Annals of Mathematics, 117 (1983), 109–146

Varieties of group representations and splittings of 3-manifolds

By Marc Culler and Peter B. Shalen

Introduction

This paper introduces a new technique in 3-dimensional topology. It is based on the interplay among hyperbolic geometry, the theory of incompressible surfaces, and the structure theory of subgroups of $SL_2(F)$, where F is a field. These same ingredients were used in the original proof of the Smith conjecture and also, for instance, in [26]; however, by taking F to be the function field of an appropriate complex algebraic curve, one can apply the structure theory of $SL_2(F)$ in a more direct way and thereby obtain answers to many purely topological questions that are inaccessible by other methods.

For example, in [6] we apply the techniques of the present paper to show that if the rational homology of a compact, orientable 3-manifold is carried by its boundary, and some component of the boundary is non-simply-connected, then either the manifold contains a non-boundary-parallel, separating, bounded, incompressible surface or it is homeomorphic to $D^2 \times S^1$ or $D^1 \times S^1 \times S^1$. This generalizes the main theorem of [26]. In [7] we apply our techniques to obtain new partial results on the conjecture that knots have property P. In [21], a variation of the method of the present paper will be used to give new proofs of two fundamental results of Thurston's: that the space of hyperbolic structures on an acylindrical 3-manifold is compact, and that the Thurston boundary of Teichmüller space consists of projective measured laminations. Purely group-theoretical applications will be given in [5].

In Section 5 of the present paper, we apply our methods to obtain alternative proofs of the Smith conjecture and the Davis-Morgan theorem [9]. These proofs provide a good deal of new information on the structure of branched covering spaces. Thus Corollary 5.1.3 implies that if N is a closed cyclic branched cover of an orientable 3-manifold M, branched over a (non-empty) link L whose complement in M is irreducible but not Seifert-fibered, then either N

003–486X/83/0117-1/0109/038$03.80/1

contains an incompressible torus or $\pi_1(N)$ has a non-trivial representation in $PSL_2(\mathbf{C})$. (An appropriate version of this, Theorem 5.1.2, holds for non-cyclic regular coverings.) This is a strong form of the generalized Smith conjecture and in addition it provides strong evidence, in the case of regular branched covers, for Thurston's conjecture (see § 4) that closed, non-sufficiently large 3-manifolds are hyperbolic or Seifert-fibered.

In Section 5 we also find sufficient conditions (see Theorems 5.2.4 and 5.2.7) for the representations given by 5.1.2 and 5.1.3 to have infinite image. This provides a new proof of Davis and Morgan's theorem [9] about group actions on homotopy 3-spheres (see our Corollary 5.2.6). It also allows one to prove many hard conjectures about general closed 3-manifolds, previously accessible only for Haken manifolds, for large classes of regular branched coverings. As examples we prove an analogue (Proposition 5.3.1) of Waldhausen's "center" theorem [36], and also an analogue (Proposition 5.3.2) of Evans and Jaco's theorem [11] about free subgroups of 3-manifold groups.

Incidentally, we should like to call attention to Theorem 4.2.2 below, which we feel deserves to be better known. It asserts in effect that for "virtual Haken manifolds," i.e. closed, irreducible, oriented 3-manifolds that have finite-sheeted covering spaces containing incompressible surfaces, the above-mentioned conjecture of Thurston's is true *up to homotopy equivalence*. Although this follows easily from work done by Meeks-Simon-Yau, Mostow, Scott, Thurston, and Waldhausen, it does not seem to have been generally recognized that the pieces fit together in this way. We include the result here for some applications in Section 5, but it is obviously of more general importance.

Before describing the basic technique of the paper, let us give some background. A recurrent theme in 3-manifold theory is the relationship between a manifold and its fundamental group. The idea originated with Stallings [28] that a free-product decomposition of a fundamental group of a 3-manifold gives rise to a system of 2-spheres in the manifold. This is exploited in [29], where the Sphere Theorem of Papakyriakopoulos and Whitehead is derived as a corollary of the deep algebraic result that a torsion-free group with infinitely many ends is a non-trivial free product. More generally, it was pointed out by Epstein [10] and Waldhausen [36] that a decomposition of the fundamental group as a free product with amalgamation or HNN group gives rise (noncanonically) to a system of incompressible surfaces. (See the end of § 2 below.) One algebraically deep method for producing amalgamated-free-product or HNN decompositions of a group is to exploit the theory of Tits, Bass, and Serre (see § 2) on the structure of subgroups of $SL_2(F)$, where F is a field. If M is a hyperbolic 3-manifold (see § 3), $\pi_1(M)$ is isomorphic to a subgroup of $SL_2(\mathbf{C})$, and the Tits-Bass-Serre theory, together with the Stallings-Epstein-Waldhausen construction, often permits one to find incompressible surfaces in M. These observations, in combination with Thurston's work on the existence of hyperbolic metrics (see § 4), were used, for example, in the original proof of the Smith conjecture [3]; there the Tits-Bass-Serre theory was exploited via Bass's GL_2-subgroup theorem [1], [2].

The technique of the present paper depends on considering the set of all representations of $\pi_1(N)$ in $SL_2(\mathbf{C})$, where N is a compact 3-manifold. This set can be regarded as a complex affine algebraic set R (§ 1). For each complex algebraic curve $C \subset R$, there is a canonical representation of $\pi_1(N)$ in $SL_2(F)$, where F is the field of all rational functions on C. Each point of the smooth projective model $\tilde{C}$ of C (cf. § 1) determines a valuation of F, and by the Tits-Bass-Serre theory, each such valuation determines a (possibly trivial) "graph product" decomposition or "splitting" of $\pi_1(N)$ (see § 2; this is a straightforward generalization of an amalgamated-free-product or HNN decomposition). These decompositions in turn can be used to produce incompressible surfaces in N by the Stallings-Epstein-Waldhausen construction.

Actually it is technically better to work with the set of characters of representations of $\pi_1(N)$ in $SL_2(\mathbf{C})$, which can again be regarded as a complex affine algebraic set X (§ 1). To each curve C in X, and each point of the smooth projective model of C, one can associate a splitting of $\pi_1(N)$ (see our "Fundamental Theorem" 2.2.1). Besides being more invariant, this point of view has the advantage that the splittings associated with *ideal* points of $\tilde{C}$ are automatically non-trivial, which means in particular that the associated incompressible surfaces are not boundary-parallel.

To make these ideas useful one must have a way of producing curves of characters of representations of $\pi_1(N)$ in $SL_2(\mathbf{C})$. Thurston has shown that if N is a "simple Haken manifold" with torus boundary components, then $\mathring{N}$ has a hyperbolic metric, that the hyperbolic metric gives rise to a representation of $\pi_1(N)$ in $SL_2(\mathbf{C})$ and, moreover, that the character of this representation lies in an irreducible component X_0 of X whose (complex) dimension is at least the number of boundary tori of N. (We give proofs of the last two statements in § 3.) Thus if $\partial N \neq \varnothing$, X always contains curves.

The applications depend on the use of certain complex-valued functions on the variety X_0 described above. For each $g \in \pi_1(N)$ define $I_g(\chi) = \chi(g)$ where χ is a character in X_0. The functions I_g turn out to be regular on X_0; if the restriction of some I_g to a curve $C \subset X_0$ is non-constant, then its extension I_g to the smooth projective model $\tilde{C}$ of C must have a pole at some ideal point $\tilde{x}$. If g is the homotopy class of a simple closed curve γ in ∂N, the Fundamental Theorem guarantees that γ cannot be a boundary component of one of the incompressible surfaces associated with $\tilde{x}$.

To prove the theorem of [6] on the existence of separating, incompressible surfaces, in the special case of a simple knot space, one observes that a surface separates N if it does not have a longitude as a boundary component. Thus one needs only to find a curve $C \subset X_0$ such that $I_g \mid C$, where g is the homotopy class of a longitude, is non-constant.

To prove Theorem 5.1.2 on regular branched coverings in the case where the branch set is a knot $K \subset M$ and $M - K$ is hyperbolic, we find a curve C in the variety of characters of $\pi_1(M - K)$, on which the function I_g, where g is the homotopy class of a meridian of K, is non-constant. Thus given a complex number τ, either (a) I_g takes the value τ at a point of C or (b) $\tilde{I}_g$ takes the value τ at an ideal point of $\tilde{C}$. We apply this by taking τ to be the trace of an element of order $2r$ in $SL_2(\mathbf{C})$, where r is the ramification index of the branched cover over K. If (a) holds, we construct a non-trivial representation of the fundamental group of the branched cover in $PSL_2(\mathbf{C})$. If (b) holds, the fundamental theorem turns out to give us an incompressible surface in the complement of a tubular neighborhood of K in M, whose boundary curves are meridians; one then applies the methods of Gordon and Litherland's paper [13] to construct an incompressible surface in the branched cover.

Here is the plan of our paper. In Section 1 we introduce the algebraic sets of representations and characters of representations of a finitely generated group into $SL_2(\mathbf{C})$. (The proof that the characters form a closed set is surprisingly difficult, and requires the Burnside Lemma from representation theory.) In Section 2 we review the Tits-Bass-Serre theory, state and prove our Fundamental Theorem, and show how graph-product decompositions give rise to incompressible surfaces. In Section 3 we review some elementary facts about hyperbolic 3-manifolds and prove some surprising theorems due to Thurston, including the above lower bound on the dimension of the variety X_0. In Section 4 we state Thurston's uniformization theorem, which asserts the existence of hyperbolic metrics on simple Haken manifolds, and the Mostow rigidity theorem; then we use them to prove the result on virtual Haken manifolds which was described above. The application to the study of regular branched coverings occupies Section 5.

We shall take for granted the contents of Hempel's treatise [14] on 3-manifolds, and the elementary facts about complex algebraic varieties that are discussed in the first three chapters of [23]. Following [23], we define a *closed algebraic set* in $\mathbf{C}^N$ to be the locus of zeroes of a set of polynomials, and an *affine variety* to be a closed algebraic set which is irreducible. Manifolds (and submanifolds) are understood to be C^∞. (Thus we are really using the smooth analogues of the results in [14], but these follow easily from the PL versions.) By a *surface* we mean a connected 2-manifold. An incompressible surface in a 3-manifold is understood to be bi-collared. A *link* in a closed 3-manifold is a non-empty, closed, 1-dimensional submanifold.

We would like to thank Hyman Bass, Cameron Gordon, William Jaco, Charles Livingston, and John Morgan for helping us to understand what we were doing.

1. Characters of representations in $SL_2(\mathbf{C})$

Some of the results of this section appear to be new, others are classical results for which no convenient reference exists. Some recent papers on the same general subject as this section are [17] and [4].

We will be considering representations of a finitely generated group in $SL_2(\mathbf{C})$. We begin by showing how to parametrize these representations and their characters as points of affine algebraic sets: this is preparation for the Fundamental Theorem, proved in Section 2, which derives information about the structure of the group from its space of characters. The most difficult result here is Proposition 1.4.4, which implies that the space of characters is a closed algebraic set.

1.1 *The space of representations.* Throughout this section Π denotes a finitely generated group. We are concerned with representations of Π in $SL_2(\mathbf{C})$ (i.e. homomorphisms $\rho : \Pi \to SL_2(\mathbf{C})$). Recall that two such representations, ρ and ρ', are *equivalent* if $\rho' = J\rho$, where J is an inner automorphism of $GL_2(\mathbf{C})$. The *character* of a representation ρ is the function $\chi_\rho : \Pi \to \mathbf{C}$ defined by $\chi_\rho(g) = \operatorname{tr}(\rho(g))$; equivalent representations obviously have the same character.

In terms of a set of generators $g_1, \ldots, g_n$ for Π, we define a set $R(\Pi) \subset SL_2(\mathbf{C})^n \subset \mathbf{C}^{4n}$ to be the set of all points $(\rho(g_1), \ldots, \rho(g_n))$ where ρ is a representation of Π. It is easy to show that $R(\Pi)$ is an affine algebraic set in $\mathbf{C}^{4n}$. (The defining equations of $R(\Pi)$ arise from the defining relations of Π; thus *a priori* there may be infinitely many equations, but the Hilbert basis theorem allows us to replace them by a finite set.) There is a natural 1-1 correspondence between the points of $R(\Pi)$ and the representations of Π in $SL_2(\mathbf{C})$, and we shall identify points with the corresponding representations and refer to $R(\Pi)$ as the space of representations of Π in $SL_2(\mathbf{C})$. Given two finite sets of generators for Π, the unique bijection between the corresponding "spaces of representations" which preserves the above identification is an isomorphism of algebraic sets. Thus $R(\Pi)$ is well-defined up to canonical isomorphism.

1.1.1. PROPOSITION. *Let V be an irreducible component of $R(\Pi)$. Then any representation equivalent to a representation in V must itself belong to V.*

Proof. The set $V \times SL_2(\mathbf{C}) \subset SL_2(\mathbf{C})^{n+1}$ is a product of two affine varieties (i.e. irreducible affine algebraic sets) and is therefore an affine variety. The map $f: V \times SL_2(\mathbf{C}) \to R(\Pi)$ given by $f(x_1, \ldots, x_n, \alpha) = (\alpha x_1 \alpha^{-1}, \ldots, \alpha x_n \alpha^{-1})$ is a regular map since it is defined by polynomials in the coordinates. The set $\overline{f(V \times SL_2(\mathbf{C}))} \subset R(\Pi)$, as the closure of the image of a variety under a regular map, is automatically a variety. Hence $\overline{f(V \times SL_2(\mathbf{C}))}$ is contained in a component V' of $R(\Pi)$. But then $V = f(V \times \{1\}) \subset V'$, and since V is itself a full component of $R(\Pi)$ we must have $V = V'$. Thus $f(V \times SL_2(\mathbf{C})) \subset V$, and this is equivalent to the statement of the proposition. □

1.2. *Irreducibility.* Recall that a representation ρ of Π in $GL_n(F)$, where F is a field, is *irreducible* if the only subspaces of F^n which are invariant under $\rho(\Pi)$ are $\{0\}$ and F^n, and that ρ is *absolutely irreducible* if it is irreducible when regarded as a representation of Π in K^n, where K denotes the algebraic closure at F.

1.2.1. LEMMA. *Let F be a field of characteristic zero and let ρ be a representation of Π in $SL_2(F)$ with non-abelian image. The following are equivalent.*

(i) *ρ is reducible.*

(ii) *ρ is reducible over the algebraic closure of F.*

(iii) *$\chi_\rho(c) = 2$ for each element c of the commutator subgroup of Π.*

Proof. Obviously (i) implies (ii).

Suppose that ρ is reducible over the algebraic closure K of F. Then ρ is equivalent over K to a representation by upper triangular matrices. The commutator of two upper triangular matrices is upper triangular with diagonal elements equal to 1. It follows that $\chi_\rho(c) = 2$ for $c \in [\Pi, \Pi]$; thus (ii) implies (iii).

To complete the proof we will show that if $\chi_\rho(c) = 2$ for all $c \in [\Pi, \Pi]$ then $\rho([\Pi, \Pi])$ has a unique 1-dimensional invariant subspace L; since $\rho([\Pi, \Pi])$ is normal in $\rho(\Pi)$, L will automatically be invariant under ρ, and ρ will therefore be reducible. Since $\rho(\Pi)$ is non-abelian there exists $c \in [\Pi, \Pi]$ such that $\rho(c) \neq 1$. Since $\chi_\rho(c) = 2$, $\rho(c)$ has a unique 1-dimensional invariant subspace L. Now suppose that for some $c' \in [\Pi, \Pi]$, L is not invariant under $\rho(c')$. Then $\rho(c') \neq 1$, and $\rho(c')$ has a 1-dimensional invariant subspace $L' \neq L$; hence after replacing ρ by an equivalent representation and recalling that $\operatorname{tr} \rho(c) = \operatorname{tr} \rho(c') = 2$, we may assume that $\rho(c) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\rho(c') = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$, where $\alpha\beta \neq 0$ since $\rho(c)$ and $\rho(c')$ are $\neq 1$. But then $\operatorname{tr} \rho(cc') = 2 + \alpha\beta \neq 2$, a contradiction. □

1.2.2. COROLLARY. *If K is an algebraically closed field of characteristic zero then a representation ρ of Π in $SL_2(K)$ is reducible if and only if $\chi_\rho(c) = 2$ for each element c of the commutator subgroup of Π.*

Proof. Since K is algebraically closed, if $\rho(\Pi)$ is abelian then $\rho(\Pi)$ has a 1-dimensional invariant subspace. If $\rho(\Pi)$ is non-abelian then Lemma 1.2.1 applies. □

1.3. *Curves of representations.* Let C be an affine algebraic curve. Then there exists a non-singular projective curve $\tilde{C}$ whose function field is isomorphic to that of C. (See Mumford [23].) Geometrically, $\tilde{C}$ can be constructed by completing C to a projective curve and resolving the singularities. It follows from [23, Proposition 7.1 (ii)] that $\tilde{C}$ is unique up to birational equivalence. Moreover, if C and D are affine curves then, since any rational map from a smooth curve to a projective variety is regular ([23, Proposition 7.1 (i)]), any rational map $f: C \to D$ determines a regular map $\tilde{f}: \tilde{C} \to \tilde{D}$. Similarly, if $\bar{V}$ denotes the closure in $\mathbf{P}^N$ of a (concrete) affine variety $V \subset \mathbf{C}^N$, a rational map $f: C \to V$ determines a regular map $\hat{f}: \tilde{C} \to \bar{V}$.

If $\bar{C}$ is any projective completion of C, then we define the *ideal points* of $\tilde{C}$ to be the points that correspond to the elements of $\bar{C} - C$ under the birational correspondence between $\tilde{C}$ and $\bar{C}$. If $f: D \to C$ is any *regular* map between affine curves then the inverse image under $\tilde{f}: \tilde{D} \to \tilde{C}$ of each ideal point of $\tilde{C}$ will consist of ideal points of $\tilde{D}$. Thus, in particular, the definition of the set of ideal points of $\tilde{C}$ does not depend on the choice of projective completion $\bar{C}$. Also, if $f: C \to \mathbf{C}$ is a regular function then $\tilde{f}: \tilde{C} \to \mathbf{P}^1$ will be a regular function whose poles are contained in the set of ideal points of $\tilde{C}$.

Suppose now that C is an affine curve contained in $R(\Pi)$. There is a canonical construction of a representation P of Π into $\mathrm{SL}_2(F)$, where F is the field of functions on the curve $\tilde{C}$. (Of course F is isomorphic to the field of functions on C, but it is much more convenient to work with functions on a smooth projective curve.) The representation P is defined as follows. Let $g \in \Pi$ be given. A point $\rho \in C$ is a representation of Π in $\mathrm{SL}_2(\mathbf{C})$. Set

$$\rho(g) = \begin{pmatrix} a(\rho) & b(\rho) \\ c(\rho) & d(\rho) \end{pmatrix}, \quad \text{for } \rho \in C.$$

Thus a, b, c, d are well-defined complex-valued functions on C. They are clearly given by polynomials in the ambient coordinates of $\mathbf{C}^{4n} \supset R(\Pi)$, and are therefore regular functions. We now set

$$P(g) = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} \in \mathrm{SL}_2(F).$$

1.3.1. Lemma. *If the curve C contains an irreducible representation, then P is absolutely irreducible.*

Proof. Let K be the algebraic closure of F. If P is reducible over K then $\mathrm{tr}(P(c)) = 2$ for each $c \in [\Pi, \Pi]$. Now, by the definition of P, the function

$\mathrm{tr}(P(c)) \in F$ is $\tilde{f}_c$, where $f_c(\rho) = \mathrm{tr}(\rho(c))$. But the element 2 of f is the constant function whose value is 2. Thus $\mathrm{tr}(\rho(c)) = 2$ for each $c \in [\Pi, \Pi]$ and each $\rho \in C$. By Corollary 1.2.2, this implies that each representation ρ in C is reducible. □

1.4. *The space of characters.* We wish to show next that the *characters* of the representations of Π in $\mathrm{SL}_2(\mathbf{C})$ can be identified with the points of an algebraic set. For each $g \in \Pi$ we may define a regular function τ_g on $R(\Pi)$ (i.e. a map $\tau_g : R(\Pi) \to \mathbf{C}$) by $\tau_g(\rho) = \mathrm{tr}\,\rho(g) = \chi_\rho(g)$. Let T denote the ring generated by all the functions τ_g, $g \in \Pi$.

1.4.1. PROPOSITION. *The ring T is finitely generated.*

Proof. Let T_0 denote the ring generated by all functions $\tau_{g_{i_1}\cdots g_{i_r}}$, where $i_1, \ldots, i_r$ are *distinct* positive integers $\leq n$. We shall prove the lemma by showing that $\tau_g \in T_0$ for every $g \in \Pi$.

The proof will be based on the identity

$$\mathrm{tr}(x)\mathrm{tr}(y) = \mathrm{tr}(xy) + \mathrm{tr}(xy^{-1}),$$

which holds for all $x, y \in \mathrm{SL}_2(\mathbf{C})$. (It follows from the identity $y + y^{-1} = (\mathrm{tr}\, y)I$, which in turn follows from the Cayley-Hamilton theorem.) The identity implies that

$$(*) \qquad \tau_g \tau_h = \tau_{gh} + \tau_{gh^{-1}}$$

for all $g, h \in \Pi$.

We first show that $\tau_g \in T_0$ whenever $g = g_{i_1}^{m_1} \cdots g_{i_r}^{m_r}$, where $i_1, \ldots, i_r$ are distinct integers between 1 and n and $m_1, \ldots, m_r \in \mathbf{Z}$. We use induction on the integer $\nu = \sum_{j=1}^{r} K_j$, where K_j is defined to be $-m_j$ if $m_j \leq 0$, $m_j - 1$ if $m_j > 0$. If $\nu = 0$ then all the m_i are 0 or 1, and so $\tau_g \in T_0$ by definition. If $\nu > 0$ then after replacing g by a conjugate element for which ν has the same value, we may assume that $m_r \neq 0, 1$. If $m_r < 0$ then by (*) we have $\tau_g = \tau_{gg_{i_r}}\tau_{g_{i_r}^{-1}} - \tau_{gg_{i_r}^2}$, where $\tau_{gg_{i_r}}, \tau_{gg_{i_r}^2} \in T_0$ by the induction hypothesis and $\tau_{g_{i_r}^{-1}} = \tau_{g_{i_r}} \in T_0$ by definition; hence $\tau_g \in T_0$. A similar reduction works if $m_r > 1$.

Now let $g \in \Pi$ be arbitrary. We may write g in the form $g_{i_1}^{m_1} \cdots g_{i_r}^{m_r}$, where $i_1, \ldots, i_r$ are integers that are *not* necessarily distinct. We shall prove by induction on r that $\tau_g \in T_0$.

By the case already proved we may assume that $i_1, \ldots, i_r$ are not all distinct. Hence, after replacing g by a conjugate element for which ν has the same value, we may assume that $i_r = i_s$ for some $s < r$. Set

$$V = g_{i_1}^{m_1} \cdots g_{i_s}^{m_s}, \qquad W = g_{i_{s+1}}^{m_{s+1}} \cdots g_{i_r}^{m_r}.$$

Then by (*) we have $\tau_g = \tau_{VW} = \tau_V \tau_W - \tau_{VW^{-1}}$. But $\tau_V, \tau_W, \tau_{VW^{-1}} \in T_0$ by the induction hypothesis, and so $\tau_g \in T_0$. □

We now fix a finite set $\gamma_1, \dots, \gamma_m$ of elements of G such that $\tau_{\gamma_1}, \dots, \tau_{\gamma_m}$ generate T. (Such a set is furnished, for example, by the proof of Proposition 1.4.1.) Define a map $t : R(\Pi) \to \mathbf{C}^m$ by $t(\rho) = (\tau_{\gamma_i}(\rho)), \dots, \tau_{\gamma_m}(\rho))$. Set $X(\Pi) = t(R(\Pi))$.

Since $\tau_{\gamma_1}, \dots, \tau_{\gamma_m}$ generate T, the character of a representation $\rho \in R(\Pi)$ is determined by $t(\rho)$. Hence there is a natural 1-1 correspondence between the points of $X(\Pi)$ and the characters of representations of Π in $\mathrm{SL}_2(\mathbf{C})$.

1.4.2. Lemma. *The set of reducible representations of* Π *has the form* $t^{-1}(V)$ *for some closed algebraic set* V *in* $\mathbf{C}^m$.

Proof. By Corollary 1.2.2 the reducible representations are those points of $R(\Pi)$ at which τ_c takes the value 2 for each $c \in [\Pi, \Pi]$. Since T is generated by $\tau_{\gamma_1}, \dots, \tau_{\gamma_m}$, the function τ_c, for every $c \in [\Pi, \Pi]$, has the form $f \circ t$ where f is an integer polynomial function in m variables. □

We shall prove that $X(\Pi) \subset \mathbf{C}^m$ is actually a closed algebraic set. This will follow from Proposition 1.4.4, the proof of which requires the following lemma.

1.4.3. Lemma. *Let* A *be a principal ideal domain and let* F *denote the field of fractions of* A. *Let* P *be an absolutely irreducible representation of a group* Π *in* $\mathrm{GL}_n(F)$, $n > 0$. *If* $\chi_P(\Pi) \subset A$ *then* P *is equivalent to a representation whose image is contained in* $\mathrm{GL}_n(A)$.

Proof. Set $\Gamma = P(\Pi)$. According to the Burnside Lemma (see for example [1, 1.2]), the absolute irreducibility of P implies that Γ spans the K-vector space $M_n(K)$ of $n \times n$ matrices over the algebraic closure K of F, and that there is a basis $t_1, \dots, t_{n^2}$ of $M_n(K)$ such that each element of Γ has the form $\Sigma \alpha_i t_i$, where the α_i are traces of elements of Γ. By hypothesis, the α_i belong to A. Thus the A-module $A\Gamma \subset M_n(F) \subset M_n(K)$ spanned by Γ is contained in a finitely generated A-module in $M_n(K)$; since A is, in particular, Noetherian, $A\Gamma$ is itself finitely generated. On the other hand, $A\Gamma$ must span $M_n(F)$ since Γ spans $M_n(K)$.

Now pick any $x \neq 0$ in F^n. The Γ-invariant A-module $A\Gamma x = \{\mu x : \mu \in A\Gamma\} \subset F^n$ is finitely generated and spans F^n. Since A is a p.i.d., $A\Gamma x$ is free and is therefore spanned by a basis of F^n. But the existence of a basis for F^n that spans a Γ-invariant A-module is clearly equivalent to the statement of the lemma. □

1.4.4. Proposition. *If R_0 is an irreducible component of $R(\Pi)$ which contains an irreducible representation, then $t(R_0) \subset \mathbf{C}^m$ is an affine variety (and in particular a closed set).*

Proof. Since $t(R_0) \subset \mathbf{C}^m$ is the image of an affine variety under a regular map, there exist an affine variety X_0 and a proper closed algebraic subset W_0 of X_0 such that $X_0 - W_0 \subset t(R_0) \subset X_0$. We must show that $t(R_0) = X_0$. This is trivial if X_0 is a single point, and so we may assume dim $X_0 > 0$. Let $x \in X_0$ be given; we will prove that $x \in t(R_0)$.

By Lemma 1.4.2 there is a closed algebraic set $V \subset \mathbf{C}^m$ such that $t^{-1}(V)$ is the set of all reducible representations in $R(\Pi)$. Since by hypothesis R_0 contains an irreducible representation, we have $t(R_0) \not\subset V$. Thus $V_0 = V \cap X_0$ is a *proper* closed algebraic subset of X_0. Since X_0 is irreducible, $V_0 \cup W_0$ is again a proper closed algebraic subset of X_0. Hence (since dim $X_0 > 0$) there exists an affine algebraic curve $C \subset X_0$ such that $x \in C$ and $C \not\subset V_0 \cup W_0$. (We may take C to be an irreducible component of the intersection of X_0 with a generic subspace of dimension $m - \dim(X_0) + 1$ passing through $x \in \mathbf{C}^m$.)

The set $C \cap W_0$ is finite, and by the definition of W_0, $C \setminus W_0 \subset t(R(\Pi))$. Hence there is a positive dimensional component H of $t^{-1}(C) \subset R(\Pi)$ such that $t|_H$ is not constant. We may construct (again by considering intersections with affine subspaces) a curve $D \subset H$ such that $t|_D$ is not constant.

In the notation of 1.3, the inclusion map $i : D \to R_0$ defines a map $d = \hat{i} : \tilde{D} \to \overline{R_0} \subset \mathbf{P}^n$, and the rational map $t|_D : D \to C$ defines $\hat{t} : \tilde{D} \to \overline{C}$. Since t is non-constant, $\hat{t}$ maps $\tilde{D}$ *onto* $\overline{C}$. Moreover, for any non-ideal point z of $\tilde{D}$, $t(d(z)) = \hat{t}(z)$.

Let $\tilde{y} \in \tilde{D}$ be a point such that $\hat{t}(\tilde{y}) = x$. If $\tilde{y}$ is not an ideal point of $\tilde{D}$ then we are done. For then $d(\tilde{y}) \in D$ and $t(d(\tilde{y})) = \hat{t}(\tilde{y}) = x$. Otherwise we will construct a new regular map $d' : \tilde{D} \to \overline{R_0}$ such that $d'(\tilde{y}) \in R_0$ and $t(d'(\tilde{y})) = x$.

This is where we use Lemma 1.4.3. Let F be the field of functions on $\tilde{D}$, let P be the canonical representation of Π in $\mathrm{SL}_2(F)$, and let $A \subset F$ be the ring of functions which do not have poles at $\tilde{y}$. Since $\tilde{D}$ is smooth, A is a discrete valuation ring and hence a principal ideal domain. To apply Lemma 1.4.3 to P we must check that $\chi_P(\Pi) \subset A$, i.e. that $\operatorname{tr} P(g) \in A$ for all $g \in \Pi$. But we have seen that the element $\operatorname{tr} P(g)$ of F, regarded as a function on D, is the function $\rho \to \operatorname{tr} \rho(g)$. Let us denote by ρ_z the representation $d(z) \in D$ where z is not an ideal point of $\tilde{D}$. To say that $\operatorname{tr} P(g) \in A$ means that $\lim_{z \to \tilde{y}} \operatorname{tr} \rho_z(g)$ is finite. But for $g = \gamma_i$, $1 \leq i \leq m$, this follows from the fact that $\lim_{z \to \tilde{y}} \hat{t}(z) = \hat{t}(y) = x$. For arbitrary $g \in \Pi$, the assertion follows from the fact that the functions $\tau_{\gamma_1}, \ldots, \tau_{\gamma_m}$ generate the ring T.

By Lemma 1.4.3, then, P is equivalent to a representation $P_1 : \Pi \to \mathrm{SL}_2(A)$. Thus there is a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(F)$ such that $P_1(g) = MP(g)M^{-1}$ for all $g \in \Pi$. There is a Zariski open set U in D (i.e. the complement of a finite set) such that U contains no ideal points of $\tilde{D}$ and no poles of α, β, γ or δ. We define d' to be the extension to $\tilde{D}$ of the map defined on U by $z \to M\rho_z M^{-1}$. (The extension exists by [23, Proposition 7.1 (ii)].) Clearly $d'(\tilde{y})$ is the representation in $R(\Pi)$ defined by

$$g \to \begin{pmatrix} a(\tilde{y}) & b(\tilde{y}) \\ c(\tilde{y}) & d(\tilde{y}) \end{pmatrix} \quad \text{where } P_1(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F).$$

Since $d'(z)$ is equivalent to $d(z)$ for each z in U, the image of d' is contained in $\overline{R}_0$ and $\hat{t}(d'(z)) = t(z)$ for all $z \in U$. By continuity $t(d'(\tilde{y})) = x$, so that $x \in t(R_0)$ as required. □

1.4.5. Corollary. *$X(\Pi)$ is a closed algebraic set.*

Proof. Since $X(\Pi)$ is the image of a closed algebraic set under a regular map, its closure in $\mathbf{C}^m$ is a closed algebraic set; hence we need only show that $X(\Pi)$ is closed. Write $X(\Pi) = t(R(\Pi)) = t(A) \cup t(B)$, where A is the union of all those components of $R(\Pi)$ which contain irreducible representations, and B is the set of all reducible representations in $R(\Pi)$. By 1.4.4, $t(A)$ is closed; we shall prove the corollary by showing that $t(B)$ is closed as well.

Every representation ρ in B is equivalent to a representation ρ' by upper triangular matrices. Moreover, ρ' clearly has the same character as the representation ρ'' defined by

$$\rho''(g) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{where } \rho'(g) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Thus $t(B) = t(D)$, where D is the set of all diagonal representations in $R(\Pi)$. But the map $t \mid D : D \to \mathbf{C}^m$ is at once seen to be a proper map, and hence $t(D)$ is indeed closed. □

We have mentioned that there is a natural 1-1 correspondence between the characters of representations of Π in $\mathrm{SL}_2(\mathbf{C})$ and the points of (the closed algebraic set) $X(\Pi)$. From now on we shall identify the points of $X(\Pi)$ with the corresponding characters; $X(\Pi)$ is the *space of characters* of the group Π, and we have $t(\rho) = \chi_\rho$ for $\rho \in R(\Pi)$. If we consider two different families of elements of Π of the form $(\gamma_i)_{1 \le i \le m}$, where the τ_{γ_i} generate the ring T, the unique bijection between the corresponding "varieties of characters" is an isomorphism of algebraic sets; thus $X(\Pi)$ is well-defined up to canonical isomorphism.

We have seen that, since $\tau_{\gamma_1}, \ldots, \tau_{\gamma_m}$ generate T, there exists for each $g \in \Pi$ a function whose value at each character $\chi \in X(\Pi)$ is $\chi(g)$. This function will be denoted by I_g. It is the restriction of a polynomial function on $\mathbf{C}^m$, and hence a regular function on $X(\Pi)$.

1.5. *Some elementary facts.*

1.5.1. LEMMA. *If ρ is an irreducible representation of Π in $\mathrm{SL}_2(\mathbf{C})$ and $\rho(g) \neq \pm 1$ for a given $g \in \Pi$, then there exists $h \in \Pi$ such that the restriction of ρ to the subgroup generated by g and h is irreducible, and such that $\chi_\rho(h) \neq \pm 2$.*

Proof. First we show that there is an element h_0 of Π such that $\rho(g)$ and $\rho(h_0)$ have no common eigenvector. If ρ has a unique 1-dimensional invariant subspace L this is obvious, since by the irreducibility of ρ there exists $h_0 \in \Pi$ such that L is not invariant under $\rho(h_0)$. If ρ has two 1-dimensional invariant subspaces L_1 and L_2, we can find h_1, h_2 such that L_i is not invariant under $\rho(h_i)$. We may take $h_0 = h_1$ or h_2 unless L_1 and L_2 are invariant under $\rho(h_2)$ and $\rho(h_1)$ respectively. But in this case we may take $h_0 = h_1 h_2$.

If $\chi_\rho(h_0) \neq \pm 2$, we may take $h = h_0$ and the conclusion of the lemma follows. Now assume that $\chi_\rho(h_0) = \pm 2$. Observe that $\rho(h_0) \neq \pm 1$ since $\rho(g)$ and $\rho(h_0)$ have no common eigenvector. Thus we may assume that $\rho(h_0) = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\rho(g)$ cannot be upper triangular, and so $\rho(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$. For any $n \in \mathbf{Z}$ we have $\chi_\rho(g h_0^{2n}) = \chi_\rho(g) + 2nc$; hence for some n, $\chi_\rho(g h_0^{2n}) \neq \pm 2$. Set $h = g h_0^{2n}$. Then $\rho(g)$ and $\rho(h)$ cannot have a common eigenvector, because the only 1-dimensional invariant subspace of $\rho(h_0^{2n})$ is also invariant under h_0. □

1.5.2. PROPOSITION. *If ρ and ρ' are representations of Π in $\mathrm{SL}_2(\mathbf{C})$ with $\chi_\rho = \chi_{\rho'}$, and if ρ is irreducible, then ρ and ρ' are equivalent.*

Proof. By 1.5.1 we can find $g, h \in \Pi$ such that the restriction of ρ to the subgroup G generated by g and h is irreducible and $\chi_\rho(h) \neq \pm 2$. Since $\chi_{\rho|G} = \chi_{\rho'|G}$, it follows from 1.2.1 that $\rho' \mid G$ is irreducible. After changing ρ and ρ' within their equivalence classes we may assume that

$$\rho(h) = \rho'(h) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad \alpha \neq \pm 1.$$

Since $\rho \mid G$ and $\rho' \mid G$ are irreducible, neither $\rho(g)$ nor $\rho'(g)$ can be upper or lower triangular. Hence, after composing ρ and ρ' with conjugations by diagonal matrices, we may further assume that

$$\rho(g) = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \qquad \rho'(g) = \begin{pmatrix} a' & 1 \\ c' & d' \end{pmatrix}$$

for some $a, a', c, c', d, d' \in \mathbf{C}$, with $c, c' \neq 0$.

Now for any $\gamma \in \Pi$, set

$$\rho(\gamma) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \qquad \rho'(\gamma) = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}.$$

We have $p + s = \chi_\rho(\gamma) = \chi_{\rho'}(\gamma) = p' + s'$, and

$$\alpha p + \alpha^{-1}s = \chi_\rho(h\gamma) = \chi_{\rho'}(h\gamma) = \alpha p' + \alpha^{-1}s';$$

since $\alpha \neq \pm 1$ it follows that $p = p'$, $s = s'$. Replacing γ by g, we get $a = a'$, $d = d'$. Since $\rho(g)$ and $\rho'(g)$ have determinant 1 it follows that $c = c'$. On the other hand, replacing γ by $g\gamma$, we get $ap + r = a'p' + r'$, so that $r = r'$, and $cq + ds = c'q' + d's' = cq' + ds$. Since $c \neq 0$ it follows that $q = q'$; thus $\rho(\gamma) = \rho'(\gamma)$, and we have shown that $\rho = \rho'$. □

Hyman Bass informs us that 1.5.2 remains true if $SL_2(\mathbf{C})$ is replaced by $GL_n(\mathbf{C})$.

1.5.3. Corollary. *If R_0 is an irreducible component of $R(\Pi)$ which contains an irreducible representation, and if $X_0 = t(R_0)$ (which by 1.4.4 is an affine variety), then*

$$\dim X_0 = \dim R_0 - 3.$$

Proof. It is enough to prove that for every point χ in a Zariski-open set $U \subset X_0$, we have $\dim t^{-1}(\chi) = 3$. Set $U = X_0 \cap (\mathbf{C}^m - V)$, where V is the closed algebraic set given by Lemma 1.4.2; then for each $\chi \in U$, $t^{-1}(\chi)$ consists of irreducible representations. By 1.5.2, $t^{-1}(\chi)$ is a single equivalence class of irreducible representations. Fix $\rho \in t^{-1}(\chi)$. The map $p : SL_2(\mathbf{C}) \to t^{-1}(\chi)$ defined by $p(M) = J_M \circ \rho$, where J_M denotes conjugation by M, is a two-sheeted covering map. (This is because an irreducible representation has non-abelian image, and the centralizer of a non-abelian subgroup of $SL_2(\mathbf{C})$ is $\{\pm 1\}$.) Hence $\dim(t^{-1}(\chi)) = \dim SL_2(\mathbf{C}) = 3$. □

1.5.4. Proposition. *Let $\rho \in R(\Pi)$ be an irreducible representation such that $\rho(g) \neq \pm 1$ for a given $g \in \Pi$. Then there is a Zariski neighborhood U of $t(\rho) = \chi_\rho$ in $X(\Pi)$ such that $\rho'(g) \neq \pm 1$ for any representation $\rho' \in t^{-1}(U)$.*

Proof. By 1.5.1, there exists $h \in \Pi$ such that the restriction of ρ to the subgroup G of Π generated by g and h is irreducible. By 1.2.1, there is an element c of the commutator subgroup of G such that $\chi_\rho(c) \neq 2$. Define U to be the Zariski-open set $X(\Pi) - I_c^{-1}(2)$. Then $t(\rho) \in U$; and for any $\rho' \in t^{-1}(U)$, we have $\chi_{\rho'}(c) \neq 2$, so that $\rho' \mid G$ is irreducible. In particular, $\rho'(G)$ is not cyclic, and therefore $\rho'(g) \neq 1$. □

We conclude with a stronger version of 1.2.1 in a special case.

1.5.5. PROPOSITION. *Suppose that* Π *is generated by* g *and* h. *Then any representation* ρ *of* Π *in* $SL_2(\mathbf{C})$, *such that* $\chi_\rho([g, h]) = 2$, *is reducible.*

Proof. We may assume that $\rho([g, h]) \neq 1$, for otherwise $\rho(\Pi)$ is abelian and the conclusion is immediate. Additionally, we may assume without loss of generality that Π is free on the generators g and h. Let G denote the subgroup generated by $g_1 = g$ and $g_2 = hg^{-1}h^{-1}$. Then the proof of 1.4.1, with Π replaced by G, shows that the ring of functions on $R(G)$ generated by τ_{g_1}, τ_{g_2} and $\tau_{g_1g_2}$ contains τ_γ for every $\gamma \in G$; hence the character of a representation of G in $SL_2(\mathbf{C})$ is determined by its values at g_1, g_2 and g_1g_2. Since $\chi_{\rho|G}$ takes the value $\alpha = \chi_\rho(g)$ at g_1 and g_2 and the value 2 at g_1g_2, it coincides with the character of the representation $\sigma : G \to SL_2(\mathbf{C})$ defined by $\sigma(g_1) = \rho(g_1)$, $\sigma(g_2) = \rho(g_1)^{-1}$. Since σ has cyclic image, it is reducible. Thus by 1.2.2, $\rho \mid G$ is reducible. It follows that $\rho(g)$ and $\rho([g, h])$ have a common eigenvector.

By the same argument, $\rho(h)$ and $\rho([g, h])$ have a common eigenvector. But since $\rho([g, h])$ has trace 2 and is not the identity, it has a unique 1-dimensional invariant subspace. Hence $\rho(g)$ and $\rho(h)$ have a common eigenvector, and the conclusion follows. □

2. Curves of characters and splittings of groups

The main result of this section is Theorem 2.2.1, which associates a decomposition of a group Π to each ideal point of the desingularization $\tilde{C}$ of a curve C in $X(\Pi)$. This is an application of the theorems of Tits, Bass, and Serre [25] on the structure of subgroups of $SL_2(F)$ where F is a field with a discrete valuation. (In our case F is the field of functions on C, and the valuation is that associated with an ideal point of $\tilde{C}$.) The relevant aspects of the Tits-Bass-Serre theory can be summarized in two theorems, which are stated in 2.1. If Π is the fundamental group of a 3-manifold M then, via the decomposition of Π, we can associate to each ideal point of $\tilde{C}$ a family of incompressible surfaces in M. This is discussed in 2.3.

2.1. *Tits-Bass-Serre theory.* We begin by reviewing the combinatorial aspect of the Bass-Serre theory of groups acting on trees; we shall take the point of view of Tretkoff's paper [35], which is the natural one for our applications.

By a *graph* we shall mean a 1-dimensional CW-complex. A connected, simply-connected graph will be called a *tree*. A *homomorphism* of graphs is a map which sends vertices to vertices and maps edges to edges linearly.

Associated with a graph G is a category whose objects are the edges and vertices of G, with a morphism from each edge to each of its endpoints. A *graph of groups*, with underlying graph G, is a functor $\mathcal{G}$ from this category to the category of groups and monomorphisms. A graph of groups will be referred to as a pair $(G, \mathcal{G})$.

An Eilenberg-Maclane space can be constructed from a graph of groups $(G, \mathcal{G})$. For each vertex v of G let X_v be a space of type $K(\mathcal{G}(v), 1)$, and for each edge e let X_e be a space of type $K(\mathcal{G}(e), 1)$. Consider the disjoint union of all the spaces $X_e \times [0, 1]$ and X_v. Let $K(G, \mathcal{G})$ be the quotient of this union defined as follows. If e is an edge with vertices v_0 and v_1, and if $f_0 : X_e \to X_{v_0}$ and $f_1 : X_e \to X_{v_1}$ are maps inducing the monomorphisms $: \mathcal{G}(e) \to \mathcal{G}(v_0)$ and $: \mathcal{G}(e) \to \mathcal{G}(v_1)$, then we identify (x, i) to $f_i(x)$ for each $x \in X_e$ and for $i = 0, 1$.

We define the *fundamental group of the graph of groups* $(G, \mathcal{G})$ to be the group $\pi_1(G, \mathcal{G}) \equiv \pi_1(K(G, \mathcal{G}))$. We remark that the isomorphism type of $\pi_1(G, \mathcal{G})$ is independent of all choices made in its definition, and that the natural homomorphisms $: \mathcal{G}(v) \to \pi_1(G, \mathcal{G})$ and $: \mathcal{G}(e) \to \pi_1(G, \mathcal{G})$ are in fact monomorphisms whose images are well-defined up to conjugacy in $\pi_1(G, \mathcal{G})$. A subgroup which is conjugate to the image of $\mathcal{G}(v)$ for a vertex v of G is called a *vertex group*. An *edge group* is a subgroup of $\pi_1(G, \mathcal{G})$ which is conjugate to the image of $\mathcal{G}(e)$ where e is an edge of G.

An isomorphism between an abstract group Π and the fundamental group of a graph of groups will be called a *splitting* of Π. A splitting of Π gives rise to a well-defined set of vertex groups and edge groups in Π. The splitting will be *non-trivial* if all the vertex groups are proper subgroups.

We will say that a group Π acts on the graph G *without inversions* provided that Π acts by isomorphisms of G which do not reverse the orientation of any invariant edge. If Π acts without inversions on G then the quotient G/Π is a graph, and the projection $: G \to G/\Pi$ is a homomorphism.

Suppose that the group Π acts without inversions on the tree T. Then T will contain a sub-tree T' which contains exactly one edge from each orbit of the action of Π on the edges of T. If c is an edge or vertex of T/Π, let $\tilde{c}$ denote the edge or a vertex of T' which projects to c under the quotient map $T \to T/\Pi$. Let $(T/\Pi, \mathcal{G})$ be the graph of groups defined by setting $\mathcal{G}(C)$ equal to the stabilizer of $\tilde{c}$ for each edge or vertex c of T/Π, with monomorphisms defined as follows. If e is an edge of T/Π, with vertices v_0 and v_1, then there exist elements γ_0 and γ_1 of Π such that $\gamma_0\tilde{v}_0$ and $\gamma_1\tilde{v}_1$ are the endpoints of $\tilde{e}$. We embed the stabilizer of $\tilde{e}$ into the stabilizer of $\tilde{v}_i$, $i = 0$ or 1, by sending x to $\gamma_i^{-1}x\gamma_i$. (If different choices of

γ_0 and γ_1 are made, the resulting graphs of groups have isomorphic fundamental groups.)

The following result is the basic combinatorial fact in the Tits-Bass-Serre theory. It is included in Theorem 7 of [35].

2.1.1. THEOREM. *If the group* Π *acts without inversions on the tree* T, *then* Π *is isomorphic to* $\pi_1(T/\Pi, \mathcal{G})$, *where the graph of groups* $(T/\Pi, \mathcal{G})$ *is defined as above.*

Suppose now that Π is a subgroup of $SL_2(F)$ where F is a field with a discrete valuation v. The valuation ring of F defined by v is the subring $\mathcal{O} = \{\alpha \in F \mid v(\alpha) \geq 0\}$. Let π be an element with $v(\pi) = 1$. Then (π) is the unique maximal ideal in $\mathcal{O}$, and the field $k = \mathcal{O}/(\pi)$ is called the residue field of v. Let V be a two-dimensional vector space over F. An $\mathcal{O}$-lattice in V is an $\mathcal{O}$-submodule of V which spans V as a k-vector space. Any $\mathcal{O}$-lattice L has a basis $\{e_1, e_2\}$ over $\mathcal{O}$, and if $L' \subset L$ is another $\mathcal{O}$-lattice then L/L' is isomorphic to $\mathcal{O}/(\pi^a) \oplus \mathcal{O}/(\pi^b)$ for non-negative integers a and b. If α is a non-zero element of F, then αL is also an $\mathcal{O}$-lattice. Define two $\mathcal{O}$-lattices L and L' to be equivalent if $L = \alpha L'$ for some $\alpha \in F^*$. We may now define a graph by taking as vertices the equivalence classes of $\mathcal{O}$-lattices in V. Two $\mathcal{O}$-lattices Λ and Λ' are joined by an edge whenever there exist $\mathcal{O}$-lattices L and L' in Λ and Λ' such that $L' \subset L$ and $L/L' \approx k$. It is shown in [25] that this graph is a tree. This is a special case of a theorem due to Tits.

Now let ρ be a representation of a group Π in $SL_2(F)$. Then ρ determines in an obvious way an action of Π on the tree defined above. Moreover, the stabilizers of the vertices under this action are precisely those subgroups of Π whose images under ρ are contained in conjugates of $SL_2(\mathcal{O})$ in $GL_2(F)$. Hence 2.1.1 yields the following result.

2.1.2. THEOREM. *If* ρ *is a representation of a group* Π *in* $SL_2(F)$, *where* F *is a discretely valued field, then* Π *has a splitting in which the vertex groups are precisely those subgroups of* Π *whose images under* ρ *are contained in conjugates of* $SL_2(\mathcal{O})$ *in* $GL_2(F)$. □

2.2. *The Fundamental Theorem.* The following theorem will be our fundamental tool for obtaining topological information from the space of characters of the fundamental group of a 3-manifold.

2.2.1. THEOREM. *Let* C *be an affine curve contained in* $X(\Pi)$. *To each ideal point* $\tilde{x}$ *of* $\tilde{C}$ *one can associate a splitting of* Π *with the property that an element* g *of* Π *is contained in a vertex group if and only if* $\tilde{I}_g$ *does not have a pole at* $\tilde{x}$. *Thus, in particular, the splitting is non-trivial.*

Proof. As in the proof of 1.4.4, there exists a curve $D \subset t^{-1}(C) \subset R(\Pi)$ such that the restriction of t to D is not constant. We thus have the following commutative diagram of rational maps:

$$\begin{array}{ccccc} \tilde{D} & \to & D & \subset & R(\Pi) \\ \downarrow \widetilde{t|_D} & & \downarrow t|_D & & \downarrow t \\ \tilde{C} & \to & C & \subset & X(\Pi). \end{array}$$

The map $\widetilde{t|_D}$ is a regular map for which the inverse image of an ideal point of $\tilde{C}$ consists of ideal points of $\tilde{D}$. Let $\tilde{y}$ be an ideal point of $\tilde{D}$ in the inverse image of $\tilde{x}$.

We now consider the canonical representation P of Π into $\mathrm{SL}_2(F)$ where F is the field of functions on $\tilde{D}$. The point $\tilde{y}$ of $\tilde{D}$ determines a discrete valuation on F for which the valuation ring $\mathcal{O}$ consists of all functions on $\tilde{D}$ which do not have poles at $\tilde{y}$. Thus we may apply Theorem 2.1.2 to conclude that the group Π has a splitting with the property that an element g of Π is contained in a vertex group if and only if $P(g)$ is conjugate in $\mathrm{GL}_2(F)$ to an element of $\mathrm{SL}_2(\mathcal{O})$.

To complete the proof we must show that $P(g)$ is conjugate to an element of $\mathrm{SL}_2(\mathcal{O})$ if and only if $\tilde{I}_g$ does not have a pole at $\tilde{x}$. If $P(g)$ is conjugate to $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha, \beta, \delta, \gamma \in \mathcal{O}$, then $\tilde{\tau}_g = \alpha + \delta$ does not have a pole at $\tilde{y}$ and hence $\tilde{I}_g$ does not have a pole at $\tilde{x}$. Conversely, suppose $\tilde{I}_g$ does not have a pole at $\tilde{x}$. Then $\tilde{\tau}_g$ does not have a pole at $\tilde{y}$, and hence is an element of $\mathcal{O}$. Let v be a vector in F^2 which is not an eigenvector for $P(g)$. In terms of the basis $\{v, P(g)v\}$, $P(g)$ is described by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & \tilde{\tau}_g \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$. Thus $P(g)$ is conjugate in $\mathrm{GL}_2(F)$ to an element of $\mathrm{SL}_2(\mathcal{O})$.

Since $\tilde{x}$ is an ideal point, $\tilde{I}_g$ must have a pole at $\tilde{x}$ for some $g \in \Pi$. Hence the splitting is non-trivial. □

2.3. *Incompressible surfaces.* The Fundamental Theorem 2.2.1 will be applied in the subsequent sections to the case that Π is the fundamental group of a compact, orientable 3-manifold. In this case the following result, whose proof is essentially due to Stallings, Epstein, and Waldhausen, will permit one to construct incompressible surfaces in the 3-manifold from the splittings given by 2.2.1.

2.3.1. PROPOSITION. *Let N be a compact, orientable 3-manifold. For any non-trivial splitting of $\pi_1(N)$ there exists a nonempty system $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_m$ of incompressible surfaces in N, none of which is boundary-parallel, such that* $\mathrm{im}(\pi_1(\Sigma_i) \to \pi_1(N))$ *is contained in an edge group for $i = 1, \ldots, m$, and*

$\mathrm{im}(\pi_1(R) \to \pi_1(N))$ *is contained in a vertex group for each component* R *of* $N - \Sigma$. *Moreover, if* $\mathcal{K} \subset \partial N$ *is a subcomplex such that* $\mathrm{im}(\pi_1(K) \to \pi_1(N))$ *is contained in a vertex group for each component* K *of* $\mathcal{K}$, *we may take* Σ *to be disjoint from* $\mathcal{K}$.

Proof. We are given an isomorphism of $\pi_1(N)$ with $\pi_1(G, \mathcal{G}) = \pi_1(K(G, \mathcal{G}))$, where $(G, \mathcal{G})$ is a graph of groups. Since $K(G, \mathcal{G})$ is an aspherical space, there is a map $f: N \to K(G, \mathcal{G})$ inducing the isomorphism. After a homotopy we may assume that f is transverse to the bi-collared subcomplex $X_e \times \{\frac{1}{2}\}$ of $K(G, \mathcal{G})$ for each edge e of G, and that for each component K of $\mathcal{K}$, $f(K)$ is contained in X_v for some vertex v of G. After a further homotopy (cf. [14, Lemma 6.5]) we may assume that each component of $f^{-1}(\bigcup X_e \times \{\frac{1}{2}\})$ is incompressible, where the union is taken over all edges e of G. Define Σ to be the union of all components of $f^{-1}(\bigcup X_e \times \{\frac{1}{2}\})$ which are not boundary-parallel. Then Σ is easily seen to have all the properties asserted in the statement of the proposition. (If Σ were empty, f would be homotopic to a map with image disjoint from all the X_e; this would imply that $\pi_1(N)$ is a vertex group, contradicting the nontriviality of the splitting.) □

3. Hyperbolic 3-manifolds

The Fundamental Theorem 2.2.1 and Proposition 2.3.1, taken together, show how the ideal points of a curve in $X(\pi_1(M))$, where M is a 3-manifold, can give rise to systems of incompressible surfaces in M. In this section we show how a hyperbolic structure on M can give rise to curves of characters to which the results of Section 2 can be applied. The substance of this section is largely due to Thurston [30]. However, since the point of view of this paper is rather different from that of [30], we have found it most convenient to give a self-contained account of the material.

3.1 *Representations and hyperbolic structures.* A (complete, oriented) *hyperbolic 3-manifold* is the quotient of the hyperbolic 3-space H^3 by a discrete, torsion-free group Γ of orientation-preserving isometries. The group of orientation-preserving isometries of H^3 can be identified with $\mathrm{PSL}_2(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})/\{\pm 1\}$, and any discrete, torsion-free subgroup of $\mathrm{PSL}_2(\mathbf{C})$ acts freely and properly discontinuously on H^3. Two hyperbolic 3-manifolds $\Gamma \setminus H^3$, $\Gamma' \setminus H^3$ are isometric by an orientation-preserving isometry (and will be identified) if and only if Γ and Γ' are conjugate in $\mathrm{PSL}_2(\mathbf{C})$. Thus with a hyperbolic 3-manifold M we can associate a representation of $\pi_1(M)$ in $\mathrm{PSL}_2(\mathbf{C})$ which is canonical up to equivalence; this representation is faithful and its image is discrete and torsion-free. It follows from results proved in [12] that M can be identified diffeomorphically with the interior of a possibly noncompact manifold-with-boundary $\overline{M}$, whose boundary components are tori, in such a way that every rank-two free abelian subgroup of $\pi_1(\overline{M}) \approx \pi_1(M)$ is contained in a *peripheral torus subgroup*, i.e. a conjugate of the image under inclusion of $\pi_1(T)$ for some incompressible component T of $\partial\overline{M}$. (Since M must clearly be irreducible, T cannot have a compressible component unless $M \approx \mathbf{R}^2 \times S^1$.) A peripheral torus subgroup of $\pi_1(M) \subset \mathrm{PSL}_2(\mathbf{C})$ is conjugate to a group of cosets of matrices of the form $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda \in \mathbf{C}$; in fact, it is easily seen that every discrete rank-two free abelian subgroup of $\mathrm{PSL}_2(\mathbf{C})$ has this form. In particular, the traces (defined up to sign) of the elements of such a subgroup are ± 2. The volume of M, as a Riemannian manifold, is finite if and only if $\overline{M}$ is compact and $\pi_1(M)$ contains no abelian subgroup of finite index.

3.1.1. Proposition. (Thurston). *Let M be a hyperbolic 3-manifold. The canonical representation of $\pi_1(M)$ in* $\mathrm{PSL}_2(\mathbf{C})$ *may be lifted to a representation in* $\mathrm{SL}_2(\mathbf{C})$.

Proof. Let $\mathcal{P}$, $\tilde{\mathcal{P}}$ denote the principal bundles of unit tangent frames to the oriented manifolds M and H^3, respectively. The action of $\mathrm{PSL}_2(\mathbf{C})$ on H^3 induces, via differentiation, an action on $\tilde{\mathcal{P}}$, which is easily seen to be simply transitive; thus the choice of a "base frame" $\tilde{\Phi} \in \tilde{\mathcal{P}}$ gives rise to a diffeomorphic identification of $\tilde{\mathcal{P}}$ with $\mathrm{PSL}_2(\mathbf{C})$. On the other hand, the covering projection $p : H^3 \to M$ induces a local diffeomorphism $P : \tilde{\mathcal{P}} \to \mathcal{P}$, which maps the fibers of the frame bundle $\tilde{\mathcal{P}}$ diffeomorphically onto the fibers of $\mathcal{P}$. Since p is a covering map, so is P. Now the action of $\mathrm{PSL}_2(\mathbf{C})$ on $\tilde{\mathcal{P}}$ restricts to an action of $\Gamma = \pi_1(M)$ (which we identify with a subgroup of $\mathrm{PSL}_2(\mathbf{C})$) on $\tilde{\mathcal{P}}$. By the chain rule we have $P \circ \gamma = P$ for each $\gamma \in \Gamma$. Moreover, since the restriction of P to each fiber of the SO(3)-bundle $\tilde{\mathcal{P}}$ is 1-1, it is clear that Γ acts transitively on the fibers of $\tilde{\mathcal{P}}$ as a covering space of $\mathcal{P}$. Thus $\tilde{\mathcal{P}}$ is a regular covering space with covering group Γ; and by using the above identification of $\tilde{\mathcal{P}}$ with $\mathrm{PSL}_2(\mathbf{C})$ we have $\mathcal{P} \approx \Gamma \setminus \mathrm{PSL}_2(\mathbf{C})$.

Let $\tilde{\Gamma}$ denote the inverse image of Γ in $\mathrm{SL}_2(\mathbf{C})$ under the quotient homomorphism ϕ. Then $\mathcal{P} \approx \Gamma \setminus \mathrm{SL}_2(\mathbf{C})$; since $\mathrm{SL}_2(\mathbf{C})$ is simply connected, $\pi_1(\mathcal{P}) \approx \tilde{\Gamma}$. But M, as an orientable 3-manifold, is necessarily parallelizable, so that $\mathcal{P} \approx M \times \mathrm{SO}(3)$. Thus $\tilde{\Gamma} \approx \pi_1(\mathcal{P}) \approx \Gamma \times \mathbf{Z}_2$. Since Γ is torsion-free, and since the kernel of ϕ has order 2, ϕ must map the first factor in this direct-product decomposition of $\tilde{\Gamma}$ isomorphically onto Γ, and the conclusion follows. $\square$

3.2 *Dimensions of varieties of characters.* The following result appears, in quite different language, in [30].

3.2.1. PROPOSITION. (Thurston). *Let N be a compact orientable 3-manifold. Let $\rho_0 : \pi_1(N) \to SL_2(\mathbf{C})$ be an irreducible representation such that for each torus component T of ∂N, $\rho_0(\mathrm{im}(\pi_1(T) \to \pi_1(N))) \not\subset \{\pm 1\}$. Let R_0 be an irreducible component of $R(\pi_1(N))$ containing ρ_0. Then $X_0 = t(R_0)$ (which by 1.4.4 is an affine variety) has dimension $\geq s - 3\chi(N)$, where s is the number of torus components of ∂N.*

Proof (Thurston). By 1.5.3, the conclusion is equivalent to the assertion that $\dim R_0 \geq s - 3\chi(N) + 3$. We shall prove this by induction on s. First suppose that $s = 0$. We may assume that $\partial N \neq \varnothing$, for otherwise $\chi(N) = 0$ and there is nothing to prove. Now N has the homotopy type of a finite 2-dimensional CW-complex K with one 0-cell and, say, m 1-cells and n 2-cells. Thus $\pi_1(N)$ has a presentation $\langle g_1, \ldots, g_m : r_1 = \cdots = r_n = 1\rangle$. Define a regular map $f: SL_2(\mathbf{C})^m \to SL_2(\mathbf{C})^n$ by $f(x_1, \ldots, x_m) = (r_i(x_1, \ldots, x_m))_{1 \leq i \leq n}$, where $r_i(x_1, \ldots, x_m)$ is the matrix obtained by substituting x_j for g_j in the word r_i. Then $R(\pi_1(N)) = f^{-1}(1, 1, \ldots, 1)$. Thus $R(\pi_1(N))$ is the inverse image of a point under a regular map from a $3m$-dimensional affine variety to a $3n$-dimensional affine variety, and hence each of its irreducible components has dimension $\geq 3m - 3n = -3\chi(K) + 3 = -3\chi(N) + 3$, as required.

Now suppose that $s > 0$. Let T be a torus component of ∂M. Since T is incompressible, there is an element $\alpha \in \pi_1(N)$, represented by a simple closed curve in T, such that $\rho_0(\alpha) \neq \pm 1 \in SL_2(\mathbf{C})$. By 1.5.1, there is an element γ of $\pi_1(N)$ such that ρ_0 restricts to an irreducible representation of the subgroup generated by α and γ, and such that $\chi_{\rho_0}(\gamma) \neq \pm 2$.

There exists a manifold N', such that N is obtained from N' by the addition of a 2-handle to a genus 2 boundary component T' of N', and such that the following conditions hold. There is a standard basis $\alpha', \beta', \gamma', \delta'$ of $\pi_1(T')$ so that α' and γ' are mapped to α and γ under the natural surjection $i_* : \pi_1(N') \to \pi_1(N)$, and the 2-handle is attached to T' along a simple closed curve which represents the conjugacy class of δ'. The manifold N' can be constructed by removing from N a neighborhood of an embedded loop, based on T, which represents the element γ of $\pi_1(N)$. Let σ denote the element $[\alpha', \beta'] = [\gamma', \delta']$ of $\pi_1(N')$.

The representation $\rho_0' = \rho_0 \circ i_* : \pi_1(N') \to SL_2(\mathbf{C})$ is irreducible since i_* is a surjection. Since the kernel of i_* is the normal closure of δ', we may identify $R(\pi_1(N))$ with the closed algebraic set $W \subset R(\pi_1(N'))$ consisting of representations ρ of $\pi_1(N')$ such that $\rho(\delta') = 1$. (This identification implicitly uses the fact that the isomorphism type of $R(\pi_1(N))$ does not depend on the choice of generators of $\pi_1(N)$.) In particular, R_0 is an irreducible component of $R_0' \cap W$, where R_0' is an irreducible component of $R(\pi_1(N'))$ containing ρ_0'. By the induction hypothesis,

$$\dim R_0' \geq (s-1) - 3\chi(N') + 3 = s - 3\chi(N) + 5.$$

We will show that R_0 is an irreducible component of the inverse image of a point under a regular map $f: R_0' \to \mathbf{C}^2$, from which it follows that $\dim R' \geq s - 3\chi(N) + 3$.

Specifically, let $f: R_0' \to \mathbf{C}^2$ be defined by $f(\rho) = (\tau(\delta'), \tau(\sigma))$. Clearly $R_0 \subset f^{-1}(2,2)$. Let Y denote the set of all representations $\rho \in R_0'$ whose restrictions to the subgroup of $\pi_1(N')$ generated by α' and γ' is reducible. By 1.4.2, Y is a closed algebraic set. Let Z denote the closed algebraic set consisting of all $\rho \in R_0'$ such that $\tau_{\gamma'}(\rho) = \chi_\rho(\gamma') = \pm 2$. Since $i_*(\gamma') = \gamma$, the defining properties of the element γ guarantee that $\rho_0' \notin Y \cup Z$. Hence in order to prove our claim it is enough to prove that $f^{-1}(2,2) \subset (R_0' \cap W) \cup Y \cup Z$.

Let $\rho \in f^{-1}(2,2)$ be given. Then $\chi_\rho(\delta') = \chi_\rho(\sigma) = 2$. By 1.5.5, the restrictions of ρ to the subgroups of $\pi_1(N')$ generated respectively by $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are reducible. Thus $\rho(\sigma)$ has an eigenvector in common with each of the matrices $\rho(\alpha'), \rho(\gamma')$. In the case $\rho(\sigma) \neq 1$, $\rho(\sigma)$ has a unique 1-dimensional invariant subspace, and it follows that $\rho \in Y$.

Now consider the case $\rho(\sigma) = 1$. Then $\rho(\gamma')$ and $\rho(\delta')$ commute. Since $\rho(\delta')$ has trace ± 2, either $\rho(\delta') = \pm 1$ or $\rho(\gamma')$ has trace ± 2; thus either $\rho \in W$ or $\rho \in Z$. This establishes the inclusion $f^{-1}(2,2) \subset (W \cap R_0') \cup Y \cup Z$. □

Now let M denote a hyperbolic 3-manifold of finite volume. Fix a faithful representation of $\pi_1(M)$ in $\mathrm{PSL}_2(\mathbf{C})$ with discrete image. Using 3.1.1, fix a lifting $\rho_0 : \pi_1(M) \to \mathrm{SL}_2(\mathbf{C})$ of this representation. Then ρ_0 is faithful and has discrete image. Choose an irreducible component R_0 of $R(\Pi)$ containing ρ_0. The representation ρ_0 is necessarily irreducible. (In fact, the argument given in [12] shows that ρ_0 could be reducible only if Π were abelian, and this would contradict the assumption that M has finite volume.) Hence by 1.4.4, $X_0 = t(R_0) \subset X(\Pi)$ is an affine variety.

If M is not closed, it may be seen at once from 3.2.2 that $\dim X_0 > 0$; thus X_0 contains at least one curve. One can apply Theorem 2.2.1 to the ideal points of such a curve to obtain splittings of $\pi_1(M)$, which by 2.3.1 give rise to systems of incompressible surfaces in M. This is the method that will be used in Section 5 of this paper and in [6] and [7].

4. Haken manifolds and virtual Haken manifolds

We begin by reviewing some deep results, due to Thurston and Mostow, about hyperbolic manifolds. We shall then show how to combine these with results due to Scott, Waldhausen, and Meeks-Simon-Yau to get a structure theorem (4.2.2) for "almost sufficiently large" 3-manifolds up to homotopy type. Although this result follows easily from several known theorems, it does not seem to be generally understood.

4.1. *Thurston's and Mostow's theorems.* Recall that a *Haken manifold* is a compact, orientable, irreducible 3-manifold that is "sufficiently large," i.e. contains some incompressible surface. The last condition is automatic in the case of a bounded manifold. By a *virtual Haken manifold* we shall mean a compact, orientable, irreducible 3-manifold that is "almost sufficiently large," i.e. that has a finite-sheeted covering containing an incompressible surface. By a recent theorem due to Meeks, Simon and Yau [19], a finite-sheeted covering of an orientable, irreducible 3-manifold is irreducible. Thus every virtual Haken manifold is covered by a Haken manifold.

Thurston's Uniformization Theorem [31] characterizes the Haken manifolds whose interiors have hyperbolic metrics. The following version of Thurston's theorem, which is equivalent to the original version via the strong form of the torus theorem ([15], [16]) is the most convenient form for the applications in this paper.

4.1.1. THURSTON'S UNIFORMIZATION THEOREM. *Let N be a Haken manifold whose boundary components (if any) are all tori. Then either N is Seifert-fibered, or N contains an incompressible torus which is not boundary-parallel, or N is homeomorphic to $\overline{M}$ (§ 3) for some hyperbolic manifold M of finite volume.* □

Thurston has conjectured that 4.1.1 is true for every closed, orientable, irreducible 3-manifold M, even if M is not assumed to contain an incompressible surface. In 4.2.1 below we shall show that *up to homotopy type* this is true for all virtual Haken manifolds. In Section 5 we shall present some new evidence for Thurston's conjecture in the case that M is, for example, a regular branched cover of S^3.

The next result is the 3-dimensional closed case of the theorem proved by Mostow in [22]. (For another proof, due to Gromov, see [30].)

4.1.2. MOSTOW RIGIDITY THEOREM. *Let $f: M \to M'$ be a diffeomorphism between closed hyperbolic 3-manifolds. Then f is homotopic to an isometry.* □

(We have restricted attention to closed manifolds as a matter of convenience; 4.1.2 remains true if M and M' are hyperbolic 3-manifolds of finite volume.)

We shall be using Mostow's theorem in an equivalent form, Proposition 4.1.5 below. A proof of 4.1.5 is actually implicit in Mostow's proof of his rigidity theorem; however, for the reader's convenience we shall show, conversely, how to derive 4.1.5 from 4.1.2. First we review some elementary facts.

If the hyperbolic 3-space H^3 is identified with the open upper half-space in $\mathbf{R}^3$, every isometry of H^3 extends to the 1-point compactification $\overline{H}^3$ of the closed upper half-space. Topologically, $\overline{H}^3$ is a closed ball; its boundary is the "sphere at infinity" S_∞. We may identify S_∞ with the Riemann sphere so that the self-homeomorphisms of S_∞ induced by isometries of H^3 are the classical homographies (Möbius transformations) and anti-homographies. A hyperbolic isometry is uniquely determined by the induced homography or anti-homography.

4.1.3. Proposition. *Let M and M' be closed hyperbolic 3-manifolds, let $F_0, F_1 : M \to M'$ be homotopic diffeomorphisms, and suppose that F_0 is an isometry. Let $\tilde{F}_0 : H^3 \to H^3$ be a lifting of F_0, so that $\tilde{F}_0$ is an isometry and induces a homography or anti-homography $\mu : S_\infty \to S_\infty$. Then F_1 has a unique lifting $\tilde{F}_1 : H^3 \to H^3$ which extends to a homeomorphism of $\overline{H}^3$ and induces the homeomorphism μ on S_∞.*

Proof. The uniqueness is clear, for any two liftings of F_1 differ by a covering transformation; and a covering transformation, as a hyperbolic isometry of H^3, cannot induce the identity on S_∞ unless it is itself the identity. To prove existence, we fix a smooth homotopy $F : M \times I \to M$ from F_0 to F_1. By the covering homotopy property, there is a homotopy $\tilde{F} : H^3 \to H^3$ such that $(\tilde{F})_0 = \tilde{F}_0$. We shall show that $\tilde{F}_1 = (\tilde{F})_1$ has the desired properties.

It suffices to show that if $(x_i)_{i\geq 0}$ is a sequence of points in H^3 converging to $x \in S_\infty$, then $(\tilde{F}_1(x_i))_{i\geq 0}$ converges to $\mu(x)$. For $i \geq 0$, the smooth path $\tilde{\alpha}_i : I \to H^3$ defined by $\tilde{\alpha}_i(t) = \tilde{F}_t(x_i)$ has bounded length as $i \to \infty$ because $\tilde{F}$ is a lift of the smooth homotopy F between closed manifolds. Thus the hyperbolic distance between $y_i = \tilde{F}_0(x_i)$ and $z_i = \tilde{F}_1(x_i)$ is bounded. But it is not hard to show that if $(y_i)_{i\geq 0}$ and $(z_i)_{i\geq 0}$ are any sequences of points in H^3 such that the hyperbolic distance between y_i and z_i is bounded as $i \to \infty$, and if (y_i) converges in $\overline{H}^3$ to a point $y \in S_\infty$, then (z_i) also converges in $\overline{H}^3$ to y. Since $(\tilde{F}_0(x_i))$ converges to $\mu(x)$, so does $(\tilde{F}_1(x_i))$. □

4.1.4. Corollary. *The isometry in the conclusion of* 4.1.2 *is unique.* □

4.1.5. Proposition. *Let $f : M \to M'$ be a diffeomorphism of closed hyperbolic 3-manifolds. Then any lifting of f to H^3 extends to a homeomorphism of $\overline{H}^3$ which restricts to a homography or anti-homography on S_∞.*

Proof. This is immediate from 4.1.2 and 4.1.3. □

4.2. *Virtual Haken manifolds*. The following consequence of the Mostow rigidity theorem was pointed out to us by Thurston, and John Morgan supplied some help with the argument.

4.2.1. PROPOSITION. *Let M be a closed, orientable 3-manifold. Suppose that some finite-sheeted covering space of M has a hyperbolic metric. Then M is homotopy-equivalent to a hyperbolic 3-manifold.*

Proof. Since every finite-index subgroup of $\pi_1(M)$ contains a normal finite-index subgroup, M has a finite-sheeted regular covering space $\tilde{M}$ which is hyperbolic. We may identify the universal covering space of M with the universal covering of $\tilde{M}$, and hence with H^3; let Γ and $\tilde{\Gamma}$ denote the groups of covering transformations of H^3 over M and $\tilde{M}$ respectively. Every $\gamma \in \Gamma$ is a lifting of some covering transformation of $\tilde{M}$ over M; hence by 4.1.5, γ extends to a homeomorphism of $\bar{H}^3$ which restricts on S_∞ to a Möbius transformation $\rho(\gamma) \in PSL_2(\mathbf{C})$. This defines a representation ρ of Γ in $PSL_2(\mathbf{C})$. Since $\tilde{\Gamma} \subset \Gamma$ is a group of hyperbolic isometries, the homomorphism $\rho \mid \tilde{\Gamma} : \tilde{\Gamma} \to PSL_2(\mathbf{C})$ is injective. But $\tilde{\Gamma}$ has finite index in Γ, and Γ is torsion-free since it acts continuously and without fixed points on the contractible, finite-dimensional space H^3. Hence $\rho : \Gamma \to PSL_2(\mathbf{C})$ is injective. Moreover, $\rho(\Gamma) \subset PSL_2(\mathbf{C})$ is discrete since it has the discrete subgroup $\rho(\tilde{\Gamma})$ of finite index. Since $\rho(\Gamma)$ is also torsion-free, it is isomorphic to the fundamental group of a hyperbolic 3-manifold M'. Since M and M' have isomorphic fundamental groups and are both covered by the contractible manifold H^3, they are homotopy-equivalent. □

4.2.2. THEOREM. *Let M be a virtual Haken manifold without boundary. Then either M is Seifert-fibered, or M contains an incompressible torus, or M is homotopy-equivalent to a hyperbolic 3-manifold.*

Proof. In the language of [27], the *torus conjecture* is said to hold in a closed irreducible 3-manifold M if every essential singular torus in M, i.e. every map $f: T^2 \to M$ which induces a monomorphism of fundamental groups, is homotopic to a map whose image is contained in a Seifert-fibered submanifold of M whose boundary tori are incompressible in M. Theorem 1.2 of [27], which follows from work done by Scott, Waldhausen, and Meeks-Simon-Yau, asserts that the torus conjecture holds in every virtual Haken manifold M. Hence if there exists an essential singular torus in M, either M is Seifert-fibered or M contains an incompressible torus.

Now consider the case that there is no essential singular torus in M. Since M is a virtual Haken manifold, some finite-sheeted covering space $\tilde{M}$ of M is a Haken manifold. Clearly there is no essential singular torus in $\tilde{M}$. In particular $\tilde{M}$ contains no incompressible torus; and $\tilde{M}$ cannot be Seifert-fibered, because in every Seifert-fibered space with infinite fundamental group there is an essential singular torus. Hence, by 4.1.1, $\tilde{M}$ is hyperbolic; and so by 4.2.1, M is homotopy-equivalent to a hyperbolic 3-manifold. □

5. Regular branched covering spaces

In this section we give new proofs of the generalized Smith conjecture [3] and of the results due to Davis and Morgan [9] on finite group actions on homotopy 3-spheres. Our proofs yield new results, apparently inaccessible by other methods, on the structure of closed, orientable 3-manifolds which are regular branched covers of other manifolds branched over non-trivial links. Theorems 5.1.2, 5.2.4, and 5.2.7 provide evidence that the conjecture of Thurston's stated in Section 4 may be true for the non-simply-connected prime factors of such manifolds. Moreover, many open questions on 3-manifolds, which pre-date Thurston's conjectures, can be proved for this class of manifolds by means of our Theorem 5.2.4; we give several examples of this in subsection 5.3.

The proof of the Smith conjecture in [3] depends on showing that a regular branched covering space N branched over a "sufficiently non-trivial" link either contains an incompressible surface or has a fundamental group that can be represented non-trivially in $\mathrm{PSL}_2(F)$ for some finite field F. In our proof, given in 5.1, the finite field F is replaced by $\mathbf{C}$. In 5.2 we show, roughly speaking, that if one of the ramification indices of the covering is greater than 5, the representation in $\mathrm{PSL}_2(\mathbf{C})$ can be taken to have infinite image. (This is strictly true if the group of covering transformations contains a normal 2-complement (Theorem 5.2.7); otherwise one gets (5.2.4) a representation with infinite image of a subgroup of $\pi_1(N)$ whose index is a power of 2.) The Davis-Morgan theorem is derived as a corollary to 5.2.4. In 5.3 we apply 5.2.4 to prove analogues for certain regular branched coverings of two known theorems on Haken manifolds: Waldhausen's theorem [36] on 3-manifolds whose fundamental groups have center, and Evans and Jaco's theorem [11] on free subgroups of 3-manifold groups.

5.1. *Representations in* $\mathrm{PSL}_2(\mathbf{C})$ *and the Smith Conjecture.* Let N denote a regular branched covering space of a closed orientable 3-manifold M, branched over a link in M. This means that M is the quotient of N under the action of a finite group Γ of orientation-preserving diffeomorphisms such that, for each $x \in N$, the stabilizer Γ_x is cyclic; and for some x, $\Gamma_x \neq \{1\}$. The subset S of N consisting of all points with non-trivial stabilizers will be called the *singular set.* It is easy to show that S is a 1-manifold. Thus $L = S/\Gamma$ is a link in M, called the *branch set*.

Suppose that K is a component of L. An element μ of $\pi_1(K - L)$ will be called a *meridian* of K if it belongs to the unique conjugacy class in $\pi_1(M - L)$ which is represented by the boundary of an embedded disk in M that meets K in one point. The *ramification index* of K is the order of the stabilizer of any point of S in the inverse image of K. (All such stabilizers are non-trivial and conjugate in Γ.)

Let $\tilde{\Gamma}$ be the group of all lifts of diffeomorphisms in Γ to the universal cover $\tilde{N}$ of N; in the language of orbifolds (see Thurston, [30]), $\tilde{\Gamma}$ would be called the fundamental group of the orbifold M. The group $\tilde{\Gamma}$ acts on $\tilde{N}$ with cyclic stabilizers, and $\pi_1(N)$ is embedded in $\tilde{\Gamma}$ as the group of lifts of the identity map. Let $\tilde{S}$ denote the singular set in $\tilde{N}$. Since $\tilde{\Gamma}$ acts freely on $\tilde{N} - \tilde{S}$ with quotient $M - L$, we may view $\tilde{\Gamma}$ as a quotient of $\pi_1(M - L)$; the stabilizers in $\tilde{\Gamma}$ are generated by the images of the meridians of components of L.

Let $\phi : \pi_1(M - L) \to \tilde{\Gamma}$ be the quotient map and let $\langle \quad \rangle$ denote normal closure. It is not difficult to check that, if $\mu_1, \ldots, \mu_s$ and $r_1, \ldots, r_s$ are the respective meridians and ramification indices of the components $K_1, \ldots, K_s$ of L, we have

(i) $\tilde{\Gamma} = \pi_1(M - L)/\langle \mu_1^{r_1}, \ldots, \mu_s^{r_s} \rangle$,

(ii) $\tilde{\Gamma}/\pi_1(N) = \Gamma$,

(iii) $$\pi_1(M) = \frac{\tilde{\Gamma}}{\langle \phi(\mu_1), \ldots, \phi(\mu_s) \rangle}.$$

We remark that if M is a rational homology sphere then $\tilde{\Gamma}$ has a canonical quotient isomorphic to $\mathbf{Z}_{r_1} \oplus \mathbf{Z}_{r_2} \oplus \cdots \oplus \mathbf{Z}_{r_s}$.

If L_i is a link in M_i for $i = 1, 2$ then we define L to be a connected sum of (M_1, L_1) and (M_2, L_2) if L is a link in $M = M_1 \# M_2$ constructed as follows. For $i = 1, 2$, let B_i be a 3-ball in M_i such that the pair $(B_i, B_i \cap L_i)$ is diffeomorphic to the standard (B^3, B^1) pair. Construct $M_1 \# M_2$ by identifying the boundaries of the deleted pairs $(M_1 - \operatorname{Int} B_1, L_1 - \operatorname{Int}(B_1 \cap L_1))$ and $(M_2 - \operatorname{Int} B_2, L_2 - \operatorname{Int}(B_2 \cap L_2))$ and let L be the union of $L_1 - \operatorname{Int}(B_1 \cap L_1)$ and $L_2 - \operatorname{Int}(B_2 \cap L_2)$. We will say that a non-trivial link is *prime* if it cannot be decomposed as a connected sum of two non-trivial links; here a *trivial* link is defined to be a link of one component which bounds a disk. Every non-trivial link in a closed 3-manifold is a finite connected sum of prime links (cf. [13, p. 3]). A link is *splittable* if its complement is the connected sum of two non-closed 3-manifolds.

Define L to be a *sufficiently large* link if $M - L$ is irreducible and if there exists a surface S in M such that, for some open tubular neighborhood $\mathcal{N}(L)$ of

L, $S - (S \cap \mathfrak{N}(L))$ is a surface which is incompressible and not boundary-parallel in $M - \mathfrak{N}(L)$.

Throughout this section, N will denote a regular branched cover of a closed orientable 3-manifold M, branched over a link L with components $K_1, \ldots, K_s$; $\tilde{N}$, S, Γ, $\tilde{\Gamma}$, μ_i, and r_i will be defined as above.

We will need the following theorem, which is essentially proved by Gordon and Litherland in [13].

5.1.1. THEOREM. *Suppose that L is prime and sufficiently large. Then either N contains an incompressible surface of positive genus or both M and N contain non-separating 2-spheres.*

(In [13] the definition of a "sufficiently large link" requires that $M - L$ contain a *closed* incompressible surface. However the proof applies verbatim under the weaker hypothesis given above.) □

5.1.2. THEOREM. *Suppose that L is prime and non-splittable. Then one of the following alternatives holds.*

(I) *$M - L$ is the connected sum of a Seifert-fibered space and a closed manifold.*

(II) *N contains a closed incompressible surface of positive genus.*

(III) *Both M and N contain non-separating 2-spheres.*

(IV) *There is a representation of $\tilde{\Gamma}$ in* $PSL_2(\mathbf{C})$, *which is not diagonalizable and whose restriction to each stabilizer $\tilde{\Gamma}_x$, $x \in \tilde{N}$, is faithful.*

Proof. Since L is not splittable, the prime decomposition of $M - L$ contains only one non-closed factor, which may be regarded as the complement of a link L' in a connected summand M' of M. If the prime manifold $M' - L'$ contains a non-separating 2-sphere, then (III) holds. Hence we may assume that $M' - L'$ is irreducible.

Note that N has a connected summand N' which is a regular branched cover of M', branched over L'; and that each component of L' has the same ramification index as the corresponding component of L. If L' is a sufficiently large link in M' then by 5.1.1, either (II) or (III) holds. On the other hand, if $M' - L'$ is Seifert-fibered then (I) holds.

There remains the case in which L' is not sufficiently large and $M' - L'$ is not Seifert-fibered. Then it follows from 4.1.1 that $M' - L'$ has a hyperbolic metric of finite volume. Let $\rho_0 : \pi_1(M' - L') \to SL_2(\mathbf{C})$ be a faithful representation whose image is discrete (see 3.1.1) and let R_0 be an irreducible component of $R(\pi_1(M' - L'))$ containing ρ_0. We shall show that R_0 contains a representation ρ, which is not diagonalizable, such that $\rho(\mu_i')$ has order $2r_i$ for $i = 1, \ldots, s$, where $K'_1,\ldots,K'_s$ are the components of L' and $\mu'_1,\ldots,\mu'_s$ are the corresponding meridians. The representation $\rho' : \pi_1(M-L) \to \mathrm{PSL}_2(\mathbf{C})$, defined by composing ρ on the left with the quotient map $\mathrm{SL}_2(\mathbf{C}) \to \mathrm{PSL}_2(\mathbf{C})$ and on the right with the projection of the free product $\pi_1(M-L)$ to its free factor $\pi_1(M'-L')$, will have the property that $\rho'(\mu_i)$ has order r_i. Hence ρ' will define a representation ρ'' of $\tilde{\Gamma}$ in $\mathrm{PSL}_2(\mathbf{C})$, which is not diagonalizable; and since the stabilizers $\tilde{\Gamma}_x$ of points of $\tilde{N}$ are the images of meridians in $\pi_1(M-L)$, the restriction of ρ'' to each $\tilde{\Gamma}_x$ will be injective. Thus alternative (IV) will hold in this case.

Set $X_0 = t(R_0) \subset X(\pi_1(M'-L'))$; by 1.4.4, X_0 is an affine variety. Consider the regular map $f: X_0 \to \mathbf{C}^s$ defined by

$$f(\chi) = (\chi(\mu_1),\ldots,\chi(\mu_s)).$$

Now, the conjugacy classes of elements of trace $\neq \pm 2$ in $\mathrm{SL}_2(\mathbf{C})$ are determined by their traces. In particular, an element of trace $e^{\pi i/r} + e^{-\pi i/r}$ will have order $2r$ in $\mathrm{SL}_2(\mathbf{C})$, and its image in $\mathrm{PSL}_2(\mathbf{C})$ will have order r. Let us consider the point $u \in \mathbf{C}^s \subset \mathbf{P}^s$ defined by

$$u = \left(e^{\pi i/r_1} + e^{-\pi i/r_1},\ldots,e^{\pi i/r_s} + e^{-\pi i/r_s}\right).$$

Assume for the moment that f is *surjective*, so that there is a character $\chi_1 \in X_0$ such that $f(\chi_1) = u$. Then in order to show that (IV) holds, it is enough to show that there exists $\rho \in R_0$ with $\chi_\rho = \chi_1$, such that the image of ρ in $\mathrm{SL}_2(\mathbf{C})$ is not diagonalizable. We know by 1.5.3 that the inverse image of χ under $t \mid R_0$ has dimension ≥ 3 for each $\chi \in X_0$. Thus there is a 3-dimensional affine subvariety of R_0 consisting of representations with character χ_1. We claim that the set B of diagonalizable representations in $t^{-1}(\chi_1)$ is contained in a closed algebraic set of dimension at most 2. There are only two diagonal elements of $\mathrm{SL}_2(\mathbf{C})$ with a given trace; hence B is a finite union of equivalence classes of representations. Since a non-trivial diagonal subgroup of $\mathrm{SL}_2(\mathbf{C})$ has a one-dimensional centralizer, each equivalence class in B is the image of $\mathrm{SL}_2(\mathbf{C})$ under a map whose fibers are 1-dimensional. This proves the claim and it follows that B is a proper subset of $t^{-1}(X_0)$; this establishes the existence of the desired representation.

To complete the proof of the theorem it is now enough to show that f is surjective. Suppose to the contrary that $w \notin f(X_0)$ for some $w \in \mathbf{C}^n$. We have $\dim X_0 \geq n$ by 3.2.1. Choose a point $x \in X_0$, and let $D \subset \mathbf{C}^n$ be an affine line joining w to $v = f(x)$. Then any irreducible component of the non-empty algebraic set $f^{-1}(D)$ has dimension $\geq (\dim X_0) - (n-1) \geq 1$. (This follows from [23, Corollary 3.14] applied to a set of $n-1$ linear polynomials: the functions $l_i \circ f$, where $l_1,\ldots,l_{n-1}$ are linear polynomials on $\mathbf{C}^n$ whose zero set is D.) Hence $f^{-1}(D)$ contains an affine curve C. Now, using the notation of Section 1, consider the map $\widetilde{f|C}: \tilde{C} \to \tilde{D}$ of projective curves. It is either constant, in which case $\widetilde{f|C}(\tilde{x}) = v$ for every ideal point $\tilde{x}$ of $\tilde{C}$, or surjective, in which case $\widetilde{f|C}(\tilde{x}) = w$ for some ideal point $\tilde{x}$ of $\tilde{C}$. In either case, there is an ideal point $\tilde{x}$ of $\tilde{C}$ for which $\tilde{I}_{\mu_j}(\tilde{x})$ is finite for $j = 1, \ldots, n$. By the Fundamental Theorem 2.2.1, $\tilde{x}$ determines a non-trivial splitting of $\pi_1(M' - L')$ for which the meridian of each component of L' belongs to a vertex group. Now apply 2.3.1, taking $\mathcal{K}$ to be a union of meridian curves, one in each component of $\overline{M' - L'}$. This gives a non-boundary-parallel incompressible surface in $\overline{M' - L'}$ whose boundary curves are all meridians. Thus L' is a sufficiently large link: contradiction. □

5.1.3. Corollary. *If* Γ *is cyclic and* L *is a non-trivial link, then either* $\pi_1(N)$ *has a non-trivial representation in* $\mathrm{PSL}_2(\mathbf{C})$, *or* N *contains an incompressible torus.*

Proof. Since L is a connected sum of prime links, and N is a connected sum of cyclic branched covers of these prime links, we may assume that L is prime. Moreover, we may assume that L is a knot; because otherwise by Smith theory we have $H_1(N; \mathbf{Z}) \neq 0$, and so $\pi_1(N)$ has a representation in $\mathrm{PSL}_2(\mathbf{C})$ with non-trivial cyclic image. Thus one of the alternatives (I)–(IV) of 5.1.2 must hold.

If (I) holds then N has a canonical Seifert-fibered connected summand N'. It is well-known that by computing its Seifert invariants one can show that N' is not a 3-sphere. Thus some non-trivial quotient of $\pi_1(N')$ is a Fuchsian group and hence embeds in $\mathrm{PSL}_2(\mathbf{R}) \subset \mathrm{PSL}_2(\mathbf{C})$.

If (II) holds, then $\pi_1(N) \neq 1$; moreover, by 4.1.1, some connected summand of N is hyperbolic or Seifert-fibered or contains an incompressible torus. In either of the first two cases, $\pi_1(N)$ has a non-trivial representation in $\mathrm{PSL}_2(\mathbf{C})$.

If (III) holds then $\pi_1(N)$ has an infinite cyclic quotient which can be embedded in $\mathrm{PSL}_2(\mathbf{C})$.

If (IV) holds then, since $\tilde{\Gamma}/\pi_1(N)$ is the cyclic group Γ, the representation of $\tilde{\Gamma}$ given by (IV) restricts to a non-trivial representation of $\pi_1(N)$. □

5.1.4. Corollary (Generalized Smith Conjecture). *If* Γ *is cyclic and* L *is a non-trivial knot then* $\pi_1(N) \neq 1$. □

5.2. *Infinite representations and group actions on homotopy 3-spheres.* The following two lemmas, which will be used in the proof of Theorem 5.2.4, are consequences of theorems due to Tollefson.

5.2.1. Lemma. *Let* T *be an involution of the bounded orientable Seifert-fibered manifold* Q_0 *such that* $T \mid \partial Q_0$ *is homotopic to the identity map on* ∂Q_0. *Then the quotient of* Q_0 *by* T *has a Seifert fibration in which the branch set* $(\mathrm{Fix}\, T)/T$ *is a finite union of fibers.*

Proof. The hypothesis that $T|_{\partial Q_0}$ be homotopic to the identity implies that T preserves orientation and hence that each component of the fixed set of T is a 1-manifold. A theorem of Tollefson [34] states that T is fiber-preserving in some Seifert fibration of Q_0. Since $T|_{\partial Q_0}$ is homotopic to the identity, the cyclic subgroup of $\pi_1(Q_0)$ whose generator is represented by a non-singular fiber is fixed by the automorphism induced by T. Thus T cannot reverse the orientation of any invariant fiber. If x is any fixed point of T then T restricts to an orientation-preserving homeomorphism of the fiber through x which has order ≤ 2 and has a fixed point. Thus T is the identity on the fiber. This shows that the fixed point set of T is a finite union of fibers, and the conclusion of the lemma follows. □

5.2.2. LEMMA. *Let Γ_0 be a normal subgroup of Γ with index a power of 2. Let $M_0 = N/\Gamma_0$ and let $L_0 \subset M_0$ be the branch set of the regular branched covering of M_0 by N. If $M_0 - L_0$ is the connected sum of a Seifert-fibered space and a closed manifold, then so is $M - L$.*

Proof. Since 2-groups are nilpotent, we may reduce to the case when Γ_0 has index 2 in Γ. The covering translation for M_0 over M restricts to an involution T of $M_0 - L_0$. A theorem due to Tollefson (Lemmas 1 and 2 of [33]) implies that if $M_0 - L_0$ contains a 2-sphere which does not bound a ball, then it contains a 2-sphere Σ such that either $\Sigma = T(\Sigma)$ or $\Sigma \cap T(\Sigma) = \varnothing$. By combining this with Kneser's Theorem, one can show that $M_0 - L_0$ contains a family of disjoint 2-spheres which is invariant under T, such that the manifold Q_1, obtained by cutting along these spheres and capping off, has only irreducible connected components. Since $M_0 - L_0$ is the connected sum of a closed manifold with a necessarily irreducible Seifert fibered manifold Q_0, it follows from the uniqueness of the prime decomposition that Q_1 has only one non-closed component and that it must be homeomorphic to Q_0.

Thus T induces an involution of Q_0. By Lemma 5.2.1, the quotient of Q_0 by T will have a Seifert fibration in which the branch set is a union of fibers. It follows that $M - L$ is a connected sum of a Seifert fibered space and a closed manifold. □

We need one more lemma for the proof of Theorem 5.2.4.

5.2.3. LEMMA. *If $\pi_2(N) = 0$ then L is a prime link and $\pi_2(M - L) = 0$.*

Proof. Suppose L were not prime. Let Σ be a 2-sphere which meets L in two points, and separates L into two nontrivial links. Let $\tilde{\Sigma}$ be the inverse image of Σ in N. There must be a component N' of $N - \tilde{\Sigma}$ whose closure contains only one component Σ' of $\tilde{\Sigma}$, since otherwise some component of $\tilde{\Sigma}$ would be non-separating. The group of covering translations which map Σ' to itself is a non-trivial cyclic group, and the quotient of N' by this group is a component of $M - \Sigma$. Thus N has a connected summand which is a cyclic branched covering space, branched over a non-trivial link. By Corollary 5.1.3, this summand of N is not simply connected. Thus $N - N'$ must be simply connected. But $N - N'$ will contain a translate of N' unless Σ' is invariant under all covering translations. However this latter statement would imply that the cover was cyclic, and Corollary 5.1.3 would then show $\pi_1(N - N') \neq 1$. Thus the first assertion is established.

Let Σ be a 2-sphere in $M - L$, and let $\tilde{\Sigma}$ be the inverse image of Σ in N. Since $\pi_2(N) = 0$, the closure of some component N' of $N - \tilde{\Sigma}$ must, as before, contain only one component of $\tilde{\Sigma}$. This implies that N' cannot contain any components of the singular set in N. But since $\pi_2(N) = 0$, either N' or $N - N'$ is simply connected; since some covering translation maps N' into $N - N'$, N' must be simply connected. Thus Σ bounds the contractible image of N' in $M - L$. This shows that $\pi_2(M - L) = 0$. □

5.2.4. THEOREM. *Suppose that* $\pi_2(N) = 0$ *and that for some* $i \leq s$ *we have* $r_i > 5$. *Then either* $M - L$ *is the connected sum of a Seifert fibered space with a closed manifold, or* N *contains an incompressible torus, or* $\pi_1(N)$ *has a subgroup of index a power of* 2 *which has a representation in* $\mathrm{PSL}_2(\mathbf{C})$ *with infinite image.*

Proof. We may assume that M is a rational homology 3-sphere, for otherwise $\pi_1(N)$ admits a homomorphism onto $\mathbf{Z}$ and hence has an infinite representation in $\mathrm{PSL}_2(\mathbf{C})$. Thus, according to the discussion in 5.1, $\tilde{\Gamma}$ has a quotient isomorphic to $\mathbf{Z}_{r_1} \oplus \mathbf{Z}_{r_2} \oplus \cdots \oplus \mathbf{Z}_{r_s}$. The cyclic factors in this quotient are generated by the images of the stabilizers in $\tilde{\Gamma}$. We may assume that $r_1 = r_2 = \cdots = r_k = 2$ and that $r_i \neq 2$ for $i = k + 1, \ldots, s$. (By hypothesis, $k < s$.) Let $\tilde{\Gamma}_0$ be the inverse image of $\mathbf{Z}_{r_{k+1}} \oplus \cdots \oplus \mathbf{Z}_{r_s}$ under the quotient map. Then $M_0 = \tilde{N}/\tilde{\Gamma}_0$ has a regular branched cover $N_0 = \tilde{N}/\tilde{\Gamma}_0 \cap \pi_1(N)$ branched over a link L_0 in M_0. The ramification indices of this covering are all greater than 2, and at least one is greater than 5. Note that N_0 is a finite-sheeted covering space of N and that the number of sheets is a power of 2.

By Lemma 5.2.3, applied to the branched covering space N_0 of M_0, the branch set L_0 is a prime link and $\pi_2(M_0 - L_0) = 0$. In particular L_0 is not splittable. Thus one of the four alternatives of Theorem 5.1.2 must hold with L_0 and M_0 in place of L and M.

If $M_0 - L_0$ is the connected sum of a Seifert fibered space and a closed manifold, then by Lemma 5.2.2, so is $M - L$. (One can show that the closed summand of $M - L$ is a homotopy 3-sphere, but we do not need this.)

If N_0 contains an incompressible surface of positive genus then since $\pi_2(N) = 0$, N will be the connected sum of a virtual Haken manifold with a homotopy 3-sphere. By 4.2.2, either N contains an incompressible torus, or N is homotopy equivalent to a Seifert fibered space or a hyperbolic 3-manifold.

If N is homotopy equivalent to a Seifert fibered space, then, since $\pi_1(N)$ is infinite, $\pi_1(N)$ has a quotient which is an infinite Fuchsian group in $PSL_2(\mathbf{R}) \subset PSL_2(\mathbf{C})$. If N is homotopy equivalent to a hyperbolic manifold then $\pi_1(N)$ obviously has an infinite representation in $PSL_2(\mathbf{C})$.

The third alternative of Theorem 5.1.2, that M_0 and N_0 contain non-separating 2-spheres, cannot arise because $\pi_2(N_0) = 0$.

The last alternative is that N_0 contains no closed incompressible surfaces and that there exists a representation σ of $\tilde{\Gamma}_0$ in $PSL_2(\mathbf{C})$ which has non-cyclic image and which maps each (cyclic) stabilizer in $\tilde{\Gamma}_0$ to a subgroup of the same order in $PSL_2(\mathbf{C})$. Hence by the hypothesis of the theorem, some element of $\sigma(\tilde{\Gamma}_0)$ has order > 5. Moreover, by the proof of 5.1.2 and the fact that $\pi_2(M_0 - L_0) = 0$, $M_0 - L_0$ is the connected sum of a hyperbolic manifold of finite volume with a homotopy 3-sphere, and σ is defined by a representation $\rho : \pi_1(M_0 - L_0) \to SL_2(\mathbf{C})$ which lies in an irreducible component R_0 of $R(\pi_1(M_0 - L_0))$ containing a faithful representation ρ_0 with discrete image. If ρ has infinite image then the conclusion of the theorem follows, since σ will then restrict to a representation of $\pi_1(N_0) \subset \tilde{\Gamma}_0$ with infinite image.

Now suppose that σ (and hence ρ) has a finite image. The only non-cyclic finite subgroups of $PSL_2(\mathbf{C})$ are the dihedral groups and the groups of symmetries of the five Platonic solids. However, $\sigma(\tilde{\Gamma}_0)$ cannot be one of the latter groups since it contains an element of order greater than 5. Hence $\sigma(\tilde{\Gamma}_0)$ must be dihedral.

Thus there is a subgroup $\tilde{\Gamma}_1$ of index 2 in $\tilde{\Gamma}_0$ such that $\sigma|_{\tilde{\Gamma}_1}$ has cyclic image. We will show that $\tilde{\Gamma}_1$ has a representation with infinite image which has the same character as $\sigma|_{\tilde{\Gamma}_1}$. This will follow from a variation of an argument used in the proof of Theorem 5.1.2.

Let $M_1 = \tilde{N}/\tilde{\Gamma}_1$. We claim that M_1 is an unbranched two-sheeted covering space of M_0, i.e., that all the stabilizers in $\tilde{\Gamma}_0$ are contained in $\tilde{\Gamma}_1$. Indeed, if a stabilizer in $\tilde{\Gamma}_0$ were not contained in $\tilde{\Gamma}_1$, its image in the dihedral group G would have order 2; since σ preserves order of stabilizers it would follow that $\tilde{\Gamma}_0$ contained a stabilizer of order 2. This contradicts the construction of $\tilde{\Gamma}_0$, and proves that M_1 is an unbranched covering of M_0.

Let L_1 denote the branch set in M_1 for the covering of M_1 by $N_1 = \tilde{N}/\tilde{\Gamma}_1 \cap \pi_1(N_0)$. The manifold $M_1 - L_1$ is the connected sum of a hyperbolic 3-manifold of finite volume with a homotopy 3-sphere; and the restriction ρ_1 of ρ_0 to $\pi_1(M_1 - L_1)$ is a discrete faithful representation of $\pi_1(M_1 - L_1)$, which by the remarks at the end of Section 3 is irreducible. Let R_1 be an irreducible component of $R(\pi_1(M_1 - L_1))$ which contains the image of R_0 under the restriction map: $R(\pi_1(M_0 - L_0)) \to R(\pi_1(M_1 - L_1))$. Set $X_1 = t(R_1) \subset X(\pi_1(M_1 - L_1))$. Of course X_1 contains the character χ_1 of ρ_1, as well as the character χ of $\rho|_{\pi_1(M_1 - L_1)}$. Since ρ_1 is an irreducible representation, it follows, as in the proof of Theorem 5.1.2, that the inverse image of χ under $t|_{R_1}$ is 3-dimensional and hence contains a non-diagonalizable representation $\psi : \pi_1(M_1 - L_1) \to \mathrm{SL}_2(\mathbf{C})$.

We shall show that ψ determines a representation of $\tilde{\Gamma}_1$. This amounts to showing that for each meridian μ of L_1, $\psi(\mu)$ has order equal to $2r$ where r is the ramification index corresponding to μ. Note that since M_1 is an unbranched covering space of M_0, μ is also a meridian in $\pi_1(M_0 - L_0) \supset \pi_1(M_1 - L_1)$ with ramification index r. Since ρ determines the representation σ of $\tilde{\Gamma}_0$ which preserves the orders of stabilizers, we have $\operatorname{tr}\rho(\mu) = \omega + \bar{\omega}$, where ω is a primitive $2r$-th root of unity. But ψ and $\rho \mid \pi_1(M_1 - L_1)$ have the same character, and so $\operatorname{tr}\psi(\mu) = \omega + \bar{\omega}$; since $r > 1$, it follows that $\psi(\mu)$ has order $2r$. Thus we may regard ψ as a representation of $\tilde{\Gamma}_1$.

Now $\rho \mid \pi_1(M_1 - L_1)$ is a reducible representation since its image is abelian; hence ψ is reducible by 1.2.1. We may therefore assume that $\psi(\tilde{\Gamma}_1)$ consists of upper triangular matrices. Set $K = \ker(\sigma \mid \tilde{\Gamma}_1)$. Then $\psi(K)$ has finite index in $\psi(\tilde{\Gamma}_1)$; moreover, $\psi(K)$ consists of upper triangular matrices with 1's on the diagonal, and is therefore free abelian. Thus K has infinite abelianization. Since $\pi_1(N_1)$ is a subgroup of finite index in $\tilde{\Gamma}_1$, it too has a subgroup of finite index with infinite abelianization. Hence N has a finite-sheeted cover with positive first Betti number. By assumption $\pi_2(N) = 0$, so N is the connected sum of a virtual Haken manifold with a homotopy 3-sphere. Therefore either N contains an incompressible torus or $\pi_1(N)$ has a representation in $\mathrm{PSL}_2(\mathbf{C})$ with infinite image. □

5.2.5. COROLLARY. *If, for some $i \leq s$, $r_i > 5$, then either N has infinite fundamental group, or $M - L$ is the connected sum of a Seifert fibered space and a closed manifold.* □

This corollary is essentially equivalent to the following theorem of Davis and Morgan [9].

5.2.6. COROLLARY. *Let Γ be a finite group of diffeomorphisms of a homotopy 3-sphere. If all of the stabilizers in Γ are cyclic, and one has order greater than 5, then Γ acts essentially linearly.*

A group of diffeomorphisms is said to act *essentially linearly* on a homotopy 3-sphere Σ if there is an invariant family of disjoint homotopy 3-cells in Σ such that S^3 is obtained by collapsing each one to a point, and such that the induced group of homeomorphisms of S^3 is conjugate to a group of linear diffeomorphisms.

Corollary 5.2.5 implies that the quotient of Σ by Γ is the connected sum of a closed manifold with a Seifert fibered space which contains the branch set as a union of fibers. The closed manifold must be a homotopy 3-sphere, which gives rise to the invariant family of homotopy 3-cells. The induced action on S^3 preserves the fibers in a Seifert fibration of the sphere; it follows that it is conjugate to a linear action. The reader is referred to [9] for the details of this argument. □

As a final remark on Theorem 5.2.4 we note that the conclusion can be strengthened if Γ is assumed to have a normal 2-complement (i.e., a normal subgroup of index a power of 2 whose order is prime to 2).

5.2.7. THEOREM. *Suppose that $\pi_2(N) = 0$, that for some $i \leq s$ we have $r_i > 5$, and that Γ has a normal 2-complement. Then either $M - L$ is the connected sum of a Seifert fibered space and a closed manifold, or N contains an incompressible torus, or $\pi_1(N)$ has a representation in* $\mathrm{PSL}_2(\mathbf{C})$ *with infinite image.*

Proof. We modify the proof of 5.2.4 by defining $\tilde{\Gamma}_0$ to be the inverse image of the normal 2-complement in Γ under the projection: $\tilde{\Gamma} \to \Gamma$. Then $\pi_1(N) \subset \tilde{\Gamma}_0$, and hence $\pi_1(N)$ has an infinite representation if $\tilde{\Gamma}_0$ does. The only properties of $\tilde{\Gamma}_0$ needed in the proof of 5.2.4 are that it be a normal subgroup of $\tilde{\Gamma}$ whose index is a power of 2 and that no stabilizer in $\tilde{\Gamma}_0$ have order 2. □

5.3. *Applications.*

5.3.1. PROPOSITION. *Suppose that L is non-trivial and that $r_i > 5$ for some i. Then if $\pi_1(N)$ has a non-trivial center, N is the connected sum of a Seifert fibered space (with orientable decomposition surface) and a homotopy 3-sphere.*

Proof. Since $\pi_1(N)$ has a non-trivial center, it cannot be a non-trivial free product. Hence by Kneser's theorem, N is a connected sum $N_0 \# \Sigma$, where Σ is a homotopy 3-sphere and N_0 is either an irreducible manifold or $S^2 \times S^1$. Since $S^2 \times S^1$ is Seifert-fibered we may assume that N_0 is irreducible. By 5.2.4, either

(a) $M - L$ is the connected sum of a Seifert fibered space with a closed manifold, or (b) N_0 contains an incompressible torus, or (c) a subgroup H of finite index in $\pi_1(N_0)$ has a representation ρ in $PSL_2(\mathbf{C})$ with infinite image. If (a) holds then N_0 is itself Seifert-fibered. If (b) holds then in particular N_0 is a Haken manifold, and Waldhausen's theorem in [36] guarantees that N_0 is Seifert-fibered. Now suppose that (c) holds. Since in particular $\pi_1(N_0)$ is infinite, and N_0 is irreducible, $\pi_1(N_0)$ must be torsion-free; hence the center Z of H is itself non-trivial. If $\rho(Z) \neq \{1\}$, then the infinite group $\rho(H)$ is abelian, because the centralizer of every non-trivial element of $PSL_2(\mathbf{C})$ is abelian. Hence in this case H admits a homomorphism onto $\mathbf{Z}$, and so the covering space $\tilde{N}_0$ of N_0 corresponding to H contains an incompressible surface. But $\tilde{N}_0$ is irreducible by [19], and hence by Waldhausen's theorem $\tilde{N}_0$ is Seifert-fibered. It now follows from Scott's theorem in [24] that N_0 is itself Seifert-fibered.

Now suppose that $\rho(Z) = \{1\}$. Note that $\rho(H)$ has an element of infinite order; in fact, by a theorem due to Schur [8, Theorem 36.2], any finitely generated, infinite subgroup of $GL_n(\mathbf{C})$ has an element of infinite order, and it follows easily that the same is true for subgroups of $PSL_2(\mathbf{C})$. If y is an element of H such that $\rho(y)$ has infinite order, and z is a non-trivial element of Z, then y and z generate a free abelian subgroup of rank two in $H \subset \pi_1(N_0)$. Hence in N_0 there is an essential singular torus, i.e. a map $f\colon T^2 \to N_0$ inducing a monomorphism of fundamental groups.

We now apply Theorem 3.1 of [27]. Recall that a group is *residually finite* if the intersection of all its finite-index subgroups is the trivial subgroup. In [27], an infinite group is said to be "half-way residually finite" if it contains subgroups of arbitrarily large finite index. By a theorem due to Mal'cev [18], every finitely generated matrix group over a field is residually finite; thus the infinite group $\rho(H)$ is residually finite, and it follows at once that $\pi_1(N_0)$ is half-way residually finite. According to [27, Theorem 3.1], in any closed orientable 3-manifold N_0 with half-way residually finite fundamental group, we have the following "torus theorem": every essential singular torus f in N_0 is homotopic to a map g such that $g(T^2)$ is contained in a Seifert-fibered submanifold Σ of N_0 with incompressible boundary. If $\partial\Sigma = \varnothing$, then $\Sigma = N_0$, and so N_0 is Seifert-fibered. If $\partial\Sigma \neq \varnothing$, then N_0 is sufficiently large, and so N_0 is Seifert-fibered by Waldhausen's theorem. □

5.3.2. PROPOSITION. *Suppose that* $r_i > 5$ *for some* i. *Then* $\pi_1(N)$ *either is polycyclic or contains a free subgroup of rank* 2.

Proof. If $\pi_2(N) \neq 0$, then $\pi_1(N)$ is isomorphic to $\mathbf{Z}$ or to a non-trivial free product. But $\mathbf{Z}$ and $\mathbf{Z}_2 * \mathbf{Z}_2$ are polycyclic, and any other non-trivial free product contains a rank-two free subgroup. Thus we may assume that $\pi_2(N) = 0$.

One of the alternatives of 5.2.4 must hold. First consider the case that N contains an incompressible torus. Then since $\pi_2(N) = 0$, N is homotopy-equivalent to a Haken manifold. It is shown in [11] that the fundamental group of a Haken manifold either is polycyclic or contains a free subgroup of rank 2.

Next consider the case that $\pi_1(N)$ has a subgroup H of finite index which admits a representation in $\mathrm{PSL}_2(\mathbf{C})$ with infinite image.

It is a theorem due to Tits [32] that every subgroup of $\mathrm{GL}_n(\mathbf{C})$ has either a solvable subgroup of finite index or a free subgroup of rank 2. It follows easily that the same is true for subgroups of $\mathrm{PSL}_2(\mathbf{C})$. If $\rho(H)$ has a free subgroup of rank 2, so does H. Hence we may assume that $\rho(H)$ has a solvable subgroup of finite index. It must then contain a subgroup of finite index whose abelianization is infinite. Hence $\pi_1(N)$ has a subgroup of finite index which admits a homomorphism onto $\mathbf{Z}$. Since $\pi_2(N) = 0$, it follows that N is the connected sum of a virtual Haken manifold N^* with a homotopy 3-sphere.

By 4.2.2, either N^* is a Haken manifold, or is homotopy-equivalent to a hyperbolic manifold, or is Seifert-fibered (and has infinite fundamental group). We have seen that in the first case, $\pi_1(N) = \pi_1(N^*)$ is polycyclic or contains a free subgroup of rank 2. In the second case, $\pi_1(N^*)$ has a natural representation in $\mathrm{PSL}_2(\mathbf{C})$. The image of this representation cannot have a solvable subgroup of finite index, because every solvable subgroup of $\mathrm{PSL}_2(\mathbf{C})$ has a subgroup of finite index that acts reducibly on $\mathbf{C}^2$, and we saw in Section 3 that the natural representation in $\mathrm{PSL}_2(\mathbf{C})$ of the fundamental group of a hyperbolic manifold of finite volume is irreducible. Hence by Tits's theorem, $\pi_1(N)$ contains a free subgroup of rank two. If N^* is Seifert-fibered and has infinite fundamental group then $\pi_1(N^*)$ has a normal subgroup K such that $\pi_1(N^*)/K$ is an infinite Fuchsian group. It is not hard to show (by either an elementary argument or one based on Tits's theorem) that an infinite Fuchsian group either is polycyclic or contains a free subgroup of rank 2. Hence the same is true of $\pi_1(N^*)$.

There remains the case that $M - L$ is the connected sum of a Seifert fibered space Q and a closed 3-manifold Σ. Then by Lemma 5.2.3, Σ is a homotopy 3-sphere. It follows that N is the connected sum of a Seifert-fibered space N^* and a homotopy 3-sphere. If the Seifert-fibered space N^* has an infinite fundamental group, we have already seen that the conclusion of the proposition is true.

Now suppose that $\pi_1(N^*)$ is finite. Then the universal covering of N^* may be identified with S^3; and since Q is Seifert-fibered, the action of $\tilde{\Gamma}$ on S^3 induced by its action on $\tilde{N}$ is linear. Thus $\tilde{\Gamma}$ is isomorphic to a subgroup $\tilde{\Gamma}^*$ of $\mathrm{SO}(4)$ which acts on S^3 in such a way that some stabilizer has order > 5.

There is a two-sheeted covering map $p : S^3 \times S^3 \to \mathrm{SO}(4)$ defined by $p(\alpha, \beta)(x) = \alpha \times \beta^{-1}$, where S^3 is identified with the group of unit quaternions. For $i = 1, 2$, define $q_i : S^3 \times S^3 \to S^3/\{\pm 1\}$ to be the projection to the i-th factor composed with the quotient map. The condition that $\tilde{\Gamma}^*$ contain a stabilizer of order > 5 is easily seen to imply that $q_i(p^{-1}(\tilde{\Gamma}^*))$ contains an element of order > 5 for $i = 1, 2$. The only finite, nonsolvable subgroup of $S^3/\{\pm 1\} \approx \mathrm{SO}(3)$ is the group $\mathcal{Q}_5$ of symmetries of the regular icosohedron, which has no elements of order > 5. Hence the groups $q_i(p^{-1}(\tilde{\Gamma}^*))$ are solvable; therefore $\tilde{\Gamma}$ is polycyclic, and in particular so is $\pi_1(N)$. □

RICE UNIVERSITY, HOUSTON, TEXAS

REFERENCES

[1] H. BASS, Groups of integral representation type, Pacific J. Math. **86** (1980), 15–52.
[2] ______, Finitely generated subgroups of GL_2, to appear in *Proc. Symp. on the Smith Conjecture*, Academic Press.
[3] H. BASS, J. BIRMAN, AND J. MORGAN, eds., *Proc. Symp. on the Smith Conjecture*, Academic Press, to appear.
[4] H. BASS AND A. LUBOTZKY, Automorphisms of groups and of schemes of finite type, preprint, Columbia University, to appear in Israel Journal of Math.
[5] G. BAUMSLAG AND P. B. SHALEN, Group deficiencies and generalized free product decomposition, to appear.
[6] M. CULLER AND P. B. SHALEN, Separating surfaces in knot manifolds, to appear.
[7] ______, Property P and incompressible surfaces, to appear.
[8] C. W. CURTIS AND I. REINER, *Representation Theory of Finite Groups and Associative Algebras*, *Pure and Appl. Math.* XI, Interscience, 1982.
[9] M. DAVIS AND J. MORGAN, Finite group actions on homotopy 3-spheres, to appear in *Proc. Symp. on the Smith Conjecture*, Academic Press.
[10] D. B. A. EPSTEIN, Free products with amalgamation and 3-manifolds, Proc. A. M. S. **12** (1961), 669–670.
[11] B. EVANS AND W. JACO, Varieties of groups and 3-manifolds, *Topology* **12** (1973), 83–97.
[12] P. FATOU, Fonctions Automorphes, Vol. 2 of *Théorie des Fonctions Algébriques*, by P. E. Appell and E. Goursat, Gauthiers-Villars, Paris, 1930, pp. 158–160.
[13] C. MCA. GORDON AND R. A. LITHERLAND, Incompressible surfaces in branched coverings, to appear in *Proc. Symp. on the Smith Conjecture*, Academic Press, preprint, University of Texas at Austin.
[14] J. HEMPEL, *3-manifolds*, Ann. of Math. Studies **86**, Princeton Univ. Press, 1976.
[15] W. H. JACO AND P. B. SHALEN, *Seifert Fibered Spaces in 3-Manifolds*, Mem. A. M. S. **21** (1979), no. 220.
[16] K. JOHANNSON, *Homotopy Equivalences of 3-Manifolds with Boundaries*, Lecture Notes in Mathematics **761**, Springer-Verlag, 1979.
[17] W. MAGNUS, Rings of Fricke characters and automorphism groups of free groups, *Math. Zeit.* **170** (1980), 91–103.
[18] A. I. MAL'CEV, On isomorphic matrix representations of infinite groups, Mat. Sb. (N.S.), **8** (1940), 405–421.
[19] W. MEEKS, L. SIMON, AND S. T. YAU, Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature, Ann. of Math. **116** (1982), 621–659.
[20] W. MEEKS AND S. T. YAU, Topology of three-dimensional manifolds and the embedding problems in minimal surface theory, to appear in *Proc. Symp. on the Smith Conjecture*, Academic Press.

[21] J. Morgan and P. B. Shalen, to appear.
[22] G. D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, Publ. Math. I.H.E.S. **34** (1968), 53–104.
[23] D. Mumford, *Algebraic Geometry* I: *Complex Projective Varieties*, Grundlehren der Math. Wiss. **221**, Springer-Verlag, 1976.
[24] G. P. Scott, There are no fake Seifert fibre spaces with infinite π_1, Ann. of Math. **117** (1983), 35–70.
[25] J-P. Serre, *Arbres. Amalgames*, SL_2, Astérisque **46** (1977).
[26] P. B. Shalen, Separating, incompressible surfaces in 3-manifolds, Invent. Math. **52** (1979), 105–126.
[27] ______, A torus theorem for regular branched covers of S^3, Michigan Math. J. **28** (1981), 247–258.
[28] J. R. Stallings, Some topological proofs and extensions of Grushko's theorem, Thesis, Princeton Univ. (1959).
[29] ______, *Group Theory and Three-Dimensional Manifolds*, Yale Math. Monographs, **4**, Yale Univ. Press, 1971.
[30] W. Thurston, *Three Dimensional Geometry and Topology*, Ann. of Math. Series, Princeton Univ. Press, 1983.
[31] ______, Hyperbolic structures on 3-manifolds, to appear in *Proc. Symp. on the Smith Conjecture*, Academic Press.
[32] J. Tits, Free subgroups in linear groups, J. Alg. **20** (1972), 250–270.
[33] J. L. Tollefson, Involutions on $S^1 \times S^2$ and other 3-manifolds, Trans. A. M. S. **183** (1973), 139–152.
[34] ______, Involutions of Seifert fiber spaces, Pacific J. Math. **74** (1978), 519–529.
[35] M. Tretkoff, A topological approach to the theory of groups acting on trees, J. Pure Appl. Algebra **16** (1980), 323–333.
[36] F. Waldhausen, Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten, Topology **6** (1967), 505–517.

(Received June 1, 1981)