \cap A$ consists of a non-empty family of core curves of A. Another application of [26, Proposition 3.1] implies that up to isotopy, each component of $F \cap (D_n \times I)$ is of the form $D_n \times \{t\}$ for some $t \in (0,1)$. Since $|A \cap F|$ has been minimized, it also follows that each component of $F \cap V$ is parallel into $\overline{\partial V \setminus A}$. It is now simple to see that F is of the form $D_n \times \{t_1\} \cup B \cup D_n \times \{t_2\}$, where $0 < t_1 < t_2 < 1$ and $B \subset V$ is an annulus as described in the previous sentence. It follows that F is isotopic to P. □

Lemma 4.5 *Suppose that $M(\beta) \cong L_p \# L_q$ and that β is not a singular slope for a closed essential surface in M. Let $P \in \mathcal{S}_\beta(M)$ be chosen to have the minimal number of boundary components.*

(1) *If $M(\beta) \cong L_p \# L_q$, where $L_p \cong \pm L(p,1)$, then (X, P) is unknotted.*

(2) *If $M(\beta) \cong L_p \# L_q$, where $L_p \cong \pm L(p,1)$ and $L_q \cong \pm L(q,1)$, then each planar surface in $\mathcal{S}_\beta(M)$ is isotopic to P.*

Proof (1) Suppose that $M(\beta) \cong L_p \# L_q$, where $L_p \cong \pm L(p,1)$. We will follow Conventions 4.3; in particular, $\widehat{X}$ is the punctured L_p and $|\partial P| = 2n$. The desired conclusion follows from a combination of [11] and [29]. In order to make the application of these two papers clear, we must first set up some notation and recall some definitions.

Since M is hyperbolic, $n \geq 2$. The boundary of P cuts the boundary of M into $2n$ annuli $A_1, A_1', A_2, A_2', \ldots, A_n, A_n'$, occurring successively around ∂M, such that $\partial X = P \cup (\cup_{i=1}^n A_i)$ and $\partial X' = P \cup (\cup_{i=1}^n A_i')$. Let V be the attached solid torus used in forming $M(\beta)$. Then V may be considered as a union of $2n$ 2–handles $H_1, H_1', H_2, H_2', \ldots, H_n, H_n'$ with attaching regions $A_1, A_1', A_2, A_2', \ldots, A_n, A_n'$ respectively. Let $\widehat{X}$ be the manifold obtained from X by adding the 2–handles $H_1, \ldots, H_n$ along $A_1, \ldots, A_n$ respectively and similarly let $\widehat{X}'$ be the manifold obtained from X' by adding the 2–handles $H_1', \ldots, H_n'$ along $A_1', \ldots, A_n'$. Then $M(\beta) = M \cup V = \widehat{X} \cup_{\widehat{P}} \widehat{X}'$, where $\widehat{P}$ is the 2–sphere obtained from P by capping off ∂P with meridian disks of V. Let K be the core curve of the solid torus V. Then K is the union of $2n$ arcs $\alpha_1, \alpha_1', \alpha_2, \alpha_2', \ldots, \alpha_n, \alpha_n'$ such that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are properly embedded in $\widehat{X}$ with regular neighborhoods $H_1, H_2, \ldots, H_n$ and $\alpha_1', \alpha_2', \ldots, \alpha_n'$ are properly embedded arcs in $\widehat{X}'$ with regular neighborhoods $H_1', H_2', \ldots, H_n'$.

Consider the n–string tangle $(\widehat{X}; \alpha_1, \ldots, \alpha_n)$ in $\widehat{X}$ with strings $\alpha_1, \ldots, \alpha_n$. Let $P_i = P \cup A_i$ and call it the A_i–tubing surface of P. The surface P is said to be A_i–*tubing compressible* if P_i is compressible in X, and is said to be *completely A_i–tubing compressible* if P_i can be compressed in X until it becomes a set of annuli parallel to $\cup_{j \neq i} A_j$. The tangle $(\widehat{X}, \alpha_1, \ldots, \alpha_n)$ is called *completely tubing compressible* if it is completely A_i–tubing compressible for each of $i = 1, \dots, n$. Since M does not contain an essential torus, the argument of [11, 2.1.2] proves that $(\widehat{X}; \alpha_1, \dots, \alpha_n)$ is completely tubing compressible. Thus for each of $i = 1, \dots, n$, there exist disjoint properly embedded disks E_i^j in X, $j \neq i$, such that ∂E_i^j meets A_j in a single essential arc of A_j and is disjoint from A_k if $k \neq i, j$ (see [11, 2.1.2] for details). This in turn implies that if Ω is a proper subset of $\{H_1, \dots, H_n\}$, then the manifold obtained by attaching 2–handles from Ω to X is a handlebody. In particular, for each of $i = 1, \dots, n$, $X \cup (\cup_{j \neq i} H_j)$ is a solid torus. Thus each α_i is a core arc of $\widehat{X}$, ie its exterior in $\widehat{X}$ is a solid torus.

Recall from [29] that a band in a compact 3–manifold W, whose boundary is a 2–sphere, is an embedded disk D in W such that $\partial D \cap \partial W$ consists of two arcs on ∂D. A collection of properly embedded arcs in W is said to be parallel in W if there is a band D in W which contains all these arcs. It is proved in [29] that if W is homeomorphic to a once punctured lens space $L(p, 1)$ and $(W; \alpha_1, \dots, \alpha_n)$ is a completely tubing compressible tangle, then the arcs $\alpha_1, \dots, \alpha_n$ are parallel in W. Though this result is not explicitly stated in [29], its proof is explicitly dealt with in the proof of Theorem 1 of that paper. Hence in our current situation, $\alpha_1, \dots, \alpha_n$ are parallel arcs in $\widehat{X}$. Let D be a band in $\widehat{X}$ which contains all the arcs and H a regular neighborhood of D in $\widehat{X}$. We may assume that H contains every H_i. Since each α_i is a core arc of H, $V = \widehat{X} \setminus \mathrm{int}(H)$ is a solid torus. More precisely H can be considered as a 2–handle and $\widehat{X}$, a once punctured L_p, is obtained by attaching H to the solid torus V along an annulus A in ∂V. Thus (1) holds.

(2) Now suppose that $M(\beta) \cong L_p \# L_q$, where $L_p \cong \pm L(p, 1)$ and $L_q \cong \pm L(q, 1)$. Part (1) of this lemma implies that both (X, P) and (X', P) are unknotted. Fix a planar surface $F \in \mathcal{S}_\beta(M)$ whose boundary is disjoint from ∂P and which has been isotoped to be transverse to P so that $|F \cap P|$ has been minimized. Let $\mathcal{F}$ be the set of surfaces in $\mathcal{S}_\beta(M)$ isotopic to F and which satisfy the conditions of this paragraph.

If $F \cap P = \varnothing$, then Lemma 4.4 implies the desired result. Assume then that $F \cap P \neq \varnothing$ and consider a component C of $F \cap P$ which is innermost in the 2–sphere $\widehat{F}$. Let F_0 be a subset of F whose boundary is the union of C and k, say, components of ∂F. We assume that F and F_0 are chosen from all the surfaces in $\mathcal{F}$ so that k is minimized. Note that $k > 0$ by the minimality of $|F \cap P|$.

Without loss of generality we take $F_0 \subset X = V \cup_A (D_n \times I)$, where $A \subset \partial V$ wraps p times around V, and after an isotopy of F which preserves P, we may arrange for F_0 to be transverse to A and $|F_0 \cap A|$ to be minimal. The components of $F_0 \cap A$ are either core circles of A or arcs properly embedded in A.

First assume that $C \cap A = \varnothing$. Then $F_0 \cap A$ consists of core circles of A and an argument like that used in the proof of Lemma 4.4 implies that F_0 is parallel into P, contrary to the minimality of $|F \cap P|$. Thus $C \cap A \neq \varnothing$. It follows that $F_0 \cap A$ contains arc components. Choose such an arc α which is outermost in the disk $\widehat{F}_0$ and let D_0 be a planar subsurface of F_0 it subtends and whose interior is disjoint from A. Set $\alpha' = \overline{\partial D_0 \setminus \alpha}$.

If $D_0 \subset V$, then D_0 is a disk. If α' is an essential arc in the annulus $E = \overline{\partial V \setminus A}$, then it connects $D_n \times \{0\}$ to $D_n \times \{1\}$. Hence D_0 is a meridian disk of V and ∂D_0 is a dual curve on ∂V to the core of A. But this is impossible as A wraps $p > 1$ times around V. Thus α' is an inessential arc in E. It follows that α is inessential in A and it is easy to see that α can be eliminated from $F_0 \cap A$ by an isotopy of X, contrary to the minimality of $|F_0 \cap A|$.

Suppose next that $D_0 \subset D_n \times I$ so that $\alpha' \subset D_n \times \partial I$, say $\alpha' \subset D_n \times \{0\}$. Note, then, that α is inessential in A. An argument like that used in the previous paragraph shows that D_0 cannot be a disk. Thus $D_0 \cap \partial M \neq \varnothing$. By Waldhausen [26, Proposition 3.1], D_0 is parallel into $D_n \times \{0\}$, and it is now easy to see that F can be isotoped in M to reduce k, contrary to our choices. This contradiction completes the proof. $\square$

Proof of Proposition 3.3 Let $\mathcal{S}^0_\beta(M) \subset \mathcal{S}_\beta(M)$ consist of the surfaces in $\mathcal{S}_\beta(M)$ which are isotopic to P, and set $\mathcal{S}^1_\beta(M) = \mathcal{S}_\beta(M) \setminus \mathcal{S}^0_\beta(M)$. By hypothesis, $\mathcal{S}^1_\beta(M) \neq \varnothing$. Choose $F \in \mathcal{S}^1_\beta(M)$ so that $|\partial F| \leq |\partial F'|$ for all $F' \in \mathcal{S}^1_\beta(M)$ and let Y, Y' be the components of M split along F. Part (2) of Lemma 4.5 shows that F is not planar.

Let B be a component of $Y \cap \partial M$ and consider $F_0 = F \cup B$. Let $C_1, C_2, \ldots, C_k$ be the components of the inner boundary F_0^- of the maximal compression body of F_0 in Y. If any of the C_i are closed, Proposition 4.2 shows that β is a singular slope for a closed essential surface in M. Suppose, then, that no C_i is closed. If some C_i is essential, the fact that $|\partial C_i| < |\partial F|$ implies that $C_i \in \mathcal{S}^0_\beta(M)$ and therefore is isotopic to P. Since F is disjoint from C_i, we can isotope F into the complement of P. But this is impossible as Lemma 4.4 would then imply that $F \in \mathcal{S}^0_\beta(M)$. Thus each C_i is a ∂–parallel annulus. Similar arguments show that either β is a singular slope for a closed essential surface in M or for each component B' of $\partial M \cap Y'$, the inner boundary of the maximal compression body of $F \cup B'$ in Y' is a family of ∂–parallel annuli. Hence if β is not a singular slope for a closed essential surface in M, the arguments of of [11, Section 2.2] imply that $\widehat{F}$ is essential in $M(\beta) \cong L_p \# L_q$. This cannot occur since the genus of F is positive. Thus β is a singular slope for a closed essential surface in M. $\square$

Proof of Corollary 3.4 Suppose that β is not a singular slope for some closed essential surface in M. Then part (1) of Lemma 4.5 shows that both (X, P) and (X', P) are unknotted. Since $p = q = 2$, this implies that both (X, P) and (X', P) are twisted I–bundle pairs, and therefore, P is a semi-fibre. But then Proposition 3.3 shows that β cannot be a strict boundary slope. This completes the proof. □

5 Preliminary reductions

In this section we state four propositions which, together with known results, respectively imply our main theorems 1.1–1.4. Recall that M always denotes a compact, connected, orientable, simple 3–manifold, whose boundary is a torus.

If $M(\beta)$ is a reducible manifold, then it follows from Gordon–Luecke [19] that $\Delta(\alpha, \beta) \leq 1$ for any slope α such that $M(\alpha)$ is reducible. If $b_1(M) \geq 2$, then it follows from Boyer–Gordon–Zhang [4, Proposition 5.1], that $\Delta(\alpha, \beta) \leq 1$ for any slope α such that $M(\alpha)$ is not hyperbolic. The conclusions of all four of the main theorems hold when $\Delta(\alpha, \beta) \leq 1$. Thus, in the proofs of these theorems, we may assume, without loss of generality, that $M(\alpha)$ is irreducible and $b_1(M) = 1$.

Next we recall that, since β is a boundary slope, it follows from Proposition 3.5 that one of the following three possibilities occurs:

(1) β is a singular slope for a closed essential surface in M; or

(2) $M(\beta)$ is homeomorphic to $L_p \# L_q$, where $q > 2$; or

(3) $M(\beta) = S^2 \times S^1$ or $P^3 \# P^3$ and β is not a strict boundary slope.

Since the conclusion of Proposition 3.1 implies that of each of the four main theorems, we may also assume that neither α nor β is a singular slope for any closed essential surface in M.

Therefore Theorems 1.1–1.4 follow, respectively, from the following four propositions.

Proposition 5.1 *Suppose that $b_1(M) = 1$ and neither α nor β is a singular slope for a closed, essential surface in M. Assume as well that $M(\beta)$ is either a connected sum of two non-trivial lens spaces or $S^1 \times S^2$. If $M(\alpha)$ has finite fundamental group, then $\Delta(\alpha, \beta) \leq 2$. Furthermore, if $\Delta(\alpha, \beta) = 2$, then $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, $M(\beta) \cong L_2 \# L_3$ and $\pi_1(M(\alpha)) \cong O^*_{24} \times \mathbb{Z}/j$, where O^*_{24} is the binary octahedral group.*

Proposition 5.2 *Suppose that $b_1(M) = 1$ and neither α nor β is a singular slope for a closed, essential surface in M. Assume as well that $M(\alpha)$ is irreducible and $M(\beta)$ is either a connected sum of two non-trivial lens spaces or $S^1 \times S^2$. If $M(\alpha)$ is a very small manifold and β is a strict boundary slope, then $\Delta(\alpha, \beta) \leq 3$.*

Proposition 5.3 *Suppose that $b_1(M) = 1$ and neither α nor β is a singular slope for a closed, essential surface in M. Assume as well that $M(\alpha)$ is irreducible and $M(\beta)$ is either a connected sum of two non-trivial lens spaces or $S^1 \times S^2$. If $M(\alpha)$ is a very small manifold which admits a geometric decomposition, then $\Delta(\alpha, \beta) \leq 2$.*

Proposition 5.4 *Suppose that $b_1(M) = 1$ and neither α nor β is a singular slope for a closed, essential surface in M. Assume as well that $M(\beta)$ is a connected sum of two non-trivial lens spaces. If β is a strict boundary slope and $M(\alpha)$ admits a π_1–injective immersion of a torus, then $\Delta(\alpha, \beta) \leq 4$. Moreover, if $\Delta(\alpha, \beta) = 4$, then $M(\alpha)$ is a Seifert fibred space with base orbifold $\mathcal{B}$ of $M(\alpha)$ of the form $S^2(r, s, t)$, where (r, s, t) is a hyperbolic triple and 4 divides at least one of r, s, t.*

These four propositions will be proved in Sections 8, 9, 10 and 12 respectively.

6 Background results on $PSL_2(\mathbb{C})$–character varieties

In this section we gather together some background material on $PSL_2(\mathbb{C})$–character varieties that will be used in the proofs of our main results. See Culler–Shalen [12], Culler–Gordon–Luecke–Shalen [11], and Boyer–Zhang [5; 6; 9] for more details. As above, M will denote a compact, connected, orientable, hyperbolic 3–manifold with boundary a torus.

Definitions 6.1 Let π be a finitely generated group. We shall denote by $R_{PSL_2}(\pi)$ and $X_{PSL_2}(\pi)$, respectively, the $PSL_2(\mathbb{C})$–representation variety and the $PSL_2(\mathbb{C})$–character variety of π. (Note that these are affine algebraic sets, but are not necessarily irreducible.) The map $t\colon R_{PSL_2}(\pi) \to X_{PSL_2}(\pi)$ which sends a representation ρ to its character χ_ρ is a regular map. When π is the fundamental group of a path-connected space Y, we will frequently denote $R_{PSL_2}(\pi)$ by $R_{PSL_2}(Y)$ and $X_{PSL_2}(\pi)$ by $X_{PSL_2}(Y)$.

There is a unique conjugacy class of homomorphisms $\eta\colon H_1(\partial M) \to \pi_1(M)$, obtained by composing the inverse of the Hurewicz isomorphism $\pi_1(\partial M) \to H_1(\partial M)$ with some homomorphism $\pi_1(\partial M) \to \pi_1(M)$ induced by inclusion. To simplify notation, we shall often suppress η in statements that are invariant under conjugation in $PSL_2(\mathbb{C})$. For instance, given $\rho \in R_{PSL_2}(\pi)$ and $\alpha \in H_1(\partial M)$, we may write $\rho(\alpha) = \pm I$ to indicate that $\eta(\alpha)$ is contained in the kernel of ρ for every choice of η.

By a *curve* in an affine algebraic set we will mean an irreducible algebraic subset of dimension 1. Suppose that X_0 is a curve in $X_{PSL_2}(M)$ and let $\widetilde{X}_0$ denote the smooth projective model of X_0. There is a canonically defined quasi-projective curve $X_0^\nu \subset \widetilde{X}_0$

which consists of all points of $\widetilde{X_0}$ that correspond to points of X_0. In particular, there is a regular, surjective, birational isomorphism $\nu\colon X_0^\nu \to X_0$. The points of X_0^ν are called *ordinary* points and the points in the finite set $\widetilde{X_0} - X_0^\nu$ are called *ideal* points. It follows from [6, Lemma 4.1] that for every curve X_0 in $X_{PSL_2}(M)$ there exists an algebraic component $R(X_0)$ of $R_{PSL_2}(M)$ such that $t(R(X_0)) = X_0$.

To each homology class $a \in H_1(\partial M)$ we can associate a regular function $f_a\colon X_0 \to \mathbb{C}$ given by $f_a(\chi) = \chi(a)^2 - 4$. Each f_a lifts to a rational function, also denoted by f_a, on $\widetilde{X_0}$. It is shown in [11] (see also [6]) that the degrees of these functions on $\widetilde{X_0}$ vary in a coherent fashion. Indeed, there is a seminorm $\|\cdot\|_{X_0}\colon H_1(\partial M;\mathbb{R}) \to [0,\infty)$, called the *Culler–Shalen seminorm* of X_0, determined by the condition that for each $a \in H_1(\partial M)$, $\|a\|_{X_0}$ is the degree of f_a on $\widetilde{X_0}$. As in [11], we use $Z_x(f)$ to denote the order of zero of a rational function f on $\widetilde{X_0}$ at a point $x \in \widetilde{X_0}$, and use $\Pi_x(f)$ to denote the order of pole of f at a point $x \in \widetilde{X_0}$. Then

$$\|a\|_{X_0} = \sum_{x\in\widetilde{X_0}} Z_x(f_a) = \sum_{x\in\widetilde{X_0}} \Pi_x(f_a). \tag{6–1}$$

If $\|\cdot\|_{X_0} \neq 0$, we define

$$s_{X_0} = \min\{\|a\|_{X_0} \mid a \in H_1(\partial M), \|a\|_{X_0} \neq 0\} \in \mathbb{Z}\setminus\{0\}.$$

We note that $f_a = f_{-a}$. As a notational convenience, if $\alpha = \{\pm a\}$ is a slope on ∂M, then we shall set $f_\alpha \doteq f_a = f_{-a}$, and define $\|\alpha\|_{X_0} \doteq \|a\|_{X_0} = \|-a\|_{X_0}$.

It is possible that $\|\cdot\|_{X_0} \neq 0$, but $\|\beta\|_{X_0} = 0$ for some slope β on ∂M. In this case the slope β is the unique slope on ∂M of norm 0, and we shall call X_0 a *β–curve*. If X_0 is a β–curve, then for any slope α on ∂M we have

$$\|\alpha\|_{X_0} = \Delta(\alpha,\beta)s_{X_0}. \tag{6–2}$$

Hence if β^* is a *dual slope* for β, that is, a slope such that $\Delta(\beta,\beta^*) = 1$, then

$$s_{X_0} = \|\beta^*\|_{X_0}.$$

If β is any slope on ∂M, then we may regard the character variety $X_{PSL_2}(M(\beta))$ as an algebraic subset of $X_{PSL_2}(M)$. To see this, note that $R_{PSL_2}(M(\beta))$ can be identified with the Zariski closed, conjugation invariant subset $R_\beta(M) := \{\rho \in R_{PSL_2}(M) : \rho(\beta) = \pm I\}$ of $R_{PSL_2}(M)$. Newstead [22, Theorem 3.3.5(iv)] shows that the image of $R_\beta(M)$ in $X_{PSL_2}(M)$ is Zariski closed and can be identified with $X_{PSL_2}(M(\beta))$. We note that if X_0 is a curve in $X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$ such that $\|\cdot\|_{X_0} \neq 0$, then X_0 is a β–curve.

The following proposition is proved by Boyer [2].

Proposition 6.2 (Boyer [2, Proposition 6.2]) *Let $X_0 \subset X_{PSL_2}(M(\beta))$ be a β–curve for a slope β on ∂M. Let β^* be a dual slope for β and let $\alpha \neq \beta$ be a slope on ∂M. Then*

(1) *For any point $x \in X_0^\nu$ and any representation ρ such that $\chi_\rho = \nu(x)$ we have*
   (a) *If $Z_x(f_\alpha) > 0$, then $\rho(\pi_1(\partial M))$ is either parabolic, or a finite cyclic group whose order divides $\Delta(\alpha, \beta)$; and*
   (b) *$Z_x(f_\alpha) \geq Z_x(f_{\beta^*})$, with equality if and only if $\rho(\pi_1(\partial M))$ is parabolic or trivial.*

(2) *If f_{β^*} has a pole at each ideal point of $\widetilde{X}_0$, then for every divisor $d > 1$ of $\Delta(\alpha, \beta)$ there exists $x \in X_0^\nu$ such that $Z_x(f_\alpha) > Z_x(f_{\beta^*})$, and $\rho(\pi_1(\partial M))$ is a cyclic group of order d for every representation ρ such that $\chi_\rho = \nu(x)$.*

We call a subvariety X_0 of $X_{PSL_2}(M)$ *non-trivial* if it contains the character of an irreducible representation.

For some applications we need a stronger condition on X_0 than non-triviality. A character $\chi_\rho \in X_0$ is called *virtually reducible* if there is a finite index subgroup $\widetilde{\pi}$ of $\pi_1(M)$ such that $\rho|\widetilde{\pi}$ is reducible. We will say that X_0 is *virtually trivial* if every point of X_0 is a virtually reducible character. The proof of Boyer–Zhang [9, Proposition 4.2] shows that if a curve X_0 in $X_{PSL_2}(M)$ is non-trivial, but contains infinitely-many virtually reducible characters, then X_0 is virtually trivial and X_0 is a curve of characters of representations $\pi_1(M) \to \mathcal{N} \subset PSL_2(\mathbb{C})$ where

$$\mathcal{N} = \left\{ \pm\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \pm\begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} \;\middle|\; z, w \in \mathbb{C}^* \right\} \subset PSL_2(\mathbb{C}).$$

Ideal points, essential surfaces, and singular slopes

One of the key relations between 3–manifold topology and $PSL_2(\mathbb{C})$–character varieties is the construction described in Culler–Shalen [12], which associates essential surfaces in a 3–manifold M to ideal points of curves in $X_{PSL_2}(M)$.

Proposition 6.3 (Culler–Shalen [12], Culler–Gordon–Luecke–Shalen [11, Section 1.3], Boyer–Zhang [6]) *Let X_0 be a non-trivial curve in $X_{PSL_2}(M)$ and x an ideal point of X_0. One of the following mutually exclusive alternatives holds: Either*

(1) *there is a unique slope α on ∂M such that f_α is finite-valued at x; or*

(2) *f_α is finite-valued for every slope α on ∂M.*

In case (1) the slope α is a boundary slope. Moreover, if X_0 is not virtually trivial, then α must be a strict boundary slope. In case (2) M contains a closed, essential surface.

If, as in case (1) of the proposition, there is a unique slope α on ∂M such that $f_\alpha(x) \in \mathbb{C}$, we say that the boundary slope α is *associated to* x.

Proposition 6.4 (Boyer–Zhang [6, Propositions 4.10 and 4.12]) *Suppose that x is an ideal point of a non-trivial curve X_0 in $X_{PSL_2}(M)$ and that β is a slope on ∂M such that every closed, essential surface in M associated to x is compressible in $M(\beta)$. Suppose further that f_δ is finite-valued at x for every slope δ on ∂M. If either*

- $X_0 \subseteq X_{PSL_2}(M(\beta))$, *or*
- $Z_x(f_\beta) > Z_x(f_\delta)$ *for some slope δ on ∂M*

then β is a singular slope for some closed essential surface in M.

The PSL_2 character variety of $L_p \# L_q$

It was shown in Boyer–Zhang [6, Example 3.2] that $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ is a disjoint union of a finite number of isolated points and $[\frac{p}{2}][\frac{q}{2}]$ non-trivial curves, each isomorphic to a complex line. If we fix generators x and y of the two cyclic free factors of $\mathbb{Z}/p * \mathbb{Z}/q$, then each curve consists of characters of representations which send x and y to elliptic elements of orders dividing p and q respectively. Such a curve is parametrized by the complex distance between the axes of these two elliptic elements.

Explicit parametrizations of the curves in $X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ can be given as follows. For integers j, k with $1 \leq j \leq [\frac{p}{2}]$ and $1 \leq k \leq [\frac{q}{2}]$, set

$$\lambda = e^{\pi i j/p}, \mu = e^{\pi i k/q}, \tau = \mu + \mu^{-1}.$$

For $z \in \mathbb{C}$ define $\rho_z \in R_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ by

$$\rho_z(x) = \pm\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \overline{\rho}_z(y) = \pm\begin{pmatrix} z & 1 \\ z(\tau - z) - 1 & \tau - z \end{pmatrix}.$$

The characters of the representations ρ_z parameterize a curve $X(j,k) \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$. Moreover, the correspondence $\mathbb{C} \to X(j,k)$, $z \mapsto \chi_{\rho_z}$, is bijective if $j < [\frac{p}{2}]$ and $k < [\frac{q}{2}]$ and a 2-1 branched cover otherwise.

We shall denote by D_k the dihedral group of order $2k$. Recall that a finite subgroup of $PSL_2(\mathbb{C})$ is either cyclic or dihedral, or else it is isomorphic to the tetrahedral group T_{12}, the octahedral group O_{24}, or the icosahedral group I_{60}.

The following elementary, but tedious, lemma characterizes the points in the curve $X(j,k)$ which correspond to the character of a representation with finite image. We leave its verification to the reader.

Lemma 6.5 *Fix integers $2 \leq p \leq q$. Let $X_{p,q}$ be the union of all curves $X(j,k) \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$ such that j and k are relatively prime to p and q respectively. Then*

(1) *An irreducible component $X(j,k) \subset X_{p,q}$ contains exactly two reducible characters if $p > 2$, and one if $p = 2$.*

(2) *An irreducible component $X(j,k) \subset X_{p,q}$ contains the character of an irreducible representation ρ whose image lies in $\mathcal{N}$ if and only if $p = 2$. Moreover, if $p = 2$ and $q > 2$, then there is exactly one such character χ_ρ and the image of ρ is D_q.*

(3) *$X_{p,q}$ contains the character of a representation whose image is T_{12} if and only if $(p,q) \in \{(2,3),(3,3)\}$. If $(p,q) = (2,3)$ there is a unique such character and if $(p,q) = (3,3)$, then there are two.*

(4) *$X_{p,q}$ contains the character of a representation whose image is O_{24} if and only if $(p,q) \in \{(2,3),(2,4),(3,4),(4,4)\}$. If $(p,q) = (3,4)$ there are two such characters, and in the remaining cases there is only one.*

(5) *$X_{p,q}$ contains the character of a representation whose image is I_{60} if and only if $(p,q) \in \{(2,3),(2,5),(3,3),(3,5),(5,5)\}$. There are eight such characters if $(p,q) = (3,5)$ or $(p,q) = (5,5)$, four if $(p,q) = (2,5)$, and two if $(p,q) = (2,3)$ or $(p,q) = (3,3)$.*

The next result follows from Proposition 6.4 and work of Culler, Shalen and Dunfield. Recall that if $X_0 \subset X_{PSL_2}(M)$ is a β–curve and β^* is a dual class to β, then $s_{X_0} = \|\beta^*\|_{X_0} = \sum_{x \in \widetilde{X}_0} \Pi_x(f_{\beta^*})$.

Proposition 6.6 *Suppose that $M(\beta) \cong L_p \# L_q$ and let x be an ideal point of the curve $X(j,k) \subset X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$. Then either*

(1) *β is a singular slope for a closed essential surface in M, or*

(2) *$\|\cdot\|_{X(j,k)} \neq 0$ and*

$$s_{X(j,k)} \geq \Pi_x(f_{\beta^*}) \geq \begin{cases} 4 & \text{if } j \neq \frac{p}{2} \text{ and } k \neq \frac{q}{2} \\ 2 & \text{if either } j = \frac{p}{2} \text{ or } k = \frac{q}{2}. \end{cases}$$

Proof Suppose that β is not a singular slope for a closed essential surface in M. Then Proposition 6.4 implies that for each ideal point x of $X(j,k)$ and for each slope $\alpha \neq \beta$, we have $f_\alpha(x) = \infty$.

The natural surjection

$$\phi\colon \mathbb{Z}/2p * \mathbb{Z}/2q \to \mathbb{Z}/p * \mathbb{Z}/q$$

induces an inclusion

$$\phi^*\colon X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q) \to X_{PSL_2}(\mathbb{Z}/2p * \mathbb{Z}/2q).$$

Given a curve $X_0 \subset X_{PSL_2}(L_p \# L_q) = X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q)$, there is a curve $Y_0 \subset X_{SL_2}(\mathbb{Z}/2p * \mathbb{Z}/2q)$ whose image in $X_{PSL_2}(\mathbb{Z}/2p * \mathbb{Z}/2q)$ coincides with $\phi^*(X_0)$. The associated regular map $g\colon Y_0 \to X_0$ has degree 1 if $j \neq \frac{p}{2}$ and $k \neq \frac{q}{2}$ and is of degree 2 otherwise. Now Y_0 is also a complex line and so has a unique ideal point y. Extend g to a map $\widetilde{g}\colon \widetilde{Y_0} \to \widetilde{X_0}$ between the smooth projective models, and observe that $\widetilde{g}(y) = x$. If $\widetilde{\beta}^* \in \phi^{-1}(\beta^*)$ it is easy to see that $f_{\beta^*} \circ \widetilde{g} = f_{\widetilde{\beta}^*}$. It can be shown that

$$\Pi_x(f_{\beta^*}) = \begin{cases} \Pi_y(f_{\widetilde{\beta}^*}) & \text{if } j \neq \frac{p}{2} \text{ and } k \neq \frac{q}{2} \\ \frac{1}{2}\Pi_y(f_{\widetilde{\beta}^*}) & \text{if either } j = \frac{p}{2} \text{ or } k = \frac{q}{2}. \end{cases}$$

We are reduced, then, to calculating $\Pi_y(f_{\widetilde{\beta}^*})$.

According to Dunfield [13, Proposition 2.2], we may choose the simplicial tree T_y associated to y so that $\Pi_y(f_{\widetilde{\beta}^*})$ equals the translation length $l(\widetilde{\beta}^*)$ of the automorphism of T_y associated to $\widetilde{\beta}^*$. Now the action of $\mathbb{Z}/2p * \mathbb{Z}/2q$ on T_y factors through an action of $\mathbb{Z}/p * \mathbb{Z}/q$, which in turn determines an action of $\pi_1(M)$ on T_y via the surjection $\pi_1(M) \to \pi_1(M(\beta)) = \mathbb{Z}/p * \mathbb{Z}/q$. In particular, $l(\widetilde{\beta}^*) = l(\beta^*)$, where we have identified β^* with its image in $\pi_1(M)$ under one of the homomorphisms in the conjugacy class η (see Definitions 6.1).

Consider now an essential surface F properly embedded in M which is dual to the action of $\pi_1(M)$. The observation above implies that F can be chosen so that $|\partial F| = l(\beta^*)$. Let F_0 be a component of F with non-empty boundary. Note that $|\partial F_0|$ is even since F_0 is separating in M. If $|\partial F_0| = 2$, then the genus of F_0 is at least 1 since M is hyperbolic. The proof of [11, Theorem 2.0.3] then shows that β is the singular slope for some closed essential surface, contrary to our hypotheses. Hence

$$\Pi_y(f_{\widetilde{\beta}^*}) = l(\widetilde{\beta}^*) = l(\beta^*) = |\partial F| \geq 4$$

and

$$\Pi_x(f_{\beta^*}) \geq \begin{cases} 4 & \text{if } j \neq \frac{p}{2} \text{ and } k \neq \frac{q}{2} \\ 2 & \text{if either } j = \frac{p}{2} \text{ or } k = \frac{q}{2}. \end{cases}$$

□

Jumps in multiplicities of zeroes

Let X_0 be a non-trivial curve in $X_{PSL_2}(M)$ Recall that $R(X_0)$ is the unique 4–dimensional subvariety of $R_{PSL_2}(M)$ satisfying $t(R(X_0)) = X_0$. Suppose that α is a slope on ∂M such that $f_\alpha | X_0 \neq 0$. As a means to estimate $\|\alpha\|_{X_0}$, we will be interested in the set

$$J_{X_0}(\alpha) = \{x \in \widetilde{X}_0 \mid Z_x(f_\alpha) > Z_x(f_\delta) \text{ for some slope } \delta \text{ such that } f_\delta \neq 0\}.$$

Lemma 6.7 *Suppose that $x \in J_{X_0}(\alpha)$ is not an ideal point.*

(1) *If $\chi_\rho = \nu(x)$ then $\rho(\alpha) = \pm I$. [11, Proposition 1.5.4]*

(2) *If $b_1(M) = 1$, there exists a representation ρ, which is either irreducible or has non-Abelian image, such that $\chi_\rho = \nu(x)$. [2, Proposition 2.8]*

(Note that there exist irreducible $PSL_2(\mathbb{C})$ representations whose image is a Klein 4–group, and hence is Abelian.)

Lemma 6.8 *Let $X_0 \subset X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$ be a β–curve for a slope β on ∂M. Let $\beta^* = \{\pm b^*\}$ be a dual slope for β. Suppose that α is a slope on ∂M such that $\Delta(\alpha, \beta) > 1$. For any non-ideal point $x \in J_{X_0}(\alpha)$ and any representation ρ such that $\chi_\rho = \nu(x)$ we have that $\rho(b^*)$ is an elliptic element with order d for some divisor $d > 1$ of $\Delta(\alpha, \beta)$.*

Proof First observe that for any slope δ on ∂M we have $f_\delta = f_{\Delta(\delta,\beta)\beta^*}$ and so $Z_x(f_\delta) = \Delta(\delta, \beta) Z_x(f_{\beta^*})$. In particular, since $x \in J_{X_0}(\alpha)$, we must have $Z_x(f_{\beta^*}) > 0$. Thus $Z_x(f_\alpha) = \Delta(\alpha, \beta) Z_x(f_{\beta^*}) > Z_x(f_{\beta^*})$. It now follows from Proposition 6.2 that $\rho(\pi_1(\partial M))$ is a cyclic group of order $d > 1$, where d divides $\Delta(\alpha, \beta)$. Since this cyclic group is generated by $\rho(b^*)$, the lemma follows. □

Proposition 6.9 *Let $X_0 \subset X_{PSL_2}(M)$ be a non-trivial curve and let α be a slope on ∂M such that $f_\alpha | X_0 \neq 0$. Suppose that there is no closed, essential surface in M which remains essential in $M(\alpha)$. If $x \in J_{X_0}(\alpha)$ is an ideal point, then either*

(1) *α is a singular slope for a closed, essential surface in M, or*

(2) *for any slope $\beta \neq \alpha$, f_β has a pole at x. In particular, α is a boundary slope and X_0 is not a β–curve. Moreover, if $b_1(M) = 1$, then $M(\alpha)$ is either a Haken manifold, $S^1 \times S^2$, or a connected sum of two non-trivial lens spaces.*

Proof Suppose that α is not a singular slope for a closed, essential surface in M. It then follows from Proposition 6.4 that for any slope $\beta \neq \alpha$ the function f_β has a pole at x. Hence Proposition 6.3 shows that α is a boundary slope. Finally if $b_1(M) = 1$, we can apply Theorem 3.2 to deduce that $M(\alpha)$ is either Haken, $S^1 \times S^2$, or is a connected sum of two non-trivial lens spaces. $\square$

Proposition 6.10 *Let X_0 be a non-trivial curve in $X_{PSL_2}(M)$ and α a slope on ∂M such that $f_\alpha | X_0 \neq 0$. Suppose that $J_{X_0}(\alpha)$ contains an ordinary point x of X_0^ν and that there exists a representation ρ, which is either irreducible or has non-Abelian image, such that $\chi_\rho = \nu(x)$. If either*

(i) *$H^1(M(\alpha); sl_2(\mathbb{C})_\rho) = 0$ and $\rho(\pi_1(\partial M)) \neq \{\pm I\}$, or*

(ii) *there is a slope β such that $X_0 \subset X_{PSL_2}(M(\beta))$ and $H^1(M; sl_2(\mathbb{C})_\rho) \cong \mathbb{C}$ (for instance the latter holds when $M(\beta) \cong L_p \# L_q$),*

then

$$Z_x(f_\alpha) = \begin{cases} Z_x(f_\beta) + 1 & \text{if } \rho \text{ is conjugate into } \mathcal{N}; \\ Z_x(f_\beta) + 2 & \text{otherwise.} \end{cases}$$

Moreover, in case (i) *$\nu(x)$ is a simple point of $X_{PSL_2}(M)$ and in case* (ii) *$\nu(x)$ is a simple point of $X_{PSL_2}(M(\beta))$.*

Proof If hypothesis (i) holds the conclusion follows from Ben Abdelghani–Boyer [1, Theorem 2.1].

Assume that hypothesis (ii) holds. Let β^* be a dual slope to β and fix simple closed curves a, b and b^* on ∂M such that $\alpha = \{\pm[a]\}$, $\beta = \{\pm[b]\}$ and $\beta^* = \{\pm[b]^*\}$. We also identify $[a]$, $[b]$ and $[b^*]$ with their images under a homomorphism in the conjugacy class η (see Definitions 6.1).

Observe that Proposition 6.2 implies that $\rho(\pi_1(\partial M))$ is a non-trivial, finite cyclic group. Thus, $\rho(\pi_1(\partial M))$ is generated by $\rho([b^*])$. After possibly replacing ρ by a conjugate representation, we may assume that

$$\rho([b^*]) = \pm \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

where $t \neq \pm 1$.

Since $X_0 \subset X_{PSL_2}(M(\beta))$ and $H^1(M; sl_2(\mathbb{C})_\rho) \cong \mathbb{C}$, Boyer [2, Theorem A] holds in our situation. In particular, the Zariski tangent space of X_0 at χ_ρ can be identified with $H^1(M(\beta); sl_2(\mathbb{C})_\rho) \cong \mathbb{C}$. We can therefore find a 1–cocycle $u \in Z^1(M(\beta); sl_2(\mathbb{C})_\rho)$ such that $\bar{u} \neq 0 \in H^1(M(\beta); sl_2(\mathbb{C})_\rho)$ and an analytic curve χ_{ρ_s} in X_0 of the form

$\rho_s = \exp(su + O(s^2))\rho$ defined for $|s|$ small. Applying the arguments of [1, Section 1.1.1 and Section 1.2.1] to this curve, modified to the $PSL_2(\mathbb{C})$ setting (cf [1, Section 2]), shows that the identities

$$Z_x(f_\alpha) = \begin{cases} Z_x(f_\beta) + 1 & \text{if } \rho \text{ is conjugate into } \mathcal{N}; \\ Z_x(f_\beta) + 2 & \text{otherwise.} \end{cases}$$

hold as long as we can prove that $u([a]) \neq 0$.

Suppose that $u([a]) = 0$ in order to arrive at a contradiction. We also have $u([b]) = 0$, since $u \in Z^1(M(\beta); sl_2(\mathbb{C})_\rho)$, and thus $u(m[a] + n[b]) = 0$ for each pair of integers m, n. Let $u([b^*]) = \left(\begin{smallmatrix} p & q \\ r & -p \end{smallmatrix}\right)$. We have assumed that $f_\alpha|X_0 \neq 0$, and therefore $[a]$ and $[b]$ span a subgroup of index $k < \infty$ of $H_1(\partial M)$. Then

$$\begin{aligned} 0 = u([b^*])^k &= \sum_{j=0}^{k-1} \rho([b^*])^j u([b^*]) \rho([b^*])^{-j} \\ &= \begin{pmatrix} kp & (1 + t^2 + \cdots + t^{2(k-1)})q \\ (1 + t^{-2} + \cdots + t^{-2(k-1)})r & -kp \end{pmatrix}, \end{aligned}$$

and therefore $p = 0$. Consider the coboundary $\delta^0\colon sl_2(\mathbb{C}) \to Z^1(M(\beta); sl_2(\mathbb{C})_\rho)$ given by $(\delta^0(A))(w) = A - \rho(w)A\rho(w)^{-1}$ and set

$$u_1 = u - \delta^0\left(\begin{pmatrix} 0 & \frac{q}{1-t^2} \\ \frac{r}{1-t^{-2}} & 0 \end{pmatrix}\right).$$

Since $\rho([b]) = \pm I$ we have $u_1([b]) = u([b]) = 0$, while the fact that $\rho([b^*]) = \pm\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ implies that $u_1([b^*]) = 0$ also. Hence $u_1 = 0$, which is impossible as $0 \neq \overline{u} = \overline{u}_1 = 0$.

Finally, if $M(\beta) \cong L_p * L_q$ we have $\pi_1(M(\beta)) \cong \mathbb{Z}/p * \mathbb{Z}/q$. A simple calculation shows that the space of 1–cocycles $Z^1(M(\beta); sl_2(\mathbb{C})_\rho)$ is isomorphic to $\mathbb{C}^4$. Thus $H^1(M(\beta); sl_2(\mathbb{C})_\rho) \cong \mathbb{C}$. This completes the proof. □

7 $PSL_2(\mathbb{C})$–representations of fundamental groups of very small 3–manifolds

We begin by considering a 3–manifold W which fibres over S^1 with fibre a torus T and monodromy A. It is known that W is a Sol manifold if and only if $|\operatorname{tr}(A)| > 2$ and a Seifert fibred space otherwise. Similarly, if W semi-fibres over the interval with semi-fibre a torus T and gluing map $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then W is a Sol manifold if and only if $ad \neq 0, 1$, and a Seifert fibred space otherwise.

Proposition 7.1 *Suppose that W either fibres over the circle with torus fibre or semi-fibres over the interval with torus semi-fibre. If $\rho\colon \pi_1(W) \to PSL_2(\mathbb{C})$ is irreducible, then up to conjugation, the image of ρ is T_{12}, or O_{24}, or lies in $\mathcal{N}$. Moreover,*

- *if the image is T_{12}, then $\rho(\pi_1(T)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and W fibres over S^1;*
- *if the image is O_{24}, then $\rho(\pi_1(T)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and W semi-fibres over the interval.*

Proof Let T denote the (semi-)fibre and consider the normal subgroup $G = \rho(\pi_1(T))$ of $\rho(\pi_1(W))$. We can conjugate G so that it equals $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \subset \mathcal{N}$, or it is contained in either $\mathcal{P}$, the group of upper-triangular parabolic matrices, or $\mathcal{D}$, the group of diagonal matrices.

If $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, a simple calculation implies that $\rho(\pi_1(W))$ is finite. The only finite subgroups of $PSL_2(\mathbb{C})$ which contain such a normal subgroup are T_{12}, O_{24}, and the dihedral group $D_2 \subset \mathcal{N}$. The first possibility is ruled out when T separates $M(\alpha)$ into two twisted I–bundles over the Klein bottle, since otherwise ρ would induce a surjection of $\mathbb{Z}/2 * \mathbb{Z}/2 = \pi_1(W)/\pi_1(T)$ onto $T_{12}/(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = \mathbb{Z}/3$, which is impossible. Similarly if T does not separate, then the image of ρ cannot be O_{24}.

Next we can rule out the possibility that $\{\pm I\} \neq G \subset \mathcal{P}$ since if this case did arise, the normality of G in $\rho(\pi_1(W))$ would then imply that ρ is reducible.

Finally assume that $G \subset \mathcal{D}$. If $G = \{\pm I\}$, then ρ factors through $\pi_1(W)/\pi_1(T)$, which is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$. The irreducibility of ρ excludes the former possibility while the lemma clearly holds in the latter. If $\{\pm I\} \neq G \subset \mathcal{D}$ is non-trivial, then its normality in $\rho(\pi_1(W))$ implies that the latter is a subset of $\mathcal{N}$. □

Proposition 7.2 *Let W be a torus bundle over S^1 with monodromy $A \in SL_2(\mathbb{Z})$ and fibre T. Consider a representation $\rho\colon \pi_1(W) \to PSL_2(\mathbb{C})$ which is either irreducible or has non-Abelian image.*

(1) *If ρ is irreducible, then $H^1(W; sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho}) = 0$ as long as $\mathrm{tr}(A) \neq -2$.*

(2) *If ρ is reducible and W fibres over the circle and the image of ρ contains non-trivial torsion, then it is Seifert fibred. Moreover, if there is torsion of order greater than 2, then $|\,\mathrm{tr}(A)| \leq 1$.*

Proof Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and recall that there is a presentation of $\pi_1(W)$ of the form

$$\langle x, y, t \mid [x, y] = 1, txt^{-1} = x^a y^c, tyt^{-1} = x^b y^d \rangle,$$

where x, y generate $\pi_1(T)$ and t projects to a generator $\overline{t}$ of $\pi_1(S^1) \cong \mathbb{Z}$.

(1) Consider the exact sequence $1 \to \pi_1(T) \to \pi_1(W) \to \mathbb{Z} \to 1$. The Lyndon–Serre spectral sequence yields an associated exact sequence in cohomology

$$0 \longrightarrow H^1(\mathbb{Z}; (sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\pi_1(T)}) \longrightarrow H^1(\pi_1(W); sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho}) \longrightarrow H^1(T; sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\mathbb{Z}} \longrightarrow 0.$$

Since ρ is irreducible, we have either $\rho(\pi_1(T)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, or $\{\pm I\} \neq \rho(\pi_1(T)) \subset \mathcal{D}$ and $\rho(\pi_1(W)) \subset \mathcal{N}$.

If $\rho(\pi_1(T)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, then $(sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\pi_1(T)} = 0$. On the other hand, using duality with twisted coefficients and the fact that $\chi(T; \mathrm{Ad}\,\rho) = 3\chi(T) = 0$, we see that the associated Betti numbers satisfy $b_1(T; sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho}) = 2b_0(T; sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})$. But since $\rho|\pi_1(T)$ is irreducible, we have $b_0(T; sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho}) = 0$. Thus $H^1(T; sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\mathbb{Z}} = 0$, which implies the desired result.

Next suppose that $\{\pm I\} \neq \rho(\pi_1(T)) \subset \mathcal{D}$ and $\rho(\pi_1(W)) \subset \mathcal{N}$. In this case

$$(sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\pi_1(T)} = \left\{ \left(\begin{smallmatrix} z & 0 \\ 0 & -z \end{smallmatrix} \right) \mid z \in \mathbb{C} \right\} \cong \mathbb{C}.$$

The irreducibility of ρ implies that up to conjugation we may suppose that $\rho(t) = \pm \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and therefore $\mathbb{Z}$ acts on $(sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\pi_1(T)}$ by multiplication by -1. Thus the set of invariants of this action, which is isomorphic to $H^0(\mathbb{Z}; (sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\pi_1(T)})$, is 0. Duality then yields $H^1(\mathbb{Z}; (sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\pi_1(T)}) = 0$.

On the other hand, it is easy to see that $H^1(\pi_1(T); sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})$ may be identified with the set of homomorphisms of $\pi_1(T)$ into $\mathbb{C}$ in such a way that if f is such a homomorphism, then $\overline{t}$ acts on f as

$$(\overline{t} \cdot f)(x^m y^n) = -f(x^{am+bn} y^{cm+dn}) = -(am+bn) f(x) - (cm+dn) f(y).$$

Hence f is invariant under the action of $\overline{t}$ if and only if $(f(x), f(y))$ is a (-1)–eigenvector of the transpose of A. It follows that

$$H^1(\pi_1(W); sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho}) \cong H^1(T; sl_2(\mathbb{C})_{\mathrm{Ad}\,\rho})^{\mathbb{Z}} \neq 0$$

if and only if $\mathrm{tr}(A) = -2$.

(2) Write $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$. As ρ is reducible with non-Abelian image, we must have $\{\pm I\} \neq \rho(\pi_1(T)) \subset \mathcal{P} \cong \mathbb{C}$. Then the image of ρ lies in $\mathcal{U}$. Suppose that

$$\rho(x) = \pm \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}, \quad \rho(y) = \pm \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad \rho(t) = \pm \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}.$$

Since the kernel of the projection $\mathcal{U} \to \mathcal{D}$ is $\mathcal{P} \cong \mathbb{C}$, any torsion element in the image of ρ is sent to an element of the same order in $\mathcal{D}$ under this projection. On the other hand, since any element of $\pi_1(W)$ can be written as a product of the form $x^l y^m t^n$, the image of $\rho(\pi_1(W))$ under the projection to $\mathcal{D}$ is isomorphic to $\{u^n \mid n \in \mathbb{Z}\} \subset \mathbb{C}^*$. Thus $\rho(\pi_1(W))$ contains a non-trivial torsion element if and only if u is a non-trivial root of unity. Assume this occurs. The relations in the presentation for $\pi_1(W)$ imply that

$$u^2\sigma = a\sigma + c\tau, \quad u^2\tau = b\sigma + d\tau.$$

Thus u^2 is an eigenvalue of A. It is well known that these eigenvalues are roots of unity if and only if $|\operatorname{tr}(A)| \le 2$. Moreover, when $\operatorname{tr}(A) = 2$ we have $u = \pm 1$, when $\operatorname{tr}(A) = -2$ we have $u = \pm i$. Thus the proposition holds. □

Proposition 7.3 *Let W semi-fibre over the interval with semi-fibre T. If there is a representation ρ: $\pi_1(W) \to PSL_2(\mathbb{C})$ which is reducible and has non-Abelian image, then the torsion elements in the image of ρ have order 2.*

Proof Now W splits along T into two twisted I–bundles over the Klein bottle. Thus there is a presentation of $\pi_1(W)$ of the form

$$\langle x_1, y_1, x_2, y_2 \mid x_1y_1x_1^{-1} = y_1^{-1}, x_2y_2x_2^{-1} = y_2^{-1}, x_1^2 = x_2^{2a}y_2^c, y_1 = x_2^{2b}y_2^d\rangle,$$

where x_1, y_1 generate the fundamental group of one of the twisted I–bundles, x_2, y_2 generate the fundamental group of the other, and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is the gluing matrix. Note that $\pi_1(T)$ is generated by either pair x_1^2, y_1 and x_2^2, y_2.

We can suppose that either $\{\pm I\} \ne \rho(\pi_1(T)) \subset \mathcal{P}$ or $\rho(\pi_1(T)) \subset \mathcal{D}$. In the latter case, $\rho(\pi_1(T)) = \{\pm I\}$ as otherwise the normality of $\pi_1(T)$ in $\pi_1(W)$ and the reducibility of ρ imply that $\rho(\pi_1(W)) \subset \mathcal{D}$.

Assume first that $\rho(\pi_1(T)) = \{\pm I\}$. Then $\rho(x_1)^2 = \rho(y_1) = \rho(x_2)^2 = \rho(y_2) = \pm I$. Note that neither $\rho(x_1) = \pm I$ nor $\rho(x_2) = \pm I$ as otherwise the image of ρ would be Abelian. Thus up to conjugation we have $\rho(x_1) = \pm\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and $\rho(x_2) = \pm\left(\begin{smallmatrix} i & 1 \\ 0 & -i \end{smallmatrix}\right)$. Thus the only torsion elements in the image of ρ have order 2.

Next assume that $\{\pm I\} \ne \rho(\pi_1(T)) \subset \mathcal{P}$. The relation $x_1y_1x_1^{-1} = y_1^{-1}$ implies that exactly one of x_1^2, y_1 is sent to $\pm I$ by ρ. If $\rho(x_1^2) = \pm I$, then up to conjugation, $\rho(y_1) = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence, as $x_2^2 = x_1^{2d}y_1^{-c}$ and $y_2 = x_1^{-2b}y_1^a$, we have $\rho(x_2^2) = \pm\left(\begin{smallmatrix} 1 & -c \\ 0 & 1 \end{smallmatrix}\right)$, $\rho(y_2) = \pm\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$. Thus the image of ρ is generated by the images of x_1, y_1 and x_2. Projecting into $\mathcal{D}$ then shows that the only non-trivial torsion elements in the image of ρ must have order 2. If $c = 0$, then $ad = 1$ and so W is Seifert fibred. On the other hand, if $c \ne 0$, then $\rho(x_2) = \pm\begin{pmatrix} 1 & -\frac{c}{2} \\ 0 & 1 \end{pmatrix}$ and so the relation $x_2y_2x_2^{-1} = y_2^{-1}$

implies that $a = 0$. Therefore $ad = 0$ and W is Seifert fibred. A similar argument shows that the proposition holds when $\rho(y_1) = \pm I$. $\square$

Lemma 7.4 *Let W be a closed, connected, orientable, irreducible, very small 3–manifold which is not virtually Haken. Then the image of any representation $\rho\colon \pi_1(W) \to PSL_2(\mathbb{C})$ is a finite group.*

Proof Let $\rho\colon \pi_1(W) \to PSL_2(\mathbb{C})$ be a representation. The Tits alternative implies that there is a finite index subgroup G of $\rho(\pi_1(W))$ which is solvable. It suffices to show that G is finite.

If $G = \{\pm I\}$ we are done so assume otherwise. Then since G is solvable it contains a non-trivial normal subgroup A which is Abelian. Up to conjugation A is either contained in $\mathcal{D}$, or in $\mathcal{P}$, or is the Klein 4–group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ realized in $PSL_2(\mathbb{C})$ as

$$D_2 = \left\{\pm I, \quad \pm\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad \pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad \pm\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)\right\}.$$

Since $A \neq \{\pm I\}$ is normal in G, it follows that $A \subset \mathcal{N}$ if the first or third possibilities arise. In these cases let $A_0 = G \cap \mathcal{D}$ and observe that A_0 is Abelian and has index at most 2 in G. Then A_0 has finite index in $\rho(\pi_1(W))$ and, since W is not virtually Haken, must therefore be finite. But then $\rho(\pi_1(W))$ is finite and we are done.

On the other hand, suppose that $A \subset \mathcal{P}$. Then the non-triviality of A and its normality in G imply that $G \subset \mathcal{U}$, the group of upper-triangular matrices in $PSL_2(\mathbb{C})$. Since each finite degree cover $\widetilde{W}$ of W is irreducible but not Haken, it has zero first Betti number. Thus the projection of G in $\mathcal{D}$ is finite and so the kernel of this projection is of finite index in G. But this kernel lies in $\mathcal{P}_+$, the subgroup of $\mathcal{P}$ consisting of matrices of trace 2. Since this group is isomorphic to $\mathbb{C}$, and again using the fact that W is not virtually Haken we see that the kernel is trivial. Thus G is finite. $\square$

We now apply the results above to the following Proposition.

Proposition 7.5 *Suppose that $X_0 \subset X_{PSL_2}(M)$ is a non-trivial curve and α is a slope on ∂M which is not a singular slope for any closed, essential surface in M. If $M(\alpha)$ either has a finite fundamental group, or is an irreducible, very small 3–manifold which is not virtually Haken, then*

(1) $J_{X_0}(\alpha) \subset X_0^{\nu}$;

(2) *for each $x \in J_{X_0}(\alpha)$, there is an irreducible representation ρ with finite image such that $\chi_\rho = \nu(x)$, $\rho(\pi_1(\partial M)) \neq \{\pm I\}$ and $H^1(M(\alpha); sl_2(\mathbb{C})_\rho) = 0$;*

(3) *if $x \in J_{X_0}(\alpha)$, then $\nu(x)$ is a simple point of $X_{PSL_2}(M)$.*

Proof Our hypotheses imply that $b_1(M) = 1$. Thus Proposition 6.9 implies that $J_{X_0}(\alpha) \subset X_0^\nu$. Consider $x \in J_{X_0}(\alpha)$ and suppose that $\chi_\rho = \nu(x)$.

If $\pi_1(M(\alpha))$ is finite, then so is the image of ρ. The same conclusion holds when $\pi_1(M(\alpha))$ is not finite by Lemma 7.4.

Suppose next that ρ is reducible. Since its image is finite, it is conjugate to a diagonal representation and as this is true for each representation in $t^{-1}(\nu(x)$, any two representations in $t^{-1}(\nu(x))$ are conjugate. Hence the dimension of $t^{-1}(\nu(x))$ is at most 2, contrary to [12, Corollary 1.5.3]. This shows that ρ is irreducible. The fact that $\rho(\pi_1(\partial M)) \neq \{\pm I\}$ can now be proven in exactly the same way as [5, Lemma 4.2].

Next we show that $H^1(M(\alpha), sl_2(\mathbb{C})_\rho) = 0$. Let $G = \rho(\pi_1(M(\alpha))$ and consider the left $\pi_1(M(\alpha))$–module $\mathbb{C}[G]_\rho$. It is well known that $\mathbb{C}[G]$ splits as a direct sum $\oplus_\sigma V_\sigma$ of irreducible $\mathbb{C}G$–modules V_σ and each irreducible $\mathbb{C}G$–module appears at least once in this decomposition (see Serre [25]). On the other hand if $W \to M(\alpha)$ is the finite cover corresponding to the kernel of ρ, our hypotheses imply that $H^1(W;\mathbb{C}) = 0$. This is obvious if $\pi_1(M(\alpha))$ is finite and follows from the fact that W is irreducible and non-Haken otherwise. Thus

$$0 = H^1(W;\mathbb{C}) = H^1(M(\alpha);\mathbb{C}[G]_\rho) = \oplus_\sigma H^1(M(\alpha);(V_\sigma)_\rho).$$

This shows that for any irreducible $\mathbb{C}[G]$–module V, $H^1(M(\alpha);V_\rho) = 0$ and therefore, $H^1(M(\alpha), sl_2(\mathbb{C})_\rho) = 0$ as claimed.

Finally, we note that, according to [7, Theorem 3], conditions (1) and (2) imply that $\nu(x)$ is a simple point of $X_{PSL_2}(M)$. □

Proposition 7.6 *Let X_0 be a non-trivial curve in $X_{PSL_2}(M)$ and α a slope on ∂M such that $f_\alpha|X_0 \neq 0$. Suppose that α is not a singular slope for a closed, essential surface in M. Assume as well that either*

(i) *$\pi_1(M(\alpha))$ is finite or $M(\alpha)$ is an irreducible very small 3–manifold which is not virtually Haken, or*

(ii) *$M(\alpha)$ is a non-Haken Seifert manifold with base orbifold of the form $S^2(r,s,t)$ and there is a slope β on ∂M such that $M(\beta) \cong S^1 \times S^2$, or*

(iii) *$X_0 \subset X_{PSL_2}(M(\beta))$, where β is a slope on ∂M such that $M(\beta) \cong L_p \# L_q$.*

Then

$$\|\alpha\|_{X_0} = m_0 + 2|J_{X_0}(\alpha)| - A,$$

where $m_0 = \sum_{x \in \widetilde{X}_0} \min\{Z_x(\widetilde{f}_\beta) \mid \widetilde{f}_\beta|\widetilde{X}_0 \neq 0\}$, and A is the number of irreducible characters $\chi_\rho \in \nu(J_{X_0}(\alpha))$ of representations ρ which are conjugate into $\mathcal{N}$.

Proof Case (ii) is done in [1, Theorem 2.3] while the proof in case (i) is handled analogously. The idea is that by combining (6–1), Proposition 6.4, and the previous two propositions, the calculation of $\|\alpha\|_{X_0}$ reduces to a weighted count of characters of representations $\pi_1(M(\alpha)) \to PSL_2(\mathbb{C})$. Note that under our assumptions, $J_{X_0}(\alpha) \subset X_0^\nu$ and $\nu | J_{X_0}(\alpha)$ is injective.

Finally, for case (iii), Proposition 6.9 implies that $J_{X_0}(\alpha) \subset X_0^\nu$ and a calculation similar to that used in case (i) yields the desired conclusion. □

8 Proof of Proposition 5.1

We suppose in this section that $b_1(M) = 1$, that neither α nor β is a singular slope for a closed, essential surface in M, that $M(\alpha)$ has a finite fundamental group, and that $M(\beta)$ is either a connected sum of two lens spaces or $S^1 \times S^2$. Theorem 3.2 implies that α is not a boundary slope.

A finite filling slope α is either of C–type or D–type or Q–type or $T(k)$–type $(1 \le k \le 3)$ or $O(k)$–type $(1 \le k \le 4)$ or $I(k)$–type $(1 \le k \le 5, k \ne 4)$. We refer to [9, pages 93–94 and 98] for these definitions. We will show

$$\Delta(\alpha, \beta) \le \begin{cases} 2 & \text{if } M(\beta) \cong L_2 \# L_3,\ H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \text{ and } \alpha \text{ is of type } O(2); \\ 1 & \text{otherwise.} \end{cases}$$

The key relationships between Culler–Shalen seminorms and finite filling classes is contained in the following result from [9].

Proposition 8.1 *Suppose that X_0 is a non-trivial curve in $X_{PSL_2}(M)$ and that α is a finite or cyclic filling slope which is not a boundary slope associated to an ideal point of X_0.*

(1) *If α is a cyclic filling slope, then $\|\alpha\|_{X_0} = s_{X_0}$. [11]*

(2) *If α is a D–type or a Q–type filling slope and X_0 is not virtually trivial, then*
   (i) $\|\alpha\|_{X_0} \le 2s_{X_0}$;
   (ii) $\|\alpha\|_{X_0} \le \|\beta\|_{X_0}$ *for any slope β such that $\|\beta\|_{X_0} \ne 0$ and $\Delta(\alpha, \beta) \equiv 0 \pmod 2$.*

(3) *If α is a $T(k)$–type filling slope, then $k \in \{1, 2, 3\}$ and*
   (i) $\|\alpha\|_{X_0} \le s_{X_0} + 2$;
   (ii) $\|\alpha\|_{X_0} \le \|\beta\|_{X_0}$ *for any slope β such that $\|\beta\|_{X_0} \ne 0$ and $\Delta(\alpha, \beta) \equiv 0 \pmod k$.*

(4) *If α is an $O(k)$–type filling slope, then $k \in \{1, 2, 4\}$ if $H_1(M)$ has no 2–torsion, $k \in \{1, 2, 3\}$ if $H_1(M)$ has 2–torsion, and*

(i) $\|\alpha\|_{X_0} \leq s_{X_0} + 3$;

(ii) $\|\alpha\|_{X_0} \leq \|\beta\|_{X_0}$ *for any slope* β *such that* $\|\beta\|_{X_0} \neq 0$ *and* $\Delta(\alpha, \beta) \equiv 0 \pmod{k}$.

(5) *If* α *is an* $I(k)$*–type filling slope, then* $k \in \{1, 2, 3, 5\}$ *and*

(i) $\|\alpha\|_{X_0} \leq s_{X_0} + 4$;

(ii) $\|\alpha\|_{X_0} \leq \|\beta\|_{X_0}$ *for any slope* β *such that* $\|\beta\|_{X_0} \neq 0$ *and* $\Delta(\alpha, \beta) \equiv 0 \pmod{k}$.

We split the proof of Proposition 5.1 into three cases.

Case 1 $M(\beta) \neq P^3 \# P^3$ is a connected sum of two lens spaces.

Recall that $X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$ contains exactly $[\frac{p}{2}][\frac{q}{2}]$ non-trivial curves $X(j,k)$, where $1 \leq j \leq \frac{p}{2}$ and $1 \leq k \leq \frac{q}{2}$. Let X be the union of these curves and observe that since β is not a singular slope for any closed essential surface in M, Proposition 6.6 implies that

$$s_X \geq s_0 = \begin{cases} (p-1)(q-1)+1 & \text{if } p, q \text{ even} \\ (p-1)(q-1) & \text{otherwise.} \end{cases} \tag{8–1}$$

By (6–2), $\|\alpha\|_X = \Delta(\alpha, \beta) s_X$. If α is a C–type filling slope, then $\|\alpha\|_X \leq s_X$ by Proposition 8.1 (recall that α is not a boundary slope) and therefore $\Delta(\alpha, \beta) \leq 1$. If it is a D or Q–type filling slope, then all irreducible representations of $\pi_1(M(\alpha))$ conjugate into $\mathcal{N}$. Thus Proposition 7.5 and Proposition 7.6 show that for each $x \in J_X(\alpha)$, $\nu(x)$ is an irreducible character and $\|\alpha\|_X \leq s_X + |\nu(J_X(\alpha))|$. On the other hand, Lemma 6.5 shows that if $X(j,k)$ is a component of X with j and k relatively prime to p and q respectively, then it contains the character of an irreducible representation with image in $\mathcal{N}$ if and only if $p = 2$, and if $p = 2$, there is a unique such character. Hence $\Delta(\alpha, \beta) s_X = \|\alpha\|_X \leq s_X + [\frac{p}{2}][\frac{q}{2}] < 2 s_X$, and therefore $\Delta(\alpha, \beta) \leq 1$.

Next assume that α is either a T or O or I–type filling slope. Then (6–2) and Proposition 8.1 show that

$$\Delta(\alpha, \beta) \leq \begin{cases} 1 + \frac{2}{s_X} \leq 1 + \frac{2}{s_0} & \text{if } \alpha \text{ is } T\text{–type} \\ 1 + \frac{3}{s_X} \leq 1 + \frac{3}{s_0} & \text{if } \alpha \text{ is } O\text{–type} \\ 1 + \frac{4}{s_X} \leq 1 + \frac{4}{s_0} & \text{if } \alpha \text{ is } I\text{–type.} \end{cases}$$

Combining this inequality with (8–1) shows that $\Delta(\alpha, \beta) \leq 1$ unless, perhaps,

- $(p, q) = (2, 4), (2, 5), (3, 3)$, $\Delta(\alpha, \beta) \leq 2$ and α is I–type, or
- $(p, q) = (2, 3)$, $\Delta(\alpha, \beta) \leq 3$ and α is I–type, or

- $(p,q)=(2,3)$, $\Delta(\alpha,\beta)\leq 2$ and α is either T or O–type.

Assume first that one of these cases arises and α is either of type T or I. It is well-known that

$$(8\text{–}2)\qquad H_1(M(\alpha))\cong\begin{cases}\mathbb{Z}/3^k j\ \ k\geq 1 \text{ and } j \text{ relatively prime to 6, if } \alpha \text{ is } T\text{–type}\\ \mathbb{Z}/j \text{ where } j \text{ is relatively prime to 30, if } \alpha \text{ is } I\text{–type}\end{cases}$$

(see [5] for instance) so that in each of these cases, $H_1(M(\alpha))$ is cyclic. This implies that $H_1(M)\cong\mathbb{Z}\oplus\mathbb{Z}/n$, where $n\geq 1$. Then n divides $|H_1(M(\delta)|$ for each primitive $\delta\in H_1(\partial M)$. Taking $\delta=\alpha$ we see that n divides $3^k j$, where $\gcd(j,6)=1$ if α is T–type, and divides j, where $\gcd(j,30)=1$ if α is I–type. On the other hand, n also divides $|H_1(M(\beta))|\cong\mathbb{Z}/p\oplus\mathbb{Z}/q$, so given the constraints we have imposed on (p,q) we see that $n=1$. Thus $H_1(M)\cong\mathbb{Z}$, and so each Dehn filling of M has a cyclic first homology group. This rules out the possibility that $(p,q)=(2,4)$ or $(3,3)$. Consider, then, the cases where $(p,q)=(2,3)$ or $(2,5)$. There is a basis $\{\mu,\lambda\}$ of $H_1(\partial M)$ such that λ is zero homologically in M and μ generates $H_1(M)$.

If α is a T–type filling slope, then $(p,q)=(2,3)$ and so by our choice of μ and λ, (8–2) implies that there are integers a,b such that up to sign, $\alpha=\{\pm(3^k j\mu+a\lambda)\}$, and $\beta=\{\pm(6\mu+b\lambda)\}$. Then b is odd and the constraints on j,k show that $\Delta(\alpha,\beta)=|6a-3^k jb|\equiv 1 \pmod 2$. As $\Delta(\alpha,\beta)\leq 2$, we have $\Delta(\alpha,\beta)=1$.

Next suppose that α is an I–type filling class. Then $(p,q)=(2,3)$ or $(2,5)$. By (8–2) there are integers a,b such that $\alpha=\{\pm(j\mu+a\lambda)\}$, and $\beta=\{\pm(2q\mu+b\lambda)\}$. Then b is relatively prime to $2q$ and since $\gcd(j,30)=1$, $\Delta(\alpha,\beta)=|2qa-jb|$ is relatively prime to $2q$ as well. When $q=5$, this shows that $\Delta(\alpha,\beta)$ is odd, and therefore as $\Delta(\alpha,\beta)\leq 2$ in this case, we have $\Delta(\alpha,\beta)=1$. Finally when $q=3$, it shows that $\Delta(\alpha,\beta)$ is relatively prime to 6, and therefore as $\Delta(\alpha,\beta)\leq 3$, we have $\Delta(\alpha,\beta)=1$.

Finally suppose that $\Delta(\alpha,\beta)=2$, $(p,q)=(2,3)$, and α has type O. Now $H_1(M(\alpha))\cong\mathbb{Z}/2j$, where j is relatively prime to 6 [5], and we can argue as above to see that either $H_1(M)\cong\mathbb{Z}$ or $H_1(M)\cong\mathbb{Z}\oplus\mathbb{Z}/2$. When $H_1(M)\cong\mathbb{Z}$, we can find, as above, a basis μ,λ of $H_1(\partial M)$ such that λ is zero homologically in M and μ generates $H_1(M)$. There are integers a,b such that $\alpha=\{\pm(2j\mu+a\lambda)\}$, and $\beta=\{\pm(6\mu+b\lambda)\}$, where a and b are odd. Since j is odd as well, we have $2\geq\Delta(\alpha,\beta)=|6a-2bj|\equiv 0 \pmod 4$. This contradiction shows that this case does not arise.

Thus in all cases, $\Delta(\alpha,\beta)\leq 2$ and $\Delta(\alpha,\beta)\leq 1$ unless, perhaps, $H_1(M)\cong\mathbb{Z}\oplus\mathbb{Z}/2$, $M(\beta)\cong L_2\#L_3$ and $M(\alpha)$ has type $O(k)$ for some k. This completes the proof in Case 1.

Case 2 $M(\beta)=P^3\#P^3$.

We show that in this case, $\Delta(\alpha, \beta) \leq 1$. First we need some auxiliary results.

Note that there is a 2–fold cover $p\colon \widetilde{M}_\beta \to M$ obtained by restricting the cover $S^1 \times S^2 \to P^3 \# P^3 \cong M(\beta)$. Let $\phi_\beta\colon \pi_1(M) \to \mathbb{Z}/2$ be the associated homomorphism. Note also that $|\partial \widetilde{M}_\beta| \in \{1, 2\}$.

Proposition 8.2 *Suppose that $M(\beta) \cong P^3 \# P^3$ and that $\beta = \{\pm b\}$ is not a strict boundary slope. Suppose that $X_0 \subset X(M)$ is a curve which is not virtually trivial and that $\|\beta\|_{X_0} \neq 0$. Then there is an index $2/|\partial \widetilde{M}_\beta|$ sublattice $\widetilde{L}$ of $H_1(\partial M)$ containing b such that $\|\beta\|_{X_0} \leq \|\alpha\|_{X_0}$ for each slope $\alpha = \{\pm a\}$, where $a \in \widetilde{L}$ and $\|\alpha\|_{X_0} \neq 0$. In particular, $\|\beta\|_{X_0} \leq 2s_{X_0}/|\partial \widetilde{M}_\beta|$.*

Proof The proof is identical to the proof of [5, Theorem 2.1(a)]. In that result a non-strict boundary slope β_0 on ∂M was given along with a cover $\widetilde{M(\beta_0)} \to M(\beta_0)$, where $\pi_1(\widetilde{M(\beta_0)})$ is a finite cyclic group. Let $p_0\colon \widetilde{M} \to M$ be the associated cover of M and T be a boundary component of $\widetilde{M}$. It was shown in [5] that if $\widetilde{L} = (p_0|T)_*(H_1(T))$. that for any slope $\{\pm a\}$ such that $a \in \widetilde{L}$ and $\|\alpha|_{X_0} \neq 0$, we have $\|\beta_0\|_{X_0} \leq \|\alpha\|_{X_0}$. The reader can readily verify that the proof works equally well in the case where $\pi_1(\widetilde{M(\beta_0)})$ is an infinite cyclic group, the situation we are considering. Let T be a boundary component of the double cover $p\colon \widetilde{M}_\beta \to M$. If we now set $\widetilde{L} = (p|T)_*(H_1(T)$, then for any slope $\alpha = \{\pm a\}$ such that $a \in \widetilde{L}$ and $\|\alpha|_{X_0} \neq 0$, we have $\|\beta\|_{X_0} \leq \|\alpha\|_{X_0}$. The index of $p_*(H_1(T))$ in $H_1(\partial M)$ is $2/|\partial \widetilde{M}_\beta|$, so the conclusions of the proposition hold. □

Corollary 8.3 *Suppose that $M(\beta) \cong P^3 \# P^3$, β is not a singular slope for a closed essential surface in M, and let $C \subset \mathbb{Z}/2 * \mathbb{Z}/2 = \pi_1(M(\beta))$ be the unique cyclic subgroup of index 2. Then $\pi_1(\partial M)$ is sent to a non-trivial subgroup of C under the natural homomorphism $\pi_1(M) \to \pi_1(M(\beta))$. Moreover, for any curve $X_0 \subset X_{PSL_2}(M)$ which* is not virtually trivial, *we have $\|\beta\|_{X_0} \leq s_{X_0}$.*

Proof Let β^* be a dual class to β and choose elements b and b^* of $H_1(\partial M)$ with $\beta = \{\pm b\}$ and $\beta^* = \{\pm b^*\}$. Identify $\pi_1(\partial M)$ with $H_1(\partial M)$, and let γ denote the image of b^* in $\pi_1(M(\beta))$. If $\gamma^2 = 1$, then $f_{b+2nb^*} = 0$ and so $\|b + 2nb^*\|_{X_0} = 0$ for each $n \in \mathbb{Z}$. It follows that $\|\cdot\|_{X_0} = 0$, and so Proposition 6.4 implies that β is a singular slope for a closed essential surface in M, contrary to our hypotheses. Thus γ has infinite order in $\mathbb{Z}/2 * \mathbb{Z}/2 = \pi_1(M(\beta))$. It follows that $\gamma \in C$ and since $b \in \pi_1(\partial M)$ maps to the identity in $\pi_1(M(\beta))$ we see that $\pi_1(\partial M)$ is sent to C. Now C is the kernel of the homomorphism $\pi_1(M(\beta)) \to \mathbb{Z}/2$ defining the cover $S^1 \times S^2 \to P^3 \# P^3$, and thus $\pi_1(\partial M) \subset \ker(\phi_\beta)$. It follows that $|\partial \widetilde{M}_\beta| = 2$. As β

is not a strict boundary slope (cf Corollary 3.4), the previous proposition shows that $\|\beta\|_{X_0} \leq s_{X_0}$. □

Lemma 8.4 *Let $X_M \subset X_{SL_2}(M)$* [1] *be the canonical curve and suppose that the slope δ is not a strict boundary class and satisfies $\|\delta\|_{X_M} = s_M$. Suppose that α is a slope such that $\pi_1(M(\alpha))$ is either finite or cyclic or $\mathbb{Z}/2 * \mathbb{Z}/2$. Then either*

(1) *α is a singular slope for a closed essential surface in M, or*

(2) *$\Delta(\alpha, \delta) \leq 2$ and if $\Delta(\alpha, \delta) = 2$, then α is of $T(k)$, $O(k)$ or $I(k)$–type, where $k \geq 3$.*

Proof Suppose that α is not a singular slope for a closed essential surface in M. Then Theorem 3.2 and Corollary 3.4 imply that it is not a strict boundary slope and therefore we can apply [5, Proposition 7.2] to see that $\Delta(\alpha, \delta) \leq 2$ when $\pi_1(M(\alpha))$ is either finite or cyclic. When it is $\mathbb{Z}/2 * \mathbb{Z}/2$, an $SL_2(\mathbb{C})$ version of Proposition 8.2 shows that $\|\alpha\|_{X_M} \leq 2s_M$ and it follows from the basic properties of $\|\cdot\|_M$ [5] that $\Delta(\alpha, \delta) \leq 2$.

Suppose, then, that $\Delta(\alpha, \delta) = 2$ and let $\tau = \{\pm t\}$ be a dual slope to $\delta = \{\pm d\}$. Then $\alpha = \{\pm(nd + 2t)\}$ for some $n \in \mathbb{Z}$. Hence $\Delta(\alpha, \delta) \equiv 0 \pmod 2$ and thus if α is of type D or Q, or $T(k), O(k), I(k)$, where $k \leq 2$, or $\pi_1(M(\alpha)) \cong \mathbb{Z}$ or $\mathbb{Z}/2 * \mathbb{Z}/2$, then $\|\alpha\|_M \leq \|\delta\|_M = s_M$ (cf Proposition 8.1 and Proposition 8.2). But it was shown in [11, Section 1.1] that if the distance between two slopes of minimal non-zero Culler–Shalen norm is 2, then both are strict boundary slopes. Hence these cases do not arise and so α is of type $T(k), O(k)$ or $I(k)$–type, where $k \geq 3$. □

Proof of Proposition 5.1 when $M(\beta) = P^3 \# P^3$. Since neither α nor β is a singular slope for a closed essential surface in M, they are not strict boundary slopes (see Theorem 3.2, Corollary 3.4). Thus Corollary 8.3 and Lemma 8.4 show that $\|\beta\|_M = s_M$ and $\Delta(\alpha, \beta) \leq 2$ with equality implying that α has type $T(k), O(k), I(k)$, where $k \geq 3$. Since $H_1(M(\beta)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, $H_1(M; \mathbb{Z}/2) \supseteq \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and so $H_1(M(\alpha); \mathbb{Z}/2) \neq 0$. Hence α is neither T or I type (cf (8–2)). We must consider the possibility that it is of type $O(k)$, where $k = 3, 4$.

Let $X_0 \subset X_{PSL_2}(\mathbb{Z}/2 * \mathbb{Z}/2) = X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$ be the unique non-trivial curve. According to Proposition 6.6, $\|\cdot\|_{X_0} \neq 0$ and further, $s_{X_0} \geq 2$. It is easy to verify that the only irreducible representations of $\pi_1(M) \to PSL_2(\mathbb{C})$ with finite image whose character lies in X_0 are ones with dihedral image. Since O–type groups

[1] $SL_2(\mathbb{C})$–character varieties and $SL_2(\mathbb{C})$ Culler–Shalen seminorms are defined in a manner similar to their $PSL_2(\mathbb{C})$ counterparts and possess similar properties. We refer the reader to [11].

admit only one such character [5, Lemma 5.3], it follows from Proposition 7.6 that $\Delta(\alpha,\beta)s_{X_0} = \|\alpha\|_{X_0} \le s_{X_0}+1$. Thus $\Delta(\alpha,\beta)\le 1$ as claimed, which completes the proof in Case 2. □

Case 3 $M(\beta) = S^1 \times S^2$.

We prove $\Delta(\alpha,\beta) = 1$.

By Theorem 3.2, β is not a strict boundary slope and so Proposition 8.1 implies $\|\beta\|_M = s_M$. Thus Lemma 8.4 shows that $\Delta(\alpha,\beta)\le 2$, and if it equals 2, then α has type $T(q)$, $O(q)$ or $I(q)$, where $q \ge 3$. We assume below that $\Delta(\alpha,\beta)=2$ in order to arrive at a contradiction. Let $i\colon \partial M \to M$ be the inclusion.

Observation 8.5 *Let $\beta = \{\pm b\}$. There is an integer $n \ge 1$ such that $H_1(M) \cong \mathbb{Z}\oplus\mathbb{Z}/n$ in such a way that $i_*(b) = (0,1)$. Moreover, there is a dual slope $\beta^* = \{\pm b^*\}$ for β such that $i_*(b^*) = (n,0)$.*

Proof Since α is a finite filling slope, the first Betti number of M is 1. Since $H_1(M(\beta)) \cong \mathbb{Z}$, we have $H_1(M)\cong \mathbb{Z}\oplus\mathbb{Z}/n$, where $n\ge 1$ and $i_*(b)$ generates $\mathbb{Z}/n$, say $i_*(b) = (0,\bar{1}) \in \mathbb{Z}\oplus\mathbb{Z}/n$. Let b_1^* be any dual class for b and observe that since i_* has rank 1, we have $i_*(b_1^*) = (d,\bar{k})$ for some integers $d\ne 0$ and k. Then $b^* = b_1^* - kb$ is also dual to b and satisfies $i_*(b^*) = (d,\bar{0})$. Let $\xi \in H_1(M)$ correspond to $(1,\bar{0})$. By our assumptions, there is a generator $\eta \in H_2(M,\partial M)$ such that $\partial(\eta) = nb$. Lefschetz duality implies that $|\xi\cdot\eta| = 1$. Hence $|d| = |i_*(b^*)\cdot\eta| = |b^*\cdot\partial(\eta)| = n$. It follows that $i_*(b^*) = \pm n\xi$, which completes the proof of Observation 8.5. □

Since $\Delta(\alpha,\beta) = 2$, we can write $a = 2b^* + mb$ (up to sign) for some $m\in\mathbb{Z}$. A homological calculation now shows that $|H_1(M(\alpha))| = 2n^2$ and so α cannot have type T or I. Thus it has type O and so $\pi_1(M(\alpha)) \cong O^*\times\mathbb{Z}/j$, where O^* is the binary octahedral group and j is an integer relatively prime to 6. Then $\mathbb{Z}/2j \cong H_1(M(\alpha)) \cong \mathbb{Z}/2n^2$. It follows that $n^2 = j$ and therefore n is odd. [9, Lemma 3.1 (4)] now shows that α has type $O(4)$. Thus the image of $\pi_1(\partial M)$ under the representation ρ, given by composition $\pi_1(M)\to\pi_1(M(\alpha))\to O_{24}\subset PSL_2(\mathbb{C})$, has image $\mathbb{Z}/4$. As $\rho(\alpha) = \pm I$ and $\Delta(\alpha,\beta) = 2$, $\rho(\beta)$ is the square of an element of order 4 in O_{24}. Thus it lies in the kernel of the surjective homomorphism $\phi\colon O_{24}\to D_3$, which sends any element of order 4 to an element of order 2. Then $\phi\circ\rho$ induces a surjective homomorphism of $\pi_1(M(\beta))\cong\mathbb{Z}$ onto the non-Abelian group D_3, which is impossible. Thus it must be that $\Delta(\alpha,\beta)\le 1$.

9 Proof of Proposition 5.2

Here we suppose that β is a strict boundary slope but is not a singular slope for a closed, essential surface in M. It follows from Theorem 3.2 and Corollary 3.4 that $M(\beta)$ is not homeomorphic to $P^3 \# P^3$ or $S^1 \times S^2$. The proof of Proposition 5.2 is therefore a consequence of the following result which, unlike Proposition 5.3, does not assume that $M(\alpha)$ admits a geometric decomposition.

Proposition 9.1 *Suppose that $M(\beta)$ is a connected sum $L_p \# L_q$ of two lens spaces, where $2 \le p \le q$ and $2 < q$, and that $M(\alpha)$ is an irreducible very small 3–manifold. Then*

$$\Delta(\alpha,\beta) \le \begin{cases} 3 & \textit{if } (p,q) \in \{(2,3),(2,5),(3,5)\}; \\ 2 & \textit{if } (p,q) \in \{(2,4),(3,3),(3,4),(5,5)\}; \\ 1 & \textit{otherwise.} \end{cases}$$

Proof Let X_0 be one of the curves $X(j,k) \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q) = X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$, where j,k are relatively prime to p,q respectively. Suppose that $x \in J_{X_0}(\alpha)$. Proposition 7.5 shows that $x \in X_0^\nu$ and $\nu(x)$ is a simple point of $X_{PSL_2}(M)$ which is the character of an irreducible representation ρ whose image is a finite subgroup of $PSL_2(\mathbb{C})$. In particular, this implies that if $\nu(x) = \chi_\rho$, where $\rho \in \mathcal{N}$, then ρ must have dihedral image.

Let $X \subset X_{PSL_2}(M)$ be the union of the curves $X(j,k) \subset X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$, where j,k are relatively prime to p,q. If d is the number of components of X, then Proposition 6.6 shows that

$$s_X \ge \begin{cases} 2d & \text{if } p = 2 \\ 4d & \text{if } p > 2. \end{cases}$$

Recall from (6–2) that $\Delta(\alpha,\beta) = \|\alpha\|_X / s_X$. On the other hand, Proposition 7.6 and our discussion above show that $\|\alpha\|_X = s_X + 2|J_X(\alpha)| - A$, where A is the number of dihedral characters in $\nu(J_X(\alpha))$. According to Lemma 6.5(2) we have $A = d$ if $p = 2$ and $A = 0$ if $p > 2$. If we set $n = |J_X(\alpha)|$, then we have

$$\Delta(\alpha,\beta) = 1 + \frac{2n - A}{s_X} \le \begin{cases} 1 + \frac{2n-d}{2d} & \text{if } p = 2; \\ 1 + \frac{2n}{4d} & \text{if } p > 2. \end{cases} \tag{9–1}$$

We have $d = \left[\frac{p}{2}\right]\left[\frac{q}{2}\right]$, and n is determined by Lemma 6.5 since $\nu(x)$ is the character of an irreducible representation with finite image for each $x \in J_X(\alpha)$. Checking each case, we see that

$$\Delta(\alpha,\beta) \le \begin{cases} 5 & \text{if } (p,q) = (2,3); \\ 3 & \text{if } (p,q) \in \{(2,5),(3,5)\}; \\ 2 & \text{if } (p,q) \in \{(2,4),(3,3),(3,4),(5,5)\}; \\ 1 & \text{otherwise.} \end{cases}$$

Thus it will suffice to prove that $\Delta(\alpha,\beta) \le 3$ when $(p,q) = (2,3)$.

Suppose that $\Delta(\alpha,\beta) = 5$ and $(p,q) = (2,3)$. Lemma 6.5 and Inequality (9–1) imply that $s_X = 2$. Proposition 6.2 (1) shows that for every point $\chi_\rho \in J_X(\alpha)$, $\rho(\pi_1(\partial M)) = \mathbb{Z}/5$. In particular, $\rho(\pi_1(M))$ has an element of order 5. The only finite, non-cyclic subgroups of $PSL_2(\mathbb{C})$ which have such elements are I_{60} and D_k, where $k \equiv 0 \pmod 5$. Therefore Lemma 6.5 shows that $10 = 5s_X \le s_X + 5 = 7$, which is impossible.

Suppose next that $\Delta(\alpha,\beta) = 4$. Lemma 6.5 and Inequality (9–1) imply that $s_X \le 3$. Let β^* be a dual slope to β and recall that $\|\beta^*\|_X = s_X$.

If $s_X = 2$, then $8 = \Delta(\alpha,\beta)s_X = \|\alpha\|_X = 2 + 2n - A$, where $A \in \{0,1\}$. Thus $n = 3, m = 0$ and so $\nu(J_X(\alpha))$ consists of three elements, where at most two are I_{60}–characters, at most one is an O_{24}–character, and at most one is a T_{12}–character. Proposition 6.2(1) shows that for every point $\chi_\rho \in J_X(\alpha)$, $\rho(\pi_1(\partial M)) = \mathbb{Z}/2$ or $\mathbb{Z}/4$. Since only the O_{24}–character has elements of order 4, there are at least two characters χ_ρ in $J_X(\alpha)$ such that $\rho(\pi_1(\partial M)) = \mathbb{Z}/2$. This implies that $4 = 2s_X = \|2\beta^*\|_X \ge s_X + 4 = 6$, which is impossible.

Finally suppose that $s_X = 3$. Then $12 = \Delta(\alpha,\beta)s_X = \|\alpha\|_X = 3 + 2n - A$, where $A \in \{0,1\}$. Hence $n = 5$, and $\nu(J_X(\alpha))$ consists of 5 elements – two I_{60}–characters, one O_{24}–character, one T_{12}–character, and one D_3–character. A similar argument to that of the previous paragraph shows that $6 = 2s_X = \|2\beta^*\|_X \ge s_X + 7 = 10$, which is impossible. This completes the proof. □

10 Proof of Proposition 5.3

In this section we suppose that $b_1(M) = 1$, neither α nor β is a singular slope for a closed essential surface in M, $M(\alpha)$ is an irreducible, very small 3–manifold which admits a geometric decomposition, and $M(\beta)$ is either $S^1 \times S^2$ or a connected sum of lens spaces $L_p \# L_q$, where $2 \le p \le q$. We must show $\Delta(\alpha,\beta) \le 2$.

The reader will verify that given our assumptions on $M(\alpha)$, one of the following possibilities holds. Either $M(\alpha)$

- is a torus bundle over S^1 with monodromy $A \in SL_2(\mathbb{Z})$ such that $|\operatorname{tr}(A)| \geq 2$; or
- semi-fibres over I with semi-fibre a torus; or
- admits a Seifert structure with base orbifold $S^2(3,3,3)$, $S^2(2,4,4)$, or $S^2(2,3,6)$.

We treat these cases separately.

Case 1 $M(\alpha)$ fibres over the circle with monodromy A for which $|\operatorname{tr}(A)| \geq 2$.

Note that α is the rational longitudinal class in this case so that $M(\beta) \not\cong S^1 \times S^2$. Thus $M(\beta) \cong L_p \# L_q$ for some $2 \leq p \leq q$. According to Proposition 9.1 we may assume that either $\Delta(\alpha,\beta) = 3$ and $(p,q) \in \{(2,3),(2,5),(3,5)\}$ or $M(\beta) \cong P^3 \# P^3$. We consider the former case first.

Let X_0 be a curve in $X_{PSL_2}(M(\beta) \subset X_{PSL_2}(M)$. Since X_0 is a β–curve, it follows from Lemma 6.8 that, for each $x \in J_{X_0}(\alpha)$ and $\rho \in R(X_0) \cap t^{-1}(\nu(x))$, we have that $\rho(\beta^*)$ has order 3. Proposition 7.2(2) implies that there are no reducible characters in $\nu(J_{X_0}(\alpha))$. Hence if $\chi_\rho \in \nu(J_{X_0}(\alpha))$, then the image of ρ is either contained in $\mathcal{N}$ or is T_{12} (Proposition 7.1). Since $q > 2$ it follows from (6–2), Lemma 6.5, and Proposition 6.6 that

$$\Delta(\alpha,\beta) \leq \begin{cases} 1 & \text{when } (p,q) \neq (2,3),(3,3) \\ 2 & \text{when } (p,q) = (2,3),(3,3) \end{cases},$$

contradicting our assumption that $\Delta(\alpha,\beta) = 3$.

Next suppose that $M(\beta) \cong P^3 \# P^3$. It follows that $H_1(M) \cong \mathbb{Z} \oplus A$, where A is either (i) $\mathbb{Z}/2$ or (ii) $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. Now $H_1(M(\alpha))$ is infinite, so α is the slope of the rational longitude in $H_1(\partial M)$, say $\alpha = \{\pm a\}$ and $i_*(a) = \sigma \in A$, where $i\colon \partial M \to M$ is the inclusion. If $\alpha^* = \{\pm a^*\}$ is any dual slope to α we have $i_*(a^*) = d\xi + \tau$, where $d \geq 1, \xi$ generates a free factor of $H_1(M)$ and $\tau \in A$. Write $\beta = \{\pm(ma + na^*)\}$ and observe that $\Delta(\alpha,\beta) = |m|$. A simple computation shows that since $H_1(M(\beta)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, we must have $m = \pm 1$ in case (ii) and therefore $\Delta(\alpha,\beta) = 1$. Similarly in case (i) we must have $\Delta(\alpha,\beta) \leq 2$. Both cases contradict our hypotheses, so we also have $\Delta(\alpha,\beta) \leq 2$ when $q = 2$.

Case 2 $M(\alpha)$ semi-fibres over the interval.

Subcase 2.1 $M(\beta) \cong L_p \# L_q \not\cong P^3 \# P^3$.

Again, according to Proposition 9.1, we may assume that either $\Delta(\alpha,\beta) = 3$ and $(p,q) \in \{(2,3),(2,5),(3,5)\}$.

Let β^* be a dual class to β. According to Lemma 6.8, for each $x \in J_{X_0}(\alpha)$ and $\rho \in R(X_0) \cap t^{-1}(\nu(x))$, we have that $\rho(\beta^*)$ has order 3. Proposition 7.3 shows that there are no reducible characters in $\nu(J_{X_0}(\alpha))$. Thus if $\chi_\rho \in \nu(J_{X_0}(\alpha))$, the image of ρ is either contained in $\mathcal{N}$ or is O_{24} by Proposition 7.1. Since $q \geq 3$, Lemma 6.5 and Proposition 7.6 show that $\Delta(\alpha,\beta) \leq 2$, contradicting our assumption that $\Delta(\alpha,\beta) = 3$.

Subcase 2.2 $M(\beta) \cong P^3 \# P^3$.

This case follows from [21, Theorem 1.2].

Subcase 2.3 $M(\beta) = S^1 \times S^2$.

There is an exact sequence $1 \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(M(\alpha)) \to \mathbb{Z}/2 * \mathbb{Z}/2 \to 1$ and therefore a non-trivial curve $X_0 \subset X_{PSL_2}(\mathbb{Z}/2 * \mathbb{Z}/2) \subset X_{PSL_2}(M(\alpha)) \subset X_{PSL_2}(M)$. As we have assumed that α is not a singular slope for a closed, essential surface in M, Proposition 6.4 implies that $\|\cdot\|_{X_0} \neq 0$. Since we have assumed that β is not a singular slope for a closed, essential surface in M, the same proposition implies that $J_{X_0}(\beta) \subset X_0^\nu$. Thus Proposition 8.1 (1) shows that $\Delta(\alpha,\beta) = 1$.

Case 3 $M(\alpha)$ admits a Seifert structure with base orbifold $S^2(3,3,3)$, $S^2(2,4,4)$, or $S^2(2,3,6)$.

Our proof in this case depends on obtaining good estimates for the value of a Culler–Shalen seminorm on α. To that end, let $X_0 \subset X_{PSL_2}(M)$ be a non-trivial curve and suppose that χ is a character contained in $\nu(J_{X_0}(\alpha))$. Since $b_1(M) = 1$, $\chi = \chi_\rho$, where $\rho \in R(X_0)$ is either irreducible or has a non-Abelian image by Lemma 6.7 and further, $\rho(\alpha) = \pm I$. Thus ρ factors through $\pi_1(M(\alpha))$. Now apply [1, Lemma 3.1] to see that ρ factors through $\Delta(r,s,t)$, the orbifold fundamental group of the base orbifold $S^2(r,s,t)$ of $M(\alpha)$ $(a \leq b \leq c)$. The irreducible characters $\Delta(r,s,t)$ were calculated in [2, Propositions 5.2, 5.3, 5.4]. If χ_ρ is reducible, ρ induces a representation $\sigma\colon \Delta(r,s,t) \to PSL_2(\mathbb{C})$ whose image is upper-triangular and non-Abelian. Write $\Delta(r,s,t) = \langle x, y : x^a, y^b, (xy)^c \rangle$ and observe that up to conjugation, $\sigma(x)$ is diagonal of order a and the $(1,2)$ entry of $\sigma(y)$ is 1. The reader will verify that as $\sigma(xy)$ is of finite order, there is at most one possibility for the character of σ. Thus, we have proven the following Lemma.

Lemma 10.1 (1) $\Delta(3,3,3)$ *has exactly one irreducible* $PSL_2(\mathbb{C})$*–character and it is the character of a representation with image* T_{12}*. It has exactly one reducible* $PSL_2(\mathbb{C})$*–character which can lie on a non-trivial curve in* $X_{PSL_2}(M)$.

(2) $\Delta(2,4,4)$ *has exactly three irreducible* $PSL_2(\mathbb{C})$*–characters and they are the characters of representations with dihedral images* D_2, D_4 *and* D_4. *It has exactly one reducible* $PSL_2(\mathbb{C})$*–character which can lie on a non-trivial curve in* $X_{PSL_2}(M)$.

(3) $\Delta(2,3,6)$ *has exactly two irreducible* $PSL_2(\mathbb{C})$*–characters, one corresponding to a representation with image* D_3, *and the other to a representations with image* T_{12}. *It has exactly one reducible* $PSL_2(\mathbb{C})$*–character which can lie on a non-trivial curve in* $X_{PSL_2}(M)$.

Proposition 7.6 now yields the estimates we need.

Proposition 10.2 *Suppose that* X_0 *is a non-trivial curve in* $X_{PSL_2}(M)$ *and that* α *is a slope on* ∂M *such that* $M(\alpha)$ *admits a Seifert structure with base orbifold* $S^2(3,3,3)$, $S^2(2,4,4)$, *or* $S^2(2,3,6)$. *If* α *is not a boundary slope associated to an ideal point of* X_0, *then*

$$\|\alpha\|_{X_0} \leq \begin{cases} s_{X_0}+4 & \text{if } M(\alpha) \text{ has base orbifold } S^2(3,3,3) \\ s_{X_0}+5 & \text{if } M(\alpha) \text{ has base orbifold } S^2(2,3,6) \text{ or } S^2(2,4,4). \end{cases}$$

Subcase 3.1 $M(\beta) \cong L_p \# L_q$, where $2 \leq p \leq q$.

Let $X_0 = X(1,1) \subset X_{PSL_2}(\mathbb{Z}/p * \mathbb{Z}/q) = X_{PSL_2}(M(\beta)) \subset X_{PSL_2}(M)$. Since we have assumed that β is not a singular slope for any closed essential surface in M, (6–2) and Proposition 6.6 imply that

$$\Delta(\alpha,\beta) = \frac{\|\alpha\|_{X_0}}{s_{X_0}} \quad \text{where} \quad s_{X_0} \geq \begin{cases} 2 & \text{if } p=2 \\ 4 & \text{if } p>2. \end{cases} \tag{10–1}$$

Hence Proposition 10.2 yields $\Delta(\alpha,\beta) \leq 2$ when $p>2$. Similarly, if there are no irreducible characters in $\nu(J_{X_0}(\alpha))$, then $\|\alpha\|_{X_0} \leq s_{X_0}+2$ (cf Lemma 6.5), which yields the desired distance estimate. Assume, then, that $p=2$ and $\nu(J_{X_0}(\alpha))$ contains at least one irreducible character.

Subsubcase 3.1.1 $2=p=q$.

In this case, all irreducible characters in X_0 are characters of representations which conjugate into $\mathcal{N}$ and therefore the base orbifold of $M(\alpha)$ cannot be $S^2(3,3,3)$ (Proposition 10.2). When it is $S^2(2,3,6)$, we obtain $\|\alpha\|_{X_0} \leq s_{X_0}+3$ and so $\Delta(\alpha,\beta) \leq 2$ by (10–1). When it is $S^2(2,4,4)$, Corollary 8.3 implies that the natural homomorphism $\pi_1(M) \to \pi_1(M(\beta))$ sends $\pi_1(\partial M)$ to the unique index 2 cyclic subgroup C of $\mathbb{Z}/2 * \mathbb{Z}/2$ (since β is not a singular slope for a closed essential surface in M). Thus

$\pi_1(\partial M)$ is sent to $\pm I$ under the diagonal representation whose character lies on X_0. It follows that $\nu(J_{X_0}(\alpha))$ does not contain a reducible character (cf Proposition 6.2). Thus Lemma 6.5 and Lemma 10.1 show that $\|\alpha\|_{X_0} \le s_{X_0} + 3$, which yields the desired result.

Subsubcase 3.1.2 $2 = p < q$.

In this case, X_0 contains exactly one character of an irreducible representation with image contained in $\mathcal{N}$ (Lemma 6.5). Thus, when the base orbifold of $M(\alpha)$ is $S^2(2,4,4)$ we have $\|\alpha\|_{X_0} \le s_{X_0} + 3$ and therefore $\Delta(\alpha, \beta) \le 2$. When it is $S^2(3,3,3)$ we have $\|\alpha\|_{X_0} \le s_{X_0} + 4$ so that $\Delta(\alpha,\beta) \le 3$. If this distance is 3, then X_0 contains the character of a representation with image T_{12} and therefore $q = 3$ (Lemma 6.5). Then $H_1(M(\beta)) \cong \mathbb{Z}/6$ so that $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/n$, where n divides 6. There is a primitive element $\lambda \in H_1(\partial M)$, unique up to sign, which is sent to a torsion element of $H_1(M)$. Let d be its order. The argument used in the proof of Observation 8.5 shows that there is a dual class $\mu \in H_1(\partial M)$ to λ which is sent to $(d, \bar{j}) \in \mathbb{Z} \oplus \mathbb{Z}/n = H_1(M)$. If $\beta = a\mu + b\lambda$ in $H_1(\partial M)$, then a homological calculation shows that $6 = |H_1(M(\beta))| = |dan|$. As d divides n and 6 is square-free, we have $d = 1$. Hence λ is homologically trivial in M and therefore if $\alpha = s\mu + t\lambda$, $H_1(M(\alpha)) \cong \mathbb{Z}/s \oplus \mathbb{Z}/n$. Since this group surjects onto $H_1(\Delta(3,3,3)) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$, both s and n are divisible by 3. Thus t is relatively prime to 3 and the same holds for a as $6 = |H_1(M(\beta))| = |dan| = |an|$. Hence $\Delta(\alpha,\beta) = |at - bs| \not\equiv 0 \pmod 3$, and we are done in this case.

Finally assume that the base orbifold of $M(\alpha)$ is $S^2(2,3,6)$. Since $\nu(J_{X_0}(\alpha))$ contains the character of an irreducible representation, Lemma 10.1 and Lemma 6.5 imply that $q = 3$. From Proposition 10.2 we have $\|\alpha\|_{X_0} \le s_{X_0} + 5$ so that $\Delta(\alpha,\beta) \le 3$. Note moreover, that if $\Delta(\alpha,\beta) = 3$, then $s_{X_0} = 2$ and $\nu(J_{X_0}(\alpha))$ consists of a T_{12} character and a reducible character (cf Proposition 7.6 and Proposition 10.2). Now $H_1(M(\beta)) \cong \mathbb{Z}/6$, so $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/n$, where n divides 6. The argument of the last paragraph shows that there is a basis μ, λ for $H_1(\partial M)$ such that if $i\colon \partial M \to M$ is the inclusion, then $i_*(\mu)$ generates a $\mathbb{Z}$–summand of $H_1(M)$, while $i_*(\lambda) = 0$. Thus for a primitive class $\delta = s\mu + t\lambda$ we have $H_1(M(\delta)) \cong \mathbb{Z}/s \oplus \mathbb{Z}/n$. In particular, taking $\beta = p\mu + q\lambda$ we have $\mathbb{Z}/6 \cong \mathbb{Z}/p \oplus \mathbb{Z}/n$, so $\gcd(p,n) = 6$ and $pn = 6$.

There is a presentation

$$\pi_1(M(\alpha)) \cong \langle x, y, h \mid x^2 = h^{-i}, y^3 = h^{-j}, (xy)^6 = h^{-k}, h \text{ central} \rangle$$

where i, j, k are relatively prime to $2, 3, 6$ respectively. Thus $H_1(M(\alpha))$ is presented by the matrix

$$A = \begin{pmatrix} 2 & 0 & 6 \\ 0 & 3 & 6 \\ i & j & k \end{pmatrix}.$$

Since the gcd of the minors of size 1 of A is 1, as are those of size 2, while the determinant of A is $6(k-2j-3i) \equiv 0 \pmod{12}$, we have $H_1(M(\alpha)) \cong \mathbb{Z}/12l$, where $l \geq 0$. On the other hand if $\alpha = s\mu + t\lambda$, then $H_1(M(\alpha)) \cong \mathbb{Z}/s \oplus \mathbb{Z}/n$, so $\gcd(s,n) = 1$ and $sn = 12l$. These two conditions are not mutually compatible when $n \in \{2, 6\}$, so $n \in \{1, 3\}$ is odd. But then $3 = \Delta(\alpha, \beta) = |at - sb| \equiv 0 \pmod 2$, which is impossible. Hence we must have $\Delta(\alpha, \beta) \leq 2$.

Subcase 3.2 $M(\beta) = S^1 \times S^2$.

In this case, β is the slope of the rational longitude in $H_1(\partial M)$ and therefore $b_1(M(\alpha)) = 0$. It follows that $M(\alpha)$ is not Haken [20, VI.13], and therefore Theorem 3.2 implies that α is not a boundary slope. Note, moreover, that as the Euler number $e(M(\alpha)) \in \mathbb{Q}$ is the obstruction to the existence of a horizontal surface in $M(\alpha)$, and since a Seifert manifold of the form we are considering admits a horizontal surface if and only if its first Betti number is 1 [20, VI.15], we have $e(M(\alpha)) \neq 0$.

Consider the canonical curve $X_M \subset X_{PSL_2}(M)$ defined by a complete hyperbolic structure [9, Section 9]. Denote by B_M the largest $\|\cdot\|_M$–ball which contains no non-zero elements of $H_1(\partial M)$ in its interior and recall that s_M is the radius of B_M. We have assumed that β is not a singular slope associated to a closed, essential surface in M, and therefore Theorem 3.2 implies that β is not a strict boundary slope. It follows from Proposition 8.1 (1) that $\|\beta\|_M = s_M$. Indeed, [11, Section 1] implies that $Z_x(f_\beta) \leq Z_x(f_\delta)$ for each $x \in \widetilde{X}_0$ and $\delta \in H_1(\partial M)$. According to Proposition 10.2 we have

$$\text{(10–2)} \quad \|\alpha\|_{X_M} \leq \begin{cases} s_M + 4 & \text{if } M(\alpha) \text{ has base orbifold } S^2(3,3,3) \\ s_M + 5 & \text{if } M(\alpha) \text{ has base orbifold } S^2(2,3,6) \text{ or } S^2(2,4,4). \end{cases}$$

Lemma 10.3 *Let $\beta = \{\pm b\}$ and $\beta^* = \{\pm b^*\}$ Then*

(1) *$b \in \partial B_M$ but is not a vertex. No class of distance 2 from b lies on ∂B_M.*

(2) *If $\pm(c_1 b + d_1 b^*), \pm(c_2 b + d_2 b^*), \dots, \pm(c_k b + d_1 b^*) \in H_1(\partial M)$ are the primitive classes associated to the vertices of B_M, then $\sum_{i=1}^k |d_i| \leq s_M$.*

(3) *If* $s_M = 2$*, then* $\Delta(\alpha,\beta) \le \frac{\|\alpha\|_M}{s_M}$*. Further, if* $s_M \ge 3$*, then* $\Delta(\alpha,\beta) < t\frac{\|\alpha\|_M}{s_M}$*, where*

$$t = \begin{cases} \frac{6}{5} & \text{if } s_M = 3 \\ \frac{4}{3} & \text{if } s_M = 4 \\ 2 & \text{if } s_M \ge 5. \end{cases}$$

Proof As β is not a strict boundary slope and $M(\beta)$ has a cyclic fundamental group, $\|\beta\|_M = s_M$ (Proposition 8.1(1)) and β is not a vertex of B_M. It is shown in [5, Lemma 6.4] that if there is a class of distance 2 from β lies on ∂B_M, then β would be a vertex of B_M. This proves part (1).

It was shown in [11, Section 1.4] that there is a homomorphism $\phi_x\colon H_1(\partial M) \to \mathbb{Z}$ such that $\Pi_x(f_\gamma) = |\phi_x(\gamma)|$. Since $|\phi_x(\delta_x)| = 0$, it is simple to see that for each $\gamma \in H_1(\partial M)$, $|\phi_x(\gamma)| = e\Delta(\gamma,\delta_x)$ for some fixed integer $e \ge 1$. In particular, we have $e = \Pi_x(f_\beta)/\Delta(\beta,\delta_x)$. Hence $d_x = \Delta(\beta,\delta_x)$ divides $\Pi_x(f_\beta)$ and for each γ we have $\Pi_x(f_\gamma) = \frac{\Delta(\gamma,\delta_x)}{\Delta(\beta,\delta_x)}\Pi_x(f_\beta)$. Summing over all the ideal points yields

$$\|\gamma\|_M = \sum_x \frac{\Delta(\gamma,\delta_x)}{\Delta(\beta,\delta_x)}\Pi_x(f_\beta). \tag{10–3}$$

In particular, $s_M = \|\beta\|_M = \sum_x \Pi_x(f_\beta) \ge \sum_x |d_x|$. This proves part (2) of the lemma.

It follows from part (1) that if $xb + yb^* \in B_M$, then $|y| < 2$, and therefore part (3) of the lemma holds for $s_M \ge 5$. Let

$$t_0 = \sup\{y \mid xb + yb^* \in B_M\}$$

and observe that $\alpha = \{\pm(pb + qb^*)\}$, where $q = \Delta(\alpha,\beta)$. Since $s_M/\|\alpha\|_M\alpha \in B_M$ we have $\Delta(\alpha,\beta) = q \le t_0\|\alpha\|_M/s_M$. Furthermore, we have strict inequality if there is a unique $xb + yb^* \in B_M$ with $y = t_0$, since in this case equality would imply that α is the slope of a vertex of B_M and therefore a strict boundary slope. To complete the proof of (3), we must show that t_0 is given as in the statement of the lemma when $s_M \in \{2,3,4\}$.

First note that there is a vertex of B_M of the form $x_0b + t_0b^*$. Let z be an ideal point of X_M associated to a strict boundary class $c_xb + d_xb^* = v_z \in H_1(\partial M)$. The vertex of B_M associated to v_z is given by $s_M/\|v_z\|_M v_z$. We explain below how to calculate the maximum value taken on by the b^*–coordinate of $s_M/\|v_z\|_M v_z \in \partial B_M$, where z varies over all ideal points of X_M.

If X_M has k ideal points $z_1, z_2, \dots, z_k$, then $\Pi_{z_1}(f_\beta), \Pi_{z_2}(f_\beta), \dots, \Pi_{z_k}(f_\beta)$ gives a partition of $s_M = \|\beta\|_M$ into k positive integers. Let $v_{z_i} = c_ib + d_ib^*$ and recall that d_i divides $\Pi_{z_i}(f_\beta)$. If we have prior knowledge of the integers $\Pi_{z_i}(f_\beta), c_i, d_i$, then we can calculate the values $\|v_{z_i}\|_M$ using (10–3), and therefore we can determine the vertices of B_M. In general though, we are not given these values, so we proceed as follows.

Fix an integer $k \geq 2$, a partition $(\Pi_1, \Pi_2, \dots, \Pi_k)$ of s_M, and a sequence of classes $v_i = c_i b + d_i b^*$, where $d_i \geq 1$ is a divisor of Π_i. Set $\|v_i\| = \sum_{j \neq i} \frac{\Delta(v_i, v_j)}{\Delta(b, v_j)} \Pi_j$ and $v_i = \frac{s_M}{\|v_i\|} v_i$. Next we consider the polygon in $H_1(\partial M; \mathbb{R})$ whose vertices are $\pm v_1, \pm v_2, \dots, \pm v_k$. We discard all polygons which are not convex, or which contain a non-zero element of $H_1(\partial M)$ in their interior, or whose maximal b^*–coordinates are at least 2, since such polygons cannot be the boundaries of a possible B_M. In this way we obtain a list of the possibilities for B_M for each value of s_M. In particular, we can determine an upper bound for their maximal b^*–coordinate. For instance when $s_M = 2$ or 3, an $SL_2(\mathbb{C})$ version of the calculation is contained in [5, Lemma 6.5]. The case $s_M = 4$ is handled similarly from this one observes that part (3) of the lemma holds. This completes the proof of the lemma. □

Note that Inequality (10–2) and part (3) of the previous lemma show that

$$\Delta(\alpha, \beta) \leq 3 \tag{10–4}$$

We must show that this inequality is strict. Denote the base orbifold of $M(\alpha)$ by $S^2(r, s, t)$. By Observation 8.5, there is a dual slope $\beta^* = \{\pm b^*\}$ for $\beta = \{\pm b\}$, an integer $n \geq 1$, and an isomorphism $H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/n$ such that if $i \colon \partial M \to M$ is the inclusion, then

$$i_*(b^*) = (n, \bar{0}), \quad i_*(b) = (0, \bar{1}).$$

Let $\xi \in H_1(M)$ correspond to $(1, 0)$, so that $i_*(b^*) = n\xi$. Choose integers t, u such that

$$\alpha = \{\pm(tb^* + ub)\}.$$

Then $\Delta(\alpha, \beta) = |t|$.

Lemma 10.4 *There is an isomorphism $H_1(M(\alpha)) \cong \mathbb{Z}/(u, n) \oplus \mathbb{Z}/\frac{tn^2}{(u,n)}$, where $\mathbb{Z}/(u, n)$ and $\mathbb{Z}/\frac{tn^2}{(u,n)}$ are generated, respectively, by the images of $\frac{tn}{(u,n)}\xi + \frac{u}{(u,n)} i_*(\beta)$ and ξ. Furthermore,*

(1) *if $(r, s, t) = (3, 3, 3)$, then $(u, n) = 3$. Hence $\Delta(\alpha, \beta) = |t| \not\equiv 0 \pmod 3$.*

(2) *if $(r, s, t) = (2, 4, 4)$, then $(u, n) = 2$ and $n \equiv 0 \pmod 4$. Hence $\Delta(\alpha, \beta) = |t|$ is odd.*

(3) *if $(r, s, t) = (2, 3, 6)$, then $\gcd(u, n) = 1$ and tn^2 is divisible by 12.*

Proof Since $e(M(\alpha)) \neq 0$, $H_1(M(\alpha))$ is finite. Moreover, it follows from our conventions that it is presented by the matrix $\left(\begin{smallmatrix} tn & 0 \\ u & n \end{smallmatrix}\right)$. Thus

$$H_1(M(\alpha)) \cong \mathbb{Z}/(u,n) \oplus \mathbb{Z}/\tfrac{tn^2}{(u,n)},$$

where the factors are generated as claimed. Comparison of this isomorphism with the calculations of the previous lemma yields the remaining conclusions of this one. □

Part (1) of the previous lemma and Inequality (10–4) show that $\Delta(\alpha,\beta) \leq 2$ when $(r,s,t) = (3,3,3)$. In order to deal with the remaining two cases we suppose that $\Delta(\alpha,\beta) = 3$ in order to derive a contradiction. Setting $\beta = \{\pm b\}$ and $\beta^* = \{\pm b^*\}$ we have

$$\alpha = \pm(3b^* + ub)\}$$

so that $\gcd(3,u) = 1$.

Assume that $(r,s,t) = (2,3,6)$. Then Lemma 10.4 (3) implies that $3n^2$ is even and $\gcd(u,n) = 1$, so n is even and u is odd. Thus $\gcd(u,6) = 1$. Consider the representation ρ: $\pi_1(M) \to PSL_2(\mathbb{C})$ with image D_3 constructed as a composition of surjective homomorphisms $\pi_1(M) \to \pi_1(M(\alpha)) \to \pi_1^{\mathrm{orb}}(S^2(2,3,6)) = \Delta(2,3,6) \to D_3 \subset PSL_2(\mathbb{C})$. Now $\rho(\pi_1(\partial M)) \subset D_3$ is Abelian, hence cyclic of order $1,2$ or 3. It cannot have order 1, as otherwise it would factor through $\pi_1(M(\beta)) \cong \mathbb{Z}$. Thus it has order 2 or 3. Since $\rho(\alpha) = \pm I$ and $|u| = \Delta(\alpha,\beta^*)$ is relatively prime to 6, $\rho(\pi_1(\partial M)))$ is generated by $\rho(b^*)$. Thus the image of b^* generates the image of $\pi_1(\partial M)$ under the composition of ρ with the Abelianization homomorphism ϕ: $D_3 \to \mathbb{Z}/2$. Now $\phi \circ \rho$ factors through $H_1(M)$ and b^* is divisible by 2 in this group (Observation 8.5). Thus $\phi \circ \rho(b^*) = 0$ and therefore $\rho(b^*) \in [D_3, D_3] = \mathbb{Z}/3$. But then as $\Delta(\alpha,\beta) = 3$, ρ factors through $\pi_1(M(\beta)) \cong \mathbb{Z}$, which is impossible. We conclude that $\Delta(\alpha,\beta) \leq 2$.

Finally assume that $(r,s,t) = (2,4,4)$. There is a dual class $\beta_0^* = \{\pm b_0^*\}$ for β such that

$$\alpha = \{\pm(b + 3b_0^*)\}.$$

Set

$$b_1^* = b + b_0^*.$$

Lemma 10.5

(1) *b is sent to a generator of a $\mathbb{Z}/2$ factor of $H_1^{\mathrm{orb}}(S^2(2,4,4)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$ under the composition $H_1(\partial M) \to H_1(M) \to H_1^{\mathrm{orb}}(S^2(2,4,4))$.*

(2) *If $x \in J_{X_M}(\alpha)$, then $f_{\beta_1^*}(x) = 0$.*

Proof (1) Lemma 10.4 implies that in our situation,

$$H_1(M(\alpha)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/\tfrac{3n^2}{2},$$

where $\mathbb{Z}/2$ and $\mathbb{Z}/\frac{3n^2}{2}$ are generated, respectively, by the images of $\omega = \frac{3n}{2}\xi + \frac{u}{(u,n)} i_*(b)$ and ξ. It follows that ω is sent to an element of order 2 in

$$H_1^{\mathrm{orb}}(S^2(2,4,4)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4,$$

and ξ is sent to an element of order 4. Lemma 10.4 also shows that $\frac{u}{(u,n)}$ is odd, so the image of b in $H_1^{\mathrm{orb}}(S^2(2,4,4))$ coincides with that of $\omega - \frac{3\epsilon n}{2}\xi$ for some $\epsilon = \pm 1$. It follows that ξ and b generate $H_1^{\mathrm{orb}}(S^2(2,4,4))$, so the image of b is non-zero there, and since n is divisible by 4, the image of $2b$ in $H_1^{\mathrm{orb}}(S^2(2,4,4))$ is zero. Thus (1) holds.

(2) Let $x \in J_{X_M}(\alpha)$ and set $\nu(x) = \chi_\rho$, where $\rho \in R(X_M)$. As $\alpha = \{\pm(-2b + 3b_1^*)\}$, we see that

$$\rho(b_1^*)^{-3} = \rho(b)^2. \tag{10–5}$$

We observed in the opening paragraph of case 3 that ρ factors through a representation $\sigma\colon \Delta(2,4,4) \to PSL_2(\mathbb{C})$. If ρ is reducible, there is a diagonal representation $\sigma_0\colon \Delta(2,4,4) \to PSL_2(\mathbb{C})$ with the same character as σ. Since σ_0 factors through $H_1(\Delta(2,4,4)) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$, (10.5.1) shows that σ_0 sends the image of b_1^* to $\pm I$. It follows that (2) holds in this case.

Assume next that ρ is irreducible. Lemma 10.1 shows that the image of ρ is either D_2 or D_4 and so as $\pi_1(M(\beta)) \cong \mathbb{Z}$, we have $\rho(b) \neq \pm I$. On the other hand, by (10.5.1) it suffices to show that $\rho(b)^2 = \pm I$. This is obvious if χ_ρ is a D_2–character so suppose that it is a D_4–character. Write $\Delta(2,4,4) = \langle x, y \mid x^2 = y^4 = (xy)^4 = 1\rangle$ and $D_4 = \langle z, w \mid z^2 = w^4 = (zw)^2 = 1\rangle \subset PSL_2(\mathbb{C})$. There are two characters of representations $\Delta(2,4,4) \to PSL_2(\mathbb{C})$ with image D_4 and they are represented by the homomorphisms $\phi_1, \phi_2\colon \Delta(2,4,4) \to D_4$, where $\phi_1(x) = z, \phi_1(y) = w$ and $\phi_2(x) = zw, \phi_2(y) = w$. As these two representations differ by an automorphism of D_4, it suffices to prove that the image of b in D_4 under ϕ_1 has order 2.

Suppose otherwise. Then its image has order 4 and so b is sent to $y^\epsilon v \in \Delta(2,4,4)$, where $\epsilon \in \{\pm 1\}$ and $v \in \ker(\phi_1)$. Now $\ker(\phi_1)$ is normally generated in $\Delta(2,4,4)$ by $(xy)^2$, so $v = \Pi_{i=1}^k u_i (xy)^{2\theta_i} u_i^{-1}$, where $u_i \in \Delta(2,4,4)$ and $\theta_i \in \{\pm 1\}$. Now x, resp. y, projects to an element $\bar{x}$, resp. $\bar{y}$, of order 2, resp. 4, in $H_1(\Delta(2,4,4))$ and therefore as b is sent to $(\epsilon + 2\sum_i \theta_i)\bar{y}$ in this group, it also has order 4 there. But this contradicts part (1) of the lemma. Therefore b must be sent to an element of order 2 in D_4. □

Now we complete the proof of our current case. We set $a = b + 3b^*$, so $\alpha = \{\pm a\}$.

Suppose first that $s_M = 2$. Then the only roots of $\widetilde{f}_\beta$ on $\widetilde{X}_M$ are the two discrete, faithful characters of $\pi_1(M)$. It follows that $\widetilde{f}_\beta(x) \neq 0$ for each $x \in J_{X_M}(\alpha)$. But then part (2) of the previous lemma shows that $J_{X_M}(\alpha) \subset J_{X_M}(\beta_1^*)$ and so putting these observations together with Lemma 10.3 (3) we conclude that $\|\beta_1^*\|_M \geq \|\alpha\|_M \geq 3s_M$. Then $\frac{1}{3}\|\beta_1^*\|_M \geq s_M$. Now $\frac{1}{7}a$ lies on the line in $H_1(\partial M;\mathbb{R})$ which passes through b and $\frac{1}{3}b_1^*$ and consideration of its position there shows that $\|\frac{1}{7}\alpha\|_M \geq s_M = 2$. But this contradicts $\|\alpha\|_M \leq s_M + 5 = 7$, so $s_M \neq 2$.

Next suppose that $s_M = 3$. If $x \in J_{X_M}(\alpha)$ is such that $\chi_\rho = \nu(x)$ is irreducible, the image of ρ is finite and non-Abelian (Lemma 10.1), from which we deduce $f_\beta(\chi_\rho) \neq 0$. Thus part (2) of the previous lemma shows that $x \in J_{X_M}(\beta_1^*)$. It follows that $\|\beta_1^*\|_M \geq s_M + 3 = 2s_M$. Thus $[\frac{1}{2}b_1^*, b_0^*] \cap \mathrm{int}(B_M) = \varnothing$. But $\frac{1}{4}a \in [\frac{1}{2}b_1^*, b_0^*]$ so that $\frac{1}{4}\|\alpha\| \geq s_M = 3$. But then $8 = s_M + 5 \geq \|\alpha_M\| \geq 12$, which is impossible. Hence $s_M \neq 3$.

Next note that $s_M \neq 4$ since Lemma 10.3 (3) shows that $\Delta(\alpha,\beta) < \frac{4}{3}(\frac{9}{4}) = 3$.

Suppose, then, that $s_M \geq 5$ so that $\|\alpha\|_M \leq s_M + 5 \leq 2s_M$, or equivalently, $\frac{a}{2} \in B_M$. The line segment $[-b, \frac{a}{2}]$, which passes through b_0^*, is contained in B_M. Therefore it is contained in ∂B_M and hence $\|\frac{a}{2}\|_M = s_M$. But then $2s_M = \|\alpha\|_M \leq s_M + 5$. It follows that $s_M = 5$. We noted above that α is not a boundary slope, so $\frac{a}{2}$ is not a vertex of B_M, nor is $b + 2b_0^*$ by Lemma 10.3(1). Thus there is a vertex $v_0 = x_0 b + y_0 b_0^*$ of the edge of ∂B_M containing $[-b, \frac{a}{2}]$ with $2 < \frac{y_0}{x_0} < 2$. Let $c_0 b + d_0 b_0^* \in H_1(\partial M)$ be the boundary class which is a rational multiple of v_0. As $s_M = 5$, part (2) of Lemma 10.3 shows that $|d_0| \in \{3,4\}$ and therefore since $\frac{d_0}{c_0} = \frac{y_0}{x_0}$, either $\frac{3}{|c_0|} \in (2,3)$ or $\frac{4}{|c_0|} \in (2,3)$, which is impossible. This final contradiction shows that $\Delta(\alpha,\beta) \leq 2$ when $(r,s,t) = (2,4,4)$. (Lemma 10.4 (2) then shows that we have $\Delta(\alpha,\beta) = 1$ in this case).

11 Characteristic subsurfaces associated to a reducible Dehn filling

In this section we develop the background results needed to prove Proposition 5.4. We assume that M is a compact, connected, orientable, simple 3–manifold with torus boundary and $M(\beta)$ is a connected sum of two non-trivial lens spaces one of which is not P^3.

Recall that an embedded 2–sphere in a 3–manifold is called essential if it does not bound a 3–ball. Since $M(\beta)$ is a connected sum of two non-trivial lens spaces, a standard cut-paste argument shows that there is an essential 2–sphere $\widehat{F}$ in $M(\beta)$ such that $F = M \cap \widehat{F}$ is a connected properly embedded essential planar surface F in M with boundary slope β. Any such surface F is separating in M since $M(\beta)$ has zero first Betti number. Any such surface F is not a semi-fibre since otherwise $M(\beta)$ would be a connected sum of two P^3's. Among all such surfaces, we assume that F has been chosen to have the minimal number of boundary components. Set $m = |\partial F|$. Note that m is an even number since F is separating. Since M is a simple manifold, we have $m \geq 4$. The planar surface F splits M into two components, X^+ and X^-, and $\widehat{F}$ separates $M(\beta)$ as $\widehat{X}^+$ and $\widehat{X}^-$ each of which is a punctured lens space. We may and shall assume that $\widehat{X}^+$ is not P^3. We use ϵ to denote an element in $\{\pm\}$.

We call a properly embedded annulus $(A, \partial A) \subset (X^\epsilon, F)$ *essential* if its inclusion is not homotopic rel ∂A to a map whose image lies in F. The minimality of $m = |\partial F|$ has the following useful consequence.

Lemma 11.1 *Suppose that $(A, \partial A) \subset (X^\epsilon, F)$ is a properly embedded essential annulus. The boundary of A splits $\widehat{F}$ into an annulus B and two disks N, N'. Then the number of boundary components of F which lie in N equals the number of boundary components of F which lie in N'.*

Proof Since $\widehat{X}^\epsilon$ has zero first Betti number, the annulus A separates $\widehat{X}^\epsilon$ into two pieces W and V, where ∂W is a 2–sphere and ∂V is a torus.

Let n, n' and b be the number of boundary components of F which lie in N, N' and B respectively. We may suppose that $n \leq n'$. If $b = 0$, then $\partial V \subset M$ and so V is a solid torus in which the winding number of B is at least 2 (since A is an essential annulus and thus not parallel to B). It follows that a regular neighborhood in $M(\beta)$ of $N \cup V$ is a punctured lens space whose boundary S is an essential 2–sphere in $M(\beta)$. Hence the number of components of $S \cap \partial M$ is at least m. That is, $2n \geq m = n + n'$. Hence $n \geq n'$, which implies that desired result.

On the other hand if $b > 0$, then ∂W is inessential in $M(\beta)$ and thus W is a 3–ball. If the 2–sphere boundary S_1 of a regular neighborhood U in X^ϵ of $N \cup V$ is inessential in $M(\beta)$, it follows that U is also a 3–ball. But this is impossible as it would imply that $\widehat{X}^\epsilon$ is a 3–ball. Hence S_1 is essential in $M(\beta)$. Since it intersects ∂M in $2n + b$ components we have $2n + b \geq m = n + b + n'$, ie $n \geq n'$. This completes the proof. □

Each essential annulus A properly embedded in (X^ϵ, F) separates the punctured lens space $\widehat{X}^\epsilon$, and hence X^ϵ. Let $V(A)$ be the component of $\widehat{X}^\epsilon_A$ such that $V(A) \cap \widehat{F}$ is an annulus $E(A)$. We call a pair of disjoint essential annuli A and A' properly embedded in (X^ϵ, F) and $(X^{\epsilon'}, F)$ *nested* if either $\partial A' \subset E(A)$ or $\partial A \subset E(A')$.

The only Seifert fibred spaces contained in a simple manifold are solid tori. This fact has the following useful application.

Lemma 11.2 *If A and A' are disjoint essential annuli properly embedded in (X^ϵ, F) and $(X^{\epsilon'}, F)$, then they are nested.*

Proof Let c_0, c_1 be the boundary components of A and c_0', c_1' those of A'. We assume first that $\epsilon = \epsilon'$. If A, A' are not nested, then $V(A) \cap V(A') = \varnothing$ and we can number the boundary components of A and A' in such a way that they divide the 2–sphere $\widehat{F}$ into five components whose interiors are pairwise disjoint: a disk N bounded by c_1; the annulus $B = E(A)$ bounded by c_1 and c_0; an annulus E bounded by c_0 and c_0'; an annulus $B' = E(A')$ bounded by c_0' and c_1'; and a disk N' bounded by c_1'. Let $n = |N \cap \partial F|, b = |B \cap \partial F|$ and define e, b', n' similarly. According to Lemma 11.1 we have $n = e + b' + n'$ and $n' = n + b + e$. It follows that $b = e = b' = 0$ and therefore $V(A), E, V(A') \subset M$. Since M is simple, both $V(A)$ and $V(A')$ are solid tori and as A and A' are essential in (X^ϵ, F), the winding numbers of B in $V(A)$ and B' in $V(A')$ are at least 2 in absolute value. It follows that a regular neighbourhood of $V(A) \cup E \cup V(A')$ in M is Seifert fibred with an incompressible torus for boundary. But the simple manifold M does not contain such a Seifert fibred space. Thus A, A' must be nested.

Assume, then, that $\epsilon \neq \epsilon'$. The case where $E(A) \cap E(A') = \varnothing$ can be shown to be impossible as in the previous paragraph. Next suppose that $E(A) \cap E(A') \neq \varnothing$ but neither $E(A) \subset E(A')$ nor $E(A') \subset E(A)$. We number the boundary components of A and A' in such a way that they divide the 2–sphere $\widehat{F}$ into five components whose interiors are pairwise disjoint: a disk N bounded by c_1; an annulus B bounded by c_1 and c_0'; an annulus E bounded by c_0' and c_0; an annulus B' bounded by c_0 and c_1'; and a disk N' bounded by c_1'. Let $n = |N \cap \partial F|$ and define b, e, b', n' similarly. Lemma 11.1 implies that $b = b' = 0$ and thus A' may be isotoped in $(X^{\epsilon'}, F)$ so that $\partial A' = \partial A$. Then $T = A \cup A'$ is a torus in M which must be compressible as $m > 2$. As T is not contained in a 3–ball, it bounds a solid torus V in M. It is easy to see that $V = V(A) \cup V(A')$ and so $E = V(A) \cap V(A') \subset F$, ie $e = 0$. But then E is isotopic through V to either A or A', which contradicts the essentiality of these two annuli. Hence it must be that either $\partial A' \subset E(A)$ or $\partial A \subset E(A')$ and thus A, A' are nested. □

Lemma 11.3 *If A and A' are disjoint essential annuli properly embedded in (X^ϵ, F) and $(X^{\epsilon'}, F)$ such that a boundary component of A is isotopic in F to a boundary component of A', then $\epsilon = \epsilon'$ and A and A' are parallel in X^ϵ.*

Proof By the previous lemma, A and A' are nested. Without loss of generality we may suppose that $\partial A \subset E(A')$. Let c_0, c_1 be the boundary components of A, and c_0', c_1' those of A', where the indices are chosen in such a way that the four curves c_0, c_1, c_0' and c_1' divide $\widehat{F}$ into five components whose interiors are pairwise disjoint: a disk N bounded by c_0'; an annulus $E \subset F$ bounded by c_0 and c_0'; an annulus B bounded by c_0 and c_1; an annulus E' bounded by c_1 and c_1'; and a disk N' bounded by c_1'. Let n be the number of components of $N \cap \partial M$. Define b, e', n' similarly so that $n + b + e' + n' = m$. Lemma 11.1 shows that $e' = 0$. Now it must be that $\epsilon = \epsilon'$ as otherwise A' can be isotoped in $(X^{\epsilon'}, F)$ so that its boundary equals that of A. The argument in the last paragraph of the proof of the previous lemma shows that this situation cannot arise. Thus $\epsilon = \epsilon'$. If $b = 0$, then $A' \cup E \cup B \cup E'$ is a torus bounding a solid torus V in M. Since $A' \subset V$ and is not parallel into F, it must be parallel into A. Thus the lemma holds. On the other hand, if $b \neq 0$, then $S_1 = N \cup E \cup A \cup E' \cup N'$ is an inessential 2–sphere in $M(\beta)$ and therefore bounds a 3–ball W in $\widehat{X}^\epsilon$. It follows that A and A' are parallel in X^ϵ through W. This completes the proof. □

Let $(\Sigma_*^\epsilon, \Phi_*^\epsilon) \subset (X^\epsilon, F)$ be the characteristic Seifert pair of (X^ϵ, F) and $(\Sigma^\epsilon, \Phi^\epsilon) \subset (X^\epsilon, F)$ be the characteristic I–bundle pair it contains. We shall use τ_ϵ to denote the free involution on Φ^ϵ induced by I fibres of Σ^ϵ. Let Φ_j^ϵ denote the jth characteristic subsurface with respect to the pair (M, F) as defined in [3, Section 5]. Note that Φ_1^ϵ is the large part of Φ^ϵ and that the involution τ_ϵ restricts to a free involution on Φ_1^ϵ, which will still be denoted as τ_ϵ. Let $(\Sigma_1^\epsilon, \Phi_1^\epsilon)$ be the corresponding I–bundle pair.

Lemma 11.4 *(Σ^+, Φ^+) is a product I–bundle pair, ie there is no embedded Möbius band $(B, \partial B) \subset (\Sigma^+, \Phi^+)$. In particular, $\Phi^+ \neq F$.*

Proof Suppose otherwise that $(B, \partial B) \subset (\Sigma^+, \Phi^+)$ is an embedded Möbius band. Then ∂B bounds a disk N in $\widehat{F}$. The union of N and B is an embedded projective plane in $\widehat{X}^+$. A regular neighborhood of this projective plane in $\widehat{X}^+$ is a punctured P^3. This implies that $\widehat{X}^+$ itself is a punctured P^3, contrary to our assumptions. □

Lemma 11.5 *Suppose that $(B, \partial B) \subset (X^-, F)$ is a properly embedded Möbius band. Then ∂B cannot be isotoped into Φ_1^+.*

Proof Let A' be the essential annulus in (X^-, F) which is the frontier of a regular neighbourhood of B in X^-. If ∂B can be isotoped into Φ_1^+, then the previous lemma shows that there is an essential annulus A, properly embedded in (X^+, F), whose boundary contains ∂B. After a small isotopy of A rel ∂B we can assume that A and A' are disjoint. But this contradicts Lemma 11.3 since a boundary component of A is isotopic to a boundary component of A'. Thus ∂B cannot be isotoped into Φ_1^+. □

A *root torus* in (X^ϵ, F) is a solid torus $\Theta \subset X^\epsilon$ such that $\Theta \cap F$ is an incompressible annulus in $\partial\Theta$ whose winding number in Θ is at least 2 in absolute value.

Lemma 11.6

(1) *Let Θ be a component of Σ^ϵ_* and set $\Phi = \Theta \cap F$. If (Θ, Φ) is not an $(I, \partial I)$–bundle, then Θ is a root torus.*

(2) *Let Φ_1 and Φ_2 be distinct components of Φ^ϵ_* and $E \subset F$ an annulus whose boundary consists of a component c_1 of $\partial\Phi_1$ and a component c_2 of $\partial\Phi_2$. Then after possibly renumbering Φ_1, Φ_2, there are an annulus $E' \supseteq E$ in F with $c_1 \subset \partial E'$ and components Σ_1, Σ_2 of Σ^ϵ_* such that Σ_1 is a product I–bundle component of Σ^ϵ_1 containing Φ_1 and Σ_2 is a root torus such that $\Sigma_2 \cap F \subset E'$. Moreover, either*

 (i) *$E = E'$, $\Sigma_1 \cap F = \Phi_1 \cup \Phi_2$, $\tau_\epsilon(c_1) = c_2$ and $\Sigma_2 \cap F \subset int(E)$, or*

 (ii) *$E \neq E'$, $\Sigma_2 \cap F = \Phi_2 \subset int(E')$.*

In particular, there is a root torus in X^ϵ whose intersection with F lies in E.

Proof

(1) Since simple manifolds contain no Seifert submanifolds with incompressible boundaries, Θ is a solid torus. Now Φ is a disjoint union of essential annuli $B_1, B_2, \dots, B_n$. If $n > 1$, then Lemma 11.3 shows that $n = 2$ and $(\Theta, \Phi) \cong (S^1 \times I \times I, S^1 \times I \times \partial I)$, contrary to our hypotheses. Thus $n = 1$ and from the defining properties of the characteristic Seifert pair we see that the winding number of $\Phi = B_1$ in Θ is at least 2 in absolute value.

(2) Let Σ_1, Σ_2' be the components of Σ^ϵ_* which contain Φ_1, Φ_2 respectively. For $j = 1, 2$ there is a unique annulus $(A_j, \partial A_j) \subset (\mathrm{fr}_{X^\epsilon}(\Sigma^j), \partial\Phi^j)$ which is essential in (X^ϵ, F) and which contains c_j. If $A_1 = A_2$, then $\Sigma_1 = \Sigma_2'$ and so $\Sigma_1 \cap F \supseteq \Phi_1 \cup \Phi_2$ has at least two components. It follows that Σ_1 is a product I–bundle with $\Sigma_1 \cap F = \Phi_1 \cup \Phi_2$ (cf part (1) of the lemma). Clearly $\tau_\epsilon(c_1) = c_2$. Moreover, $A_2 \cup E$ is a torus in M which bounds a solid torus $V \subset X^\epsilon$. Since A_2 is essential, it is isotopic to a component Σ_2 of Σ^ϵ_* with $\Sigma_2 \cap F \subset \mathrm{int}(E)$. Thus (i) holds.
Assume, then, that $A_1 \neq A_2$. According to Lemma 11.3, A_1 and A_2 are parallel in X^ϵ. Hence there is another annulus E^* in F such that $\partial(A_1 \cup A_2) = \partial(E \cup E^*)$. Lemma 11.2 implies that at least one of the $(\Sigma_j, \Sigma_j \cap F)$, say $(\Sigma_1, \Sigma_1 \cap F)$, is an $(I, \partial I)$–bundle. Then $(\Sigma_2, \Sigma_2 \cap F)$ cannot be an $(I, \partial I)$–bundle as otherwise the product region N between A_1 and A_2 could be used

to build an $(I, \partial I)$–bundle structure on $\Sigma_1 \cup N \cup \Sigma_2$, contrary to the defining properties of Σ^ϵ_*. Thus (Σ_2, Φ_2) is a root torus. Set $E' = E \cup (\Sigma_2 \cap F) \cup E^*$ and observe that (ii) holds. □

A boundary component of Φ^ϵ or Φ^ϵ_j is called an *inner* boundary component if it is not isotopic in F to a component of ∂F, otherwise it is called an *outer* boundary component. Note that every boundary component c of Φ^ϵ_1 is a boundary component of an essential annulus in $(\Sigma^\epsilon_1, \Phi^\epsilon_1) \subset (X^\epsilon, F)$ whose boundary is c and $\tau_\epsilon(c)$. The following result is a consequence of Lemma 11.1.

Lemma 11.7 *A simple closed curve c in F is an inner, resp. outer, boundary component of Φ^ϵ if and only if $\tau_\epsilon(c)$ is an inner, resp. outer, boundary component of Φ^ϵ.*

By Lemma 11.7 and Lemma 11.3, we can and shall normalize Φ^ϵ_j to have the property that if a component of ∂F is isotopic to a boundary component of Φ^ϵ_j, then it is already contained in Φ^ϵ_j.

Recall from [3, Section 7] that a subsurface T of F is said to be *tight* if the frontier of T in F is a connected simple closed curve. Thus a component of Φ^ϵ_1 is tight if and only if it has exactly one inner boundary component. It follows from Lemma 11.7 that τ_ϵ permutes the tight components of Φ^ϵ_1. Note also that a component Φ_0 of Φ^ϵ_1 left invariant by the free involution τ_ϵ has an even number of inner boundary components since $\tau_\epsilon|\Phi_0$ reverses orientation. In particular, no tight component of Φ^ϵ_1 is invariant under τ_ϵ. Thus they are paired by this involution.

Lemma 11.8 *If $\Phi^\epsilon_1 \neq F$ and $\chi(F) = \chi(\Phi^\epsilon_1)$, then Φ^ϵ_1 consists of a pair of tight components T_1, T_2 and it contains ∂F. Moreover, $\tau_\epsilon(T_1) = T_2$.*

Proof Note that we also have $\chi(F) = \chi(\Phi^\epsilon)$ and $\Phi^\epsilon \neq F$. Obviously Φ^ϵ has at least two tight components T_1, T_2 with $\tau_\epsilon(T_1) = T_2$. If c_j denotes the inner boundary component of T_j, then we also have $\tau_\epsilon(c_1) = c_2$. Since $\chi(F) = \chi(\Phi^\epsilon)$, there is an annulus $E \subset \mathrm{int}(F)$ such that $E \cap \Phi^\epsilon = \partial E$ and $E \cap T_1 = c_1$. According to Lemma 11.6 (2), there is a product I–bundle component of Σ^ϵ which intersects F in $T_1 \cup \tau_\epsilon(T_1) = T_1 \cup T_2$ and $\partial E = c_1 \cup \tau_\epsilon(c_1) = c_1 \cup c_2$. It follows that $F = T_1 \cup_{c_1} E \cup_{c_2} T_2$ as claimed by the lemma. □

Suppose that c is a simple closed curve in F. We will say that c *sweeps out* an essential annulus in (X^ϵ, F) if there is an essential annulus in (X^ϵ, F) having a boundary component isotopic to c.

Lemma 11.9 *Let c be an essential simple closed curve contained in Φ^ϵ. If c sweeps out an essential annulus A in (X^ϵ, F), then A is isotopic in (X^ϵ, F) to an essential annulus in the component of Σ^ϵ which contains c. In particular, ∂A is isotopic in F to $c \cup \tau_\epsilon(c)$.*

Proof Let Φ_0 be the component of Φ^ϵ which contains c and Σ_0 the component of Σ^ϵ containing Φ_0. The annulus $(A, \partial A)$ is homotopic in (X^ϵ, F) into a component Θ of the characteristic Seifert pair $(\Sigma^\epsilon_*, \Phi^\epsilon_*)$. If $\Theta = \Sigma_0$, then it is easy to see that the lemma holds. On the other hand if $\Theta \neq \Sigma_0$, then c is isotopic in F to the core of an annulus $E \subset F$ whose boundary consists of a component of $\partial\Phi_0$ and a component of $\partial(\Theta \cap F)$. Without loss of generality we can suppose that $c = \partial E \cap \Phi_0$. Then c sweeps out an annulus $A_1 \subset \mathrm{fr}_{X^\epsilon}(\Sigma_0)$ which is essential in (X^ϵ, F). Set $c' = \partial E \setminus c \subset \partial(\Theta \cap F)$ and let A_2 be the essential annulus contained in $\mathrm{fr}_{X^\epsilon}(\Theta)$ which is swept out by c'. By Lemma 11.3, A_1 is parallel to A_2 in X^ϵ and by Lemma 11.6, $(\Theta, \Theta \cap F)$ is a root torus. Since A is homotopic into Θ but not into F, it is isotopic to A_2, and therefore to $A_1 \subset \Sigma^\epsilon$. This completes the proof of the lemma. □

Lemma 11.10 *Let c be an essential simple closed curve in F. The following conditions are equivalent:*

1. *c sweeps out an essential annulus in (X^ϵ, F);*
2. *c is isotopic in F to a simple closed curve c' in Φ^ϵ such that the geometric intersection number of c' and $\tau_\epsilon(c')$ is 0.*

Proof From Lemma 11.9 it is clear that (1) implies (2).

If condition (2) holds for c, then, by choosing a negatively curved metric on F, we may assume that either c' and $\tau_\epsilon(c')$ are disjoint or that c' is invariant under τ_ϵ. In the first case, there is an essential annulus in (X^ϵ, F) with boundary curves c' and $\tau_\epsilon(c')$. In the second case there is an embedded Möbius band $(B, \partial B) \subset (X^\epsilon, F)$ with boundary curve c'. The frontier of a regular neighborhood of B in X^ϵ is an essential annulus with both boundary curves isotopic to c', and hence to c. □

Lemma 11.11 *If c is an inner boundary component of Φ^ϵ_j which is isotopic to a simple closed curve in $\Phi^{-\epsilon}_{j+1}$, then c sweeps out an essential annulus in $(X^{-\epsilon}, F)$.*

Proof Let c' be a simple closed curve in $\Phi^{-\epsilon}_{j+1}$ which is isotopic to c. Since $\tau_{-\epsilon}(c')$ lies in Φ^ϵ_j, and since c' is isotopic to a boundary curve of Φ^ϵ_j, it follows that the geometric intersection number of c' and $\tau_{-\epsilon}(c')$ is zero. Now apply Lemma 11.10. □

Recall from [3, Proposition 5.3.1] that for each $\epsilon \in \{\pm\}$ and $j \geq 0$, there is a homeomorphism $h_j^\epsilon\colon \Phi_j^\epsilon \to \Phi_j^{(-1)^{j+1}\epsilon}$, unique up to isotopy, which satisfies some useful properties. In particular,

$$h_{2j}^\epsilon\colon \Phi_{2j}^\epsilon \xrightarrow{\cong} \Phi_{2j}^{-\epsilon} \text{ for each } \epsilon \in \{\pm\} \text{ and each } j \geq 0. \tag{11–1}$$

Moreover,

$$h_{2j+1}^\epsilon\colon \Phi_{2j+1}^\epsilon \xrightarrow{\cong} \Phi_{2j+1}^\epsilon \text{ is a free involution for each } \epsilon \in \{\pm\} \text{ and each } j \geq 0.$$

For any compact surface S, $\chi(S)$ denotes the Euler characteristic of S.

Proposition 11.12 *Suppose that $j \geq 2$ and that $\chi(\Phi_j^\epsilon) = \chi(\Phi_{j+1}^\epsilon)$. Then $\Phi_j^\epsilon = \Phi_{j+1}^\epsilon$.*

Proof If $\Phi_j^\epsilon \neq \Phi_{j+1}^\epsilon$, there is an annulus $(E, \partial E) \subset (\Phi_j^\epsilon \setminus \mathrm{int}(\Phi_{j+1}^\epsilon), \partial\Phi_{j+1}^\epsilon)$. We show that this leads to a contradiction.

Consider the homeomorphism $h_j^\epsilon\colon \Phi_j^\epsilon \to \Phi_j^{(-1)^{j+1}\epsilon}$. The image of Φ_{j+1}^ϵ under this homeomorphism is, by [3, Proposition 5.3.5], $\Phi_1^{(-1)^j\epsilon} \wedge \Phi_j^{(-1)^{j+1}\epsilon}$. Thus the image E_0 of E under this map satisfies

$$E_0 \subset (F \setminus \mathrm{int}(\Phi_1^{(-1)^j\epsilon})) \wedge \Phi_j^{(-1)^{j+1}\epsilon}.$$

Let c_0 be a boundary component of E_0. Then c_0 is a boundary component of $\Phi_1^{(-1)^j\epsilon}$ and thus is a boundary component of an annulus A which is properly embedded and essential in $(X^{(-1)^j\epsilon}, F)$. On the other hand, c_0 is isotopic in F to a curve in $\Phi_j^{(-1)^{j+1}\epsilon}$ and so since $j \geq 2$, Lemma 11.11 implies that c_0 is a boundary component of an essential annulus $(A_1, \partial A_1) \subset (X^{(-1)^{j+1}\epsilon}, F)$. But this contradicts Lemma 11.3. Hence $\Phi_j^\epsilon = \Phi_{j+1}^\epsilon$. □

Corollary 11.13 *Fix $\epsilon \in \{\pm 1\}$ and suppose that $\chi(\Phi_{2k+1}^\epsilon) < 0$ for some $k \geq 1$. Then*

$$\chi(\Phi_3^\epsilon) < \chi(\Phi_5^\epsilon) < \cdots < \chi(\Phi_{2k+3}^\epsilon).$$

Proof Apply Proposition 11.12 and [3, Proposition 5.3.9]. □

Lemma 11.14 *Suppose that (X^ϵ, F) is not a twisted I–bundle pair. Then $\chi(F) < \chi(\Phi_3^\epsilon)$.*

Proof Suppose otherwise that $\chi(F) = \chi(\Phi_3^\epsilon)$. According to the previous lemma we have $\Phi_2^\epsilon = \Phi_3^\epsilon$ and therefore [3, Proposition 5.3.9] implies that $\Phi_1^\epsilon \neq \Phi_2^\epsilon$. But since $\chi(\Phi_1^\epsilon) = \chi(\Phi_2^\epsilon)$, there is an annulus $(E, \partial E) \subset (\Phi_1^\epsilon \setminus \mathrm{int}(\Phi_2^\epsilon), \partial\Phi_2^\epsilon)$. Let

$E_1 = \tau_\epsilon(E) \subset \Phi_1^\epsilon$ and observe that $E_1 \subset F \setminus \mathrm{int}(\Phi_1^{-\epsilon})$ while $\partial E_1 \subset \partial \Phi_1^{-\epsilon}$. By Lemma 11.6 there is a root torus $V_1 \subset X^{-\epsilon}$ such that $V_1 \cap F \subset E_1$. Let A_1 be the essential annulus in $(X^{-\epsilon}, F)$ given by $\partial V_1 \setminus (V_1 \cap F)$.

Next observe that since X^ϵ is not a twisted I–bundle but $\chi(F) = \chi(\Phi_1^\epsilon)$, there is an annulus $E_2 \subset F \setminus \mathrm{int}(\Phi_1^\epsilon)$ such that $\partial E_2 \subset \partial \Phi_1^\epsilon$. Another application of Lemma 11.6 produces a root torus $V_2 \subset X^\epsilon$ such that $V_2 \cap F \subset E_2$. Let A_2 be the essential annulus in $(X^{-\epsilon}, F)$ given by $\partial V_2 \setminus (V_2 \cap F)$. Since $V_1 \cap F \subset E_1 \subset \Phi_1^\epsilon$ and $V_2 \cap F \subset E_2 \subset F \setminus \mathrm{int}(\Phi_1^\epsilon)$, we may suppose that $V_1 \cap V_2$ is empty. But then, A_1, A_2 are disjoint essential annuli which are not nested, contrary to Lemma 11.2. Thus we must have $\chi(F) < \chi(\Phi_3^\epsilon)$. □

Lemma 11.15 *Any reduced homotopy in (M, F) has length at most $m-2$ if (X^-, F) is not a twisted I–bundle pair or has length at most $m-1$ if (X^-, F) is a twisted I–bundle pair. Furthermore, if a reduced homotopy in (M, F) has length $m-1$, then it starts and ends on the X^- side.*

Proof First note that, by [3, Corollary 5.3.8], $\chi(\Phi_j^\epsilon)$ is even for each $j \geq 1$ odd. Applying this together with Corollary 11.13 and Lemma 11.14 we see that Φ_{m-1}^ϵ is the empty set if (X^ϵ, F) is not a twisted I–bundle pair. So the length of a reduced homotopy in (M, F) is at most $-\chi(F) = m-2$ if the homotopy starts on a side which is not a twisted I–bundle pair, and at most $1 - \chi(F) = m-1$ if the homotopy starts on a side which is a twisted I–bundle. In the latter case, the homotopy starts on the X^- side by Lemma 11.4, and finishes there since m is even. □

It follows from the definition of Φ_j^ϵ that if (X^-, F) is a twisted I–bundle pair, then $\Phi_{2j}^- = \Phi_{2j+1}^-$ and $\Phi_{2j+1}^+ = \Phi_{2j+2}^+$ for each $j \geq 0$.

Lemma 11.16 *If (X^-, F) is a twisted I–bundle pair and Φ_1^+ is not empty, then $\chi(\Phi_1^+) < \chi(\Phi_3^+)$.*

Proof Suppose otherwise. Then $\chi(\Phi_2^+) = \chi(\Phi_3^+)$. By Proposition 11.12, we have $\Phi_2^+ = \Phi_3^+$. Thus $\Phi_1^+ = \Phi_2^+ = \Phi_3^+$. But this is impossible as it contradicts [3, Proposition 5.3.9]. □

Proposition 11.17 *If $\chi(F) < \chi(\Phi_1^+)$, then any reduced homotopy in (M, F) has length at most $m-3$.*

Proof First assume that (X^-, F) is a twisted I–bundle pair. It follows from Lemma 11.16, Corollary 11.13 and the assumption $\chi(F) < \chi(\Phi_1^+)$ that Φ_{m-3}^+ is the empty set. Hence a reduced homotopy in (M, F) has length at most $m-4$ if it starts on X^+ side and length at most $m-3$ if it starts on X^- side.

Suppose, then, that (X^-, F) is not a twisted I–bundle pair. If $\chi(\Phi_1^+) < \chi(\Phi_3^+)$, then arguing as in the previous paragraph yields the desired conclusion. Suppose, then, that $\chi(\Phi_1^+) = \chi(\Phi_3^+)$. Then $\Phi_2^+ = \Phi_3^+$ by Proposition 11.12. It follows from the definition of the characteristic subsurfaces that $\Phi_{2j+1}^- = \Phi_{2j+2}^-$ and $\Phi_{2j}^+ = \Phi_{2j+1}^+$ for each $j \geq 1$. Now Corollary 11.13 and the condition $\chi(F) < \chi(\Phi_1^+)$ imply that Φ_{m-1}^+ is the empty set. Since $m-1$ is an odd number, $\Phi_{m-2}^+ = \Phi_{m-1}^+$ is the empty set. But Φ_{m-2}^- is homeomorphic to Φ_{m-2}^+ (cf (11–1)) and thus $\Phi_{m-3}^- = \Phi_{m-2}^-$ is the empty set. Therefore the length of a reduced homotopy in (M, F) is at most $m-4$ if the homotopy starts on the X^- side and therefore at most $m-3$ in general. $\square$

Corollary 11.18 *If there is a reduced homotopy in (M, F) with length at least $m-2$, then Φ_1^+ consists of a pair of tight components and contains ∂F. Further, Φ_1^- is either a twisted I–bundle or consists of a pair of tight components and contains ∂F.*

Proof By Proposition 11.17, we have $\chi(F) = \chi(\Phi_1^+)$. By Lemma 11.4 $\Phi_1^+ \neq F$. Now apply Lemma 11.8 to see that Φ_1^+ consists of a pair of tight components and contains ∂F.

If Φ_1^- is not a twisted I–bundle, we may exchange X_+ and X_- $\square$

12 Proof of Proposition 5.4

Recall that we are assuming that β is a strict boundary slope, $M(\beta)$ is a connected sum of two non-trivial lens spaces, one of which is not P^3, and $M(\alpha)$ admits a π_1–injective immersion of a torus. We will use the method of [3] to show that $\Delta(\alpha, \beta) \leq 4$.

Let V_α be the filling solid torus used in forming $M(\alpha)$. As in [3] we obtain a map $h\colon T \to M(\alpha)$ from a torus T to $M(\alpha)$ such that

(1) $h^{-1}(V_\alpha)$ is a non-empty set of embedded disks in T and h is an embedding when restricted on $h^{-1}(V_\alpha)$;

(2) $h^{-1}(F)$ is a set of arcs or circles properly embedded in the punctured torus $Q = T \setminus h^{-1}(V_\alpha)$, where F is the planar surface given in Section 11;

(3) If e is an arc component of $h^{-1}(F)$, then $h\colon e \to F$ is an essential (immersed) arc;

(4) If c is a circle component of $h^{-1}(F)$, then c does not bound a disk in Q and $h\colon c\to F$ is an essential (immersed) 1–sphere.

For any subset s of T, we use s^* denote its image under the map h. Denote the components of $\partial(h^{-1}(V_\alpha))$ by $a_1,\dots,a_n$ so that $a_1^*,\dots,a_n^*$ appear consecutively on ∂M. Note again that $a_1,\dots,a_n$ are embedded in ∂M and each of these curves has slope α. Denote the components of ∂F by $b_1,\dots,b_m$ so that they appear consecutively in ∂M. We fix an orientation on Q and let each component a_i of ∂Q have the induced orientation. Two components a_i and a_j are said to have the same orientation if a_i^* and a_j^* are homologous in ∂M. Otherwise, they are said to have different orientations. Similar definitions are defined for the components of ∂F. Since Q, F and M are all orientable, one has the following rule.

Parity rule An arc component e of $h^{-1}(F)$ in Q connects components of ∂Q with the same orientation (resp. opposite orientations) if and only if the corresponding e^* in F connects components of ∂F with opposite orientations (resp. the same orientation).

We define a graph Γ on the torus T by taking $h^{-1}(V_\alpha)$ as (fat) vertices and taking arc components of $h^{-1}(F)$ as edges. Note that Γ has no trivial loops, ie no 1–edge disk faces. Also note that each a_i^* intersects each component b_j in ∂M in exactly $\Delta(\alpha,\beta)$ points. If e is an edge in Γ with an endpoint at the vertex a_i, then the corresponding endpoint of e^*, is in $a_i^*\cap b_j$ for some b_j, and the endpoint of e is thus given the label j. So when we travel around a_i in some direction, we see the labels of the endpoints of edges appearing in the order $1,\dots,m,\dots,1,\dots,m$ (repeated $\Delta(\alpha,\beta)$ times). It also follows that each vertex of Γ has valence $m\Delta(\alpha,\beta)$.

Suppose that e and e' are two adjacent parallel edges of Γ. Let R be the bigon face between them, realizing the parallelism. Then $(R, e\cup e')$ is mapped into (X^ϵ, F) by the map h for some ϵ. Moreover, $h|_R$ provides a basic essential homotopy between the essential paths $h|_e$ and $h|_{e'}$ (cf [3]). We may and shall assume that $R^*=h(R)$ is contained in the characteristic I–bundle pair $(\Sigma_1^\epsilon,\Phi_1^\epsilon)$ of (X^ϵ,F). We may consider R as $e\times I$ and assume that the map $h\colon R\to\Sigma_1^\epsilon$ is I–fibre preserving.

A face f of Γ is said to lie on the X^ϵ side if f^* is contained in X^ϵ. Every face of Γ lies on either the X^+ side or the X^- side. Since F separates M, if two faces of Γ share a common edge, then the two faces will lie on different sides of F.

The torus ∂M is cut by ∂F into $m=2g$ parallel annuli. We denote these annuli by $B_1,\dots,B_m$ so that $\partial B_i=b_i\cup b_{i+1}$ for $i=1,\dots,m-1$ and $\partial B_m=b_m\cup b_1$. We may assume that B_1 is contained in X^-. Then for each odd i, B_i is contained in X^- and for each even i, B_i is contained in X^+. So $\partial X^-=F\cup B_1\cup B_3\cup\cdots\cup B_{2g-1}$ and $\partial X^+=F\cup B_2\cup B_4\cup\cdots\cup B_{2g}$, both being closed surface of genus g.

The complement of the interior of F in the essential 2–sphere $\widehat{F}$ is a set of m disjoint meridian disks of the attached solid torus V_α. These disks cut the solid torus V_α into m pieces, denoted $H_1, \dots, H_m$, such that each H_i is a 2–handle attached to X^- (when i odd) or to X^+ (when i even) along B_i.

Suppose that the characteristic I–bundle pair $(\Sigma_1^+, \Phi_1^+) \subset (X^+, F)$ is a connected trivial I–bundle containing all B_i with i even, ie Φ_1^+ is a pair of tight components T_1 and T_2 including all components of ∂F. This happens when the length of a reduced homotopy in (M, F) is at least $m-2$ by Corollary 11.18. Let $\widehat{\Sigma}_1^+$ be Σ_1^+ with all the 2–handles H_i, i even, attached along B_i. Then $\widehat{\Sigma}_1^+$ is an I–bundle over the disk $\widehat{T}_1$, where $\widehat{T}_1$ is the disk in $\widehat{F}$ whose intersection with F is the tight component T_1 of Φ_1^+. Write $\widehat{\Sigma}_1^+ = \widehat{T}_1 \times [0,1]$. Let $D_{1/2} = \widehat{T}_1 \times \{1/2\}$. Let U be the union of $\widehat{\Sigma}_1^+$ and a regular neighborhood of $\widehat{F}$ in $\widehat{X}^+$. Obviously U is a once punctured solid torus with $D_{1/2}$ as a meridian disk. The torus boundary of U must bound a solid torus V in X^+. That is, the once punctured lens space $\widehat{X}^+$ is the union of U and V along their torus boundary. Hence the core curve of U carries a generator of the first homology group of $\widehat{X}^+$ when given an orientation. We record this property in the following lemma which will be used later in the proof of Lemma 12.8.

Lemma 12.1 *If the length of a reduced homotopy in (M, F) is at least $m-2$, then the core curve of the punctured solid torus U given in the proceeding paragraph carries a generator of the first homology group of the non-trivial punctured lens space $\widehat{X}^+$ and the disk $D_{1/2}$ is a meridian disk of U.*

Definition 12.2 A pair of adjacent parallel edges $\{e, e'\}$ of Γ is called an S–cycle if

- the two edges connect two vertices v and v' with the same orientation;
- the label of the endpoint of e at v is j and the label of the endpoint of e at v' is $j+1$ (note all calculations concerning labels are defined mod (m));
- the label of the endpoint of e' at v is $j+1$ and the label of the endpoint of e' at v' is j.

An S–cycle $\{e, e'\}$ is called an extended S–cycle if the two edges e and e' are the two middle edges in a family of four adjacent parallel edges of Γ, see Figure 1.

Lemma 12.3 *Suppose that two vertices v and v' of Γ have the same orientation and are connected by a family of n parallel consecutive edges $e_1, \dots, e_n$ of Γ.*

(1) *If $n > m/2$, then there is an S–cycle in this family of edges.*

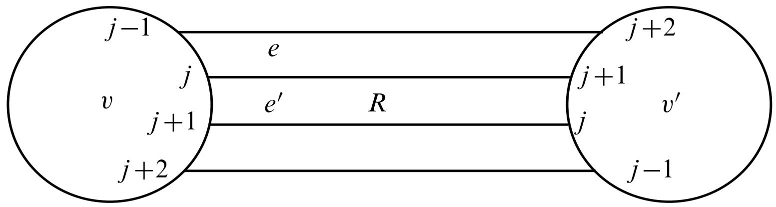

Figure 1: An extended S–cycle.

(2) *If $n > \frac{m}{2}+1$, then either there is an extended S–cycle in this family of edges or both $\{e_1, e_2\}$ and $\{e_{n-1}, e_n\}$ are S–cycles.*

(3) *If $n > \frac{m}{2}+2$, then there is an extended S–cycle in this family of edges.*

Proof Part (1) is [11, Corollary 2.6.7]. Parts (2) and (3) follow from part (1) directly. □

Lemma 12.4 *If $\{e, e'\}$ is an S–cycle in Γ, then the bigon face R between them is mapped into (Σ_1^-, Φ_1^-) under the map h. Moreover, there is properly embedded Möbius band $B \subset X^-$ such that ∂B is contained in Φ_1^-.*

Proof Assume R^* is contained in Σ_1^ϵ and that the S–cycle has labels j and $j+1$. Then e^* and e'^* are paths in F connecting the two components b_j and b_{j+1} of ∂F. Recall that B_j denotes the annulus in ∂M with boundary $b_j \cup b_{j+1}$, and τ_ϵ denotes the involution of Φ_1^ϵ. We have $\tau_\epsilon(b_j) = \tau_\epsilon(b_{j+1})$ and $\tau_\epsilon(e^*) = \tau_\epsilon(e'^*)$ and hence the connected set $b_j \cup e^* \cup e^{'*} \cup b_{j+1}$ is invariant under τ_ϵ. There is a τ_ϵ–invariant regular neighborhood N of $b_j \cup e^* \cup e^{'*} \cup b_{j+1}$ contained in Φ_1^ϵ and it is simple to see that there is a τ_ϵ–invariant essential simple closed curve in N. Thus there is a properly embedded Möbius band $B \subset X^\epsilon$ such that ∂B is contained in N. Therefore by Lemma 11.4, we have $\epsilon = -$. □

Lemma 12.5 *There is no extended S–cycle in Γ.*

Proof Suppose that $\{e, e'\}$ is an extended S–cycle of Γ as shown in Figure 1. If R denotes the bigon face between e and e', Lemma 12.4 shows that R^* is contained in Σ_1^-. From Figure 1, one easily sees that the set $b_j \cup e^* \cup e'^* \cup b_{j+1}$ is contained in Φ_1^+ and so the same may be assumed true for its regular neighborhood N used in the proof of Lemma 12.4. It follows that the boundary of the Möbius band in X^- constructed in the proof of Lemma 12.4 is contained in Φ_1^+. But Lemma 11.5 prohibits this possibility. Thus Γ contains no extended S–cycles. □

Lemma 12.6 *Suppose that $m \geq 6$. If two vertices v and v' of Γ have the same orientation, then they cannot be connected by $5m/6$ parallel edges.*

Proof By Lemma 12.5 and Lemma 12.3 (3), $5m/6 \leq \frac{m}{2}+2$, ie $m \leq 6$. So suppose $m=6$ and there are $5m/6=5=\frac{m}{2}+2$ parallel consecutive edges $e_1,\dots,e_5$ connecting two vertices with the same orientation. Then by Lemma 12.3 (2) and Lemma 12.5, we may assume that both $\{e_1,e_2\}$ and $\{e_4,e_5\}$ are S–cycles. But the bigon face R between e_1,e_2 and the bigon face R' between e_4,e_5 are on different sides of F. This contradicts Lemma 12.4. □

Suppose that Γ has m consecutively parallel edges $e_1,\dots,e_m$ connecting two vertices v and v' with different orientations. The existence of the m parallel edges implies that there is a length $m-1$ reduced homotopy in (M,F). Let R_i denote the bigon face between the adjacent parallel edges e_i and e_{i+1}, $i=1,\dots,m-1$. Then $R_1^*,\dots,R_{m-1}^*$ are contained alternatively in X^- and X^+ starting and ending on the X^- side of F by Lemma 11.15. Thus each of the bigon faces R_1, R_3, …, R_{m-1} is mapped in (Σ_1^-,Φ_1^-) and each of R_2, $R_4,\dots,$ R_{m-2} is mapped in (Σ_1^+,Φ_1^+).

Orient all the edges $e_1,\dots,e_m$ in the same direction such that their tails are in v and their heads are in v'. Up to renumbering, we may assume that the labels of the tails of $e_1,\dots,e_m$ are $1,\dots,m$ respectively. The labels of the heads of $e_1,\dots,e_m$ are $\sigma(1),\dots,\sigma(m)$ for some permutation σ of $\{1,\dots,m\}$. (Note that the indices are defined modulo m.)

Since F separates M, b_i and b_{i+1} have different orientations, for all i. Also b_i and b_j have the same orientation if and only if $i \equiv j \pmod 2$. By the parity rule, for each $i \in \{1,\dots,m\}$, the components b_i and $b_{\sigma(i)}$ of ∂F, connected by e_i^*, have the same orientation. (Note that if b_i and $b_{\sigma(i)}$ are the same component of ∂F for some i, then they are the same component for all $i=1,\dots,m$, ie σ is the trivial permutation.) It follows that b_i is different from $b_{\sigma(i+1)}$ and that $b_{\sigma(i)}$ is different from b_{i+1}, for all i.

Let d be the number of orbits of the action of the permutation σ on the set $\{b_1,\dots,b_m\}$, each of m/d elements. We may assume that indices are given as shown in Figure 2. By the parity rule, the index k in Figure 2 must be an odd number. From Figure 2, we see obviously that b_1 and b_k are in the same orbit, and b_m and b_{k-1} are in another orbit. By Corollary 11.18, Φ_1^+ is a pair of tight components, T_1 and T_2, which include all boundary components of F.

Lemma 12.7 *Suppose that $e_1,\dots,e_m$ are m consecutively parallel edges of Γ connecting two vertices v and v' with different orientations. We may assume that the*

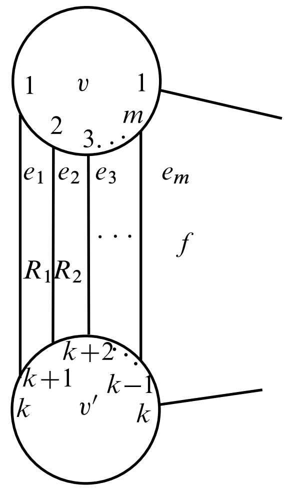

Figure 2: A pair of vertices of opposite orientations connected by m parallel edges.

permutation σ given in the preceding paragraph is as shown in Figure 2. Then $b_1 \cup b_k$ and $b_{k-1} \cup b_m$ are contained in different components of Φ_1^+; ie one in T_1 and the other in T_2.

Proof Recall that the annulus B_{k-1} in ∂M has boundary $b_{k-1} \cup b_k$ and the annulus B_m has boundary $b_m \cup b_1$, both contained in X^+. Thus b_{k-1} and b_k are contained in different components of Φ_1^+, and so are b_1 and b_m. In particular, the conclusion of the lemma follows immediately if $k = 1$, ie if the permutation σ is trivial. So we may assume that σ is non-trivial, b_1 is contained T_1, and b_m in T_2. We now only need to show that b_k is in T_1. Since $k \neq 1$, the bigon face R_{k-1} is mapped into X^+ (since k is odd) and e_k^* connects b_k to $b_{\sigma(k)}$. If $\sigma(k) = 1$, then we are done. If $\sigma(k) \neq 1$, then $R^*_{\sigma(k)-1}$ is in X^+ (since $\sigma(k)$ is odd) and $e^*_{\sigma(k)}$ connects $b_{\sigma(k)}$ to $b_{\sigma^2(k)}$ (recall that the indices here are defined mod (m)). Repeat in this way for finitely many times until $\sigma^n(k) = 1$ for some positive integer n (actually $n = \frac{m}{d} - 1$ is the number of elements in the orbit minus one). □

Let $\overline{\Gamma}$ denote the reduced graph of Γ, obtained from Γ by amalgamating parallel edges into a single edge. Then $\overline{\Gamma}$ is a graph with no 1–edge or 2–edge disk faces. If an edge $\overline{e}$ of $\overline{\Gamma}$ represents n parallel edges of Γ, we say the edge $\overline{e}$ has weight n.

Lemma 12.8 *Let $\overline{e}$ be an edge of $\overline{\Gamma}$ with weight m. Then no face f of $\overline{\Gamma}$ with ∂f containing $\overline{e}$ is a triangle face, ie a 3–edge disk face.*

Proof Suppose otherwise that there is a triangle face f in $\overline{\Gamma}$ whose boundary contains $\overline{e}$. Note that f is also a face in the graph Γ. Let v and v' be the two vertices connected by $\overline{e}$, and $e_1, \dots, e_m$ be the family of parallel edges of Γ represented by $\overline{e}$.

First we consider the case that v and v' have different orientations. We may assume that the labels of the endpoints of the edges e_i are given as in Figure 2. Now consider the two "corners" of the face f at the vertices v and v', ie the intersection arcs of the boundary of f with the boundary of the fat vertices. From Figure 2, we see that the two corners have labels $k, k-1$ and $m, 1$ respectively. If we follow ∂f in the clockwise direction, the four labels appear in the order $k, k-1, m, 1$.

Recall the setting in Lemma 12.1. The punctured solid torus U carried a generator of the first homology of the punctured lens space $\widehat{X}^+$ and the disk $D_{1/2}$ was a meridian disk of U. Note that U contains all the 2–handles H_i for i even and that the disk $D_{1/2}$ intersects each H_i, i even, in a single meridian disk of H_i.

If a disk face f' of Γ has n corners and is on the X^+ side, then it is not hard to see that $(\partial f')^*$ is contained in U and intersects $D_{1/2}$ transversely in n points (all the intersections occur precisely one each within the corner arcs of $\partial f'$). In our current situation, the triangle face f is indeed on the X^+ side since R_{m-1} is on the X^- side. Further, the algebraic intersection number of ∂f with $D_{1/2}$ is 1 or -1 because of the label orders on ∂f together with Lemma 12.7. Now we see that the existence of such a triangle face f implies that the first homology of $\widehat{X}^+$ is trivial, contradicting to the fact that $\widehat{X}^+$ is a punctured non-trivial lens space. This completes the proof for the case when the vertices v and v' have different orientation.

Now we consider the case when v and v' have the same orientation. Then by Lemma 12.6, we have $m = 4$. By Lemma 12.3 (2), Lemma 12.4 and Lemma 12.5, we may assume that four edges form two S–cycles and the labels on the tails and heads of the four edges are as shown in Figure 3. From the figure, we see directly that $b_1 \cup b_2$ is contained in one the tight components of Φ_1^+ and $b_3 \cup b_4$ is contained in the other. And the labels $2, 3, 4, 1$ appeared consecutively on ∂f (in clockwise direction). This would imply, as in the previous case, that the first homology of $\widehat{X}^+$ is trivial, giving the same contradiction. □

Now we can finish the proof of Proposition 5.4. Suppose otherwise that $\Delta(\alpha, \beta) \geq 5$. An Euler characteristic calculation shows that either the reduced graph $\overline{\Gamma}$ on the torus T has a vertex of valence less than 6 or every vertex of $\overline{\Gamma}$ has valence 6 and every face of $\overline{\Gamma}$ is a triangle face. By Lemma 11.15, every edge of $\overline{\Gamma}$ has weight at most m. So we may assume that $\overline{\Gamma}$ has no vertex of valence less than 5.

Suppose that there is a vertex of valency 5 or less. Consideration of Lemma 11.15 yields $\Delta = 5$ and the fact that every edge of $\overline{\Gamma}$ incident a valence 5 vertex has weight

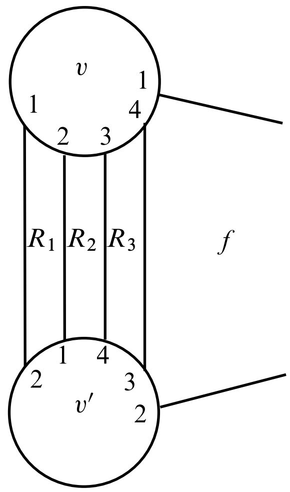

Figure 3: A pair of vertices of the same orientation connected by $m = 4$ parallel edges which form two S–cycles.

exactly equal to m. So it follows from Lemma 12.8 that the graph $\overline{\Gamma}$ in the torus T has the following properties:

- no disk face has 1 or 2 edges,
- every vertex has valence at least 5,
- no triangle face is incident to a valence 5 vertex.

For a vertex v and face f of $\overline{\Gamma}$, we write $v \in \partial f$ to signify that v is incident to f. Consider

$$\chi_f = \chi(f) + \sum_{v \in \partial f} \Big(\frac{1}{\text{valency}(v)} - \frac{1}{2} \Big).$$

By construction, if ∂f has three edges, then $\text{valency}(v) \geq 6$ for each $v \in \partial f$. Hence $\chi_f \leq 1 + 3(-\frac{1}{3}) = 0$ with equality if and only if f is a triangle face and each of its vertices has valency 6. On the other hand, if ∂f has at least four edges, then $\chi_f \leq 1 + 4(\frac{1}{5} - \frac{1}{2}) = -\frac{1}{5} < 0$. Thus since $0 = \chi(T) = \sum_f \chi_f$, each face of $\overline{\Gamma}$ is a triangle face and vertex has valency 6.

The proof of the following lemma is similar to that of Lemma 12.8.

Lemma 12.9 *The graph $\overline{\Gamma}$ cannot have an edge with weight larger than $m-2$.*

Proof Suppose otherwise that $\overline{e}$ is an edge with weight at least $m-1$. Since every face of $\overline{\Gamma}$ is a triangle face, Lemma 12.8 shows that the weight of $\overline{e}$ is exactly $m-1$.

Let v and v' be the two vertices connected by $\overline{e}$ and let $e_1, \ldots, e_{m-1}$ be the family of parallel edges of Γ represented by $\overline{e}$, oriented such that their tails are at v and heads at v'.

If v and v' have the same orientation, then we have $m = 4$ (Lemma 12.6). Since v has valence 6 while there are $4\Delta(\alpha, \beta) \geq 20$ endpoints of edges of Γ incident to v, some edge of $\overline{\Gamma}$ incident to v will have weight $m = 4$, contrary to the conclusion of Lemma 12.8.

Suppose, then, that v and v' have different orientations. We may assume that the labels of the endpoints of the edges e_i are given as in Figure 4. By the parity rule, for each of the edges $e_1, \ldots, e_{m-1}$, the two labels at its endpoints are congruent (mod 2). In particular, the label k in Figure 4 is an odd number. Denote by $R_1, \ldots, R_{m-2}$ the $m-2$ bigon faces defined by the $m-1$ edges, where R_j contains the edges e_j and e_{j+1}. By our convention R_1 lies on the X^- side since R_1^* intersects the annulus B_1 which lies on the X^- side (cf Figure 4). So the triangle face f of $\overline{\Gamma}$ which contains the edge e_1 (shown in Figure 4) lies on the X^+ side. It also follows that for each $i = 1, \ldots, m-2$, R_i lies on the X^-–side if i is odd or on the X^+ side if i is even.

By Corollary 11.18, Φ_1^+ is a pair of tight components T_1 and T_2 and contains all components of ∂F. We want to show that $b_1 \cup b_k$ is contained in one component of Φ_1^+ and $b_m \cup b_{k-1}$ is contained in the other. This is obviously true if $k = 1$ since B_m is contained in Σ_1^+. So suppose that $k > 1$. As in the proof of Lemma 12.8, by considering the orbit of the label 1 under the permutation of odd integers $\{1, 3, 5, \ldots, m-1\}$ given by the $m-1$ edges, we see that there is a sequence of odd labels $k_1 = k, k_2, \ldots, k_n \in \{3, 5, \ldots, m-1\}$ and edges $e_{i_1}, \ldots, e_{i_n} \in \{e_3, e_5, \ldots, e_{m-1}\}$ such that for $1 \leq j < n$, the edge e_{i_j} has tail label k_j and head label k_{j+1}, and the edge e_{i_n} has tail label k_n and head label 1. Since R_{i_j-1} lies on the X^+ side, we see that all $e_{i_j}^*$, $j = 1, \ldots, n$, are contained Φ_1^+. Since these n edges $e_{i_1}^*, \ldots, e_{i_n}^*$ connect $b_k = b_{k_1}, b_{k_2}, \ldots b_{k_n}$ and b_1, we see that $b_1 \cup b_k$ is contained in one component of Φ_1^+, say T_1. It follows that $b_m \cup b_{k-1}$ is contained in T_2, the other component of Φ_1^+, since the annuli B_m and B_{k-1} are contained in Σ_1^+.

From Figure 4, we see that the two corners of f at v and v' have labels $m, 1$ and $k, k-1$ respectively in clockwise direction. Now combining with Lemma 12.1, we see that the first homology of $\widehat{X}^+$ is trivial, which is a contradiction. □

We call an edge of $\overline{\Gamma}$ positive (respectively negative) if it connects two vertices of the same orientation (respectively different orientations). We call the endpoint of an edge at a vertex positive or negative if the edge is positive or negative. We define the weight

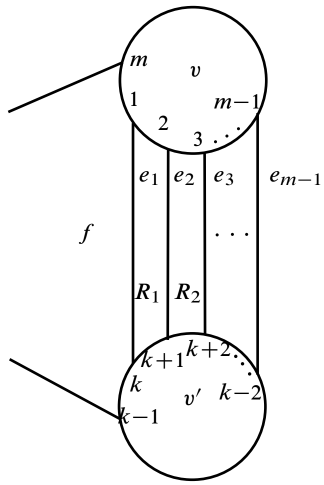

Figure 4: A pair of vertices of different orientations connected by $m-1$ parallel edges.

of an endpoint of an edge to be the weight of the edge. The sum of the weights of the endpoints at any vertex is $\Delta(\alpha,\beta)m$.

Lemma 12.10 *Let v be a vertex of $\overline{\Gamma}$. Then among the six endpoints at v, at most one is positive.*

Proof If there are two positive endpoints at v, then their weight sum is at most $m+4$ by Lemma 12.5 and Lemma 12.3. So the rest four endpoints have total weight at least $4m-4$. So at least one endpoint has weight $m-1$. This gives a contradiction with Lemma 12.9. □

Lemma 12.11 *There is a vertex of $\overline{\Gamma}$ with at least two positive endpoints.*

Proof The previous lemma implies that the graph $\overline{\Gamma}$ has no loops. Pick any vertex v_0 of $\overline{\Gamma}$ and let $p_1,\dots,p_6$ be the six endpoints at v in clockwise order. We may assume that $p_5,\dots,p_6$ are all negative endpoints. Let $\overline{e}_i$ be the edge of $\overline{\Gamma}$ with endpoint p_i and observe that they are distinct edges since there are no loops. Let v_i be the other vertex that $\overline{e}_i$ is incident to. Then the v_i have the same orientation for $i=2,\dots,6$. Now $v_5\neq v_4$ since there are no loops, and there is an edge of $\overline{\Gamma}$ connecting them. Similarly there is an edge connecting v_3 and v_4. Note that $v_3\neq v_5$ since otherwise there is either a non-triangle face of $\overline{\Gamma}$ or v_4 has valence less than 6. Thus v_4 has at least two positive endpoints. □

Proof of Proposition 5.4. The contradiction between Lemma 12.10 and Lemma 12.11 completes the proof that $\Delta(\alpha,\beta) \leq 4$. If we have equality, then [3, Proposition 8.4] and Theorem 1.1 imply that $M(\alpha)$ is Seifert fibred with base orbifold of the form $S^2(r,s,t)$, where (r,s,t) is a hyperbolic triple and $\mathrm{lcm}(r,s,t)$ divides 4. Thus Proposition 5.4 holds. □

References

[1] **L Ben Abdelghani**, **S Boyer**, *A calculation of the Culler–Shalen seminorms associated to small Seifert Dehn fillings*, Proc. London Math. Soc. (3) 83 (2001) 235–256 MR1829566

[2] **S Boyer**, *On the local structure of* $\mathrm{SL}(2,\mathbb{C})$*–character varieties at reducible characters*, Topology Appl. 121 (2002) 383–413 MR1909000

[3] **S Boyer**, **M Culler**, **P B Shalen**, **X Zhang**, *Characteristic subsurfaces and Dehn filling*, Trans. Amer. Math. Soc. 357 (2005) 2389–2444 MR2140444

[4] **S Boyer**, **C M Gordon**, **X Zhang**, *Dehn fillings of large hyperbolic* 3*–manifolds*, J. Differential Geom. 58 (2001) 263–308 MR1913944

[5] **S Boyer**, **X Zhang**, *Finite Dehn surgery on knots*, J. Amer. Math. Soc. 9 (1996) 1005–1050 MR1333293

[6] **S Boyer**, **X Zhang**, *On Culler–Shalen seminorms and Dehn filling*, Ann. of Math. (2) 148 (1998) 737–801 MR1670053

[7] **S Boyer**, **X Zhang**, *On simple points of character varieties of* 3*–manifolds*, from: "Knots in Hellas '98 (Delphi)", Ser. Knots Everything 24, World Sci. Publ., River Edge, NJ (2000) 27–35 MR1865698

[8] **S Boyer**, **X Zhang**, *Virtual Haken* 3*–manifolds and Dehn filling*, Topology 39 (2000) 103–114 MR1710995

[9] **S Boyer**, **X Zhang**, *A proof of the finite filling conjecture*, J. Differential Geom. 59 (2001) 87–176 MR1909249

[10] **A Casson**, **D Jungreis**, *Convergence groups and Seifert fibered* 3*–manifolds*, Invent. Math. 118 (1994) 441–456 MR1296353

[11] **M Culler**, **C M Gordon**, **J Luecke**, **P B Shalen**, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987) 237–300 MR881270

[12] **M Culler**, **P B Shalen**, *Varieties of group representations and splittings of* 3*–manifolds*, Ann. of Math. (2) 117 (1983) 109–146 MR683804

[13] **N M Dunfield**, *Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds*, Invent. Math. 136 (1999) 623–657 MR1695208

[14] **D Gabai**, *Foliations and the topology of* 3*–manifolds. II*, J. Differential Geom. 26 (1987) 461–478 MR910017

[15] **D Gabai**, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) 136 (1992) 447–510 MR1189862

[16] **D Gabai**, *Homotopy hyperbolic* 3*–manifolds are virtually hyperbolic*, J. Amer. Math. Soc. 7 (1994) 193–198 MR1205445

[17] **D Gabai**, **G R Meyerhoff**, **N Thurston**, *Homotopy hyperbolic* 3*–manifolds are hyperbolic*, Ann. of Math. (2) 157 (2003) 335–431 MR1973051

[18] **C M Gordon**, *Dehn filling: a survey*, from: "Knot theory (Warsaw, 1995)", Banach Center Publ. 42, Polish Acad. Sci., Warsaw (1998) 129–144 MR1634453

[19] **C M Gordon**, **J Luecke**, *Reducible manifolds and Dehn surgery*, Topology 35 (1996) 385–409 MR1380506

[20] **W Jaco**, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics 43, American Mathematical Society, Providence, R.I. (1980) MR565450

[21] **S Lee**, *Reducing and toroidal Dehn fillings on* 3*–manifolds bounded by two tori*, Math. Res. Lett. 13 (2006) 287–306 MR2231118

[22] **P E Newstead**, *Introduction to moduli problems and orbit spaces*, volume 51 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*, Tata Institute of Fundamental Research, Bombay (1978) MR546290

[23] **S Oh**, *Reducible and toroidal* 3*–manifolds obtained by Dehn fillings*, Topology Appl. 75 (1997) 93–104 MR1425387

[24] **P Scott**, *A new proof of the annulus and torus theorems*, Amer. J. Math. 102 (1980) 241–277 MR564473

[25] **J-P Serre**, *Représentations linéaires des groupes finis*, revised edition, Hermann, Paris (1978) MR543841

[26] **F Waldhausen**, *On irreducible* 3*–manifolds which are sufficiently large*, Ann. of Math. (2) 87 (1968) 56–88 MR0224099

[27] **Y Q Wu**, *Incompressibility of surfaces in surgered* 3*–manifolds*, Topology 31 (1992) 271–279 MR1167169

[28] **Y-Q Wu**, *Dehn fillings producing reducible manifolds and toroidal manifolds*, Topology 37 (1998) 95–108 MR1480879

[29] **Y-Q Wu**, *Standard graphs in lens spaces*, Pacific J. Math. 220 (2005) 389–397 MR2200380

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