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Bounded, separating, incompressible surfaces in knot manifolds

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We shall prove the following result. (See [3] or [4] for the standard 3-manifold terminology that we use. A *surface* is a connected 2-manifold.)

Theorem 1. *Let M be a compact, connected, orientable 3-manifold having a non-simply-connected boundary component. Suppose that $H_1(\partial M;\mathbf{Q})\to H_1(M;\mathbf{Q})$ is surjective and that M is not homeomorphic to $D^2\times S^1$ or $S^1\times S^1\times I$. Then M contains a separating, (properly embedded), incompressible surface which has non-empty boundary and is not boundary-parallel.*

This generalizes and strengthens the main theorem of [13]. Note that the hypothesis of Theorem 1 is satisfied whenever M is a knot manifold, i.e. the complement of an open tubular neighborhood of a nontrivial knot in S^3. (The theorem of [13] gives no information for a knot manifold.) In this case various versions of Theorem 1 have been conjectured by L.P. Neuwirth. For example, Conjecture A of [11], that every knot group is a free product of two proper subgroups amalgamated along a free group, is an immediate corollary to Theorem 1.

The following result can be derived from Theorem 1 above in the same way that [13, Theorem 2] is derived from [13, Theorem 1]. Because the proof parallels so precisely that of [13, Theorem 2], we shall omit it.

Corollary. *Let M be an irreducible 3-manifold satisfying the hypotheses of Theorem 1. There exists a finite sequence of (possibly disconnected) 3-manifolds $M=M_0,\dots,M_n$ such that* (i) *$M_j(0<j\leqq n)$ is obtained by splitting M_{j-1} along a bounded incompressible surface which separates the component of M_{j-1} containing it, and* (ii) *each component of M_n homeomorphic to $D^2\times S^1$ or $S^1\times S^1\times I$.*

In the most interesting case, when the components of ∂M are all tori, Theorem 1 can be sharpened as follows.

Theorem 2. *Let M be a compact, connected, orientable 3-manifold whose boundary is a non-empty union of tori. Suppose that $H_1(\partial M;\mathbf{Q})\to H_1(M;\mathbf{Q})$ is surjective and that M is not homeomorphic to $D^2\times S^1$ or $S^1\times S^1\times I$. Then for each* *component B of ∂M, there is a separating, non-boundary-parallel, incompressible surface in M whose boundary is non-empty and is contained in B.*

Theorem 2 is based on the following result, which should be of independent interest. Recall that a compact, orientable 3-manifold M is said to be *simple* if it is connected and irreducible and is not a Seifert fibered space, and if every incompressible torus in M is boundary-parallel.

Theorem 3. *Let M be a simple, compact, orientable 3-manifold whose boundary consists of tori $B_1, \ldots, B_n$ where $n>0$. Let k be an integer with $0<k\leqq n$, and let γ_i be a simple, non-contractible closed curve in B_i for $1\leqq i\leqq k$. Then M contains a system of disjoint, non-boundary-parallel, incompressible surfaces $\Sigma_1, \ldots, \Sigma_r$ such that* (i) $(\partial\Sigma_1 \cup \ldots \cup \partial\Sigma_r)\cap B_i=\emptyset$ *for* $k<i\leqq n$, *and* (ii) *for* $1\leqq i\leqq k$, $(\partial\Sigma_1 \cup \ldots \cup \partial\Sigma_r)\cap B_i$ *is a non-empty union of non-contractible simple closed curves in B_i whose common homotopy class in B_i is distinct from that of γ_i.*

The proof of Theorem 3, which occupies most §1 below, is based on the techniques of [1]. The starting point is Thurston's theorem that $\mathring{M}$ has a hyperbolic metric of finite volume. This is used to construct a certain complex affine curve C in the algebraic set of characters of representations of $\pi_1(M)$ in $SL_2(\mathbf{C})$. An "ideal point" of C gives rise to a "splitting" of $\pi_1(M)$, i.e. an isomorphism of $\pi_1(M)$ with the fundamental group of a graph of groups, and with a splitting one can associate a system of incompressible surfaces in M.

Each element g of $\pi_1(M)$ determines a function I_g on C defined by $I_g(\chi) = \chi(g)$, and the information provided by the theorem on the boundaries of the Σ_j is related to the behavior of the functions I_g at the given ideal point $\tilde{x}$. For example, if $k=n=1$, the Σ_j will have boundary curves non-homotopic to γ_1 provided that $I_{[\gamma_1]}$ has a pole at $\tilde{x}$. An $\tilde{x}$ with this property will exist provided that $I_{[\gamma_1]}$ is non-constant on C. In order to guarantee this we need a refinement for the case of $SL_2(\mathbf{C})$ of the theorem proved by Weil in [17]; this refinement, due to Marden, Thurston, and Lok, is discussed in §1. It is essential for Theorem 3 that M be assumed simple; we show at the end of §1 that the theorem in general becomes false if M is, for example, a Seifert fibered space.

Theorems 1 and 2 are proved in §2. The proof of Theorem 1 supplants the proof of [13, Theorem 1] in the sense that it uses neither the statement of the latter result nor any of the difficult preliminary results from [13]; however, we have found it convenient to quote some very elementary facts from §2 of [13].

We are very grateful to Allen Hatcher for pointing out a proof of the present version of Theorem 1, which is considerably stronger than our original version, and thus leading us to the discovery of Theorem 2. We would also like to thank Hatcher for pointing out that a result like Theorem 3 is of independent interest and should be formulated as a separate theorem.

§1. Boundaries of incompressible surfaces

We shall begin the proof of Theorem 3 by discussing a refinement of a theorem proved by Weil in [17]. Let N be an orientable 3-manifold of finite volume. Thus $N = \Gamma \backslash H^3$, where H^3 is hyperbolic 3-space and Γ is a discrete, torsion-free subgroup of the group of orientation-preserving isometries of H^3. Since the latter group can be identified with $PSL_2(\mathbf{C}) = SL_2(\mathbf{C})/\{\pm\mathbf{1}\}$, $\Pi = \pi_1(N)$ has, up to equivalence (i.e. conjugacy in $PSL_2(\mathbf{C})$), a canonical representation $\bar{\rho}_0$ in $PSL_2(\mathbf{C})$; it is a faithful representation whose image is a discrete group.

The set $PR(\Pi)$ of all representations of Π in $PSL_2(\mathbf{C})$ has a natural topology. (For example, in terms of a set of m generators for Π, $PR(\Pi)$ becomes a subset of $PSL_2(\mathbf{C})^m$, from which it inherits a topology which is easily seen to be independent of the choice of a generating set.) The theorem of [17] implies that if N is closed then $\bar{\rho}_0$ has a neighborhood in $PR(\Pi)$ consisting of faithful representations with discrete images. By the Mostow rigidity theorem [9], these are all equivalent to $\bar{\rho}_0$.

Now suppose that N is open. It is well known [2, pp. 158–160] that since N has finite volume, it may be identified homeomorphically with the interior of a compact 3-manifold $\bar{N}$ whose boundary components are tori; and that the peripheral elements of $\pi_1(\bar{N}) = \Pi$, i.e. those carried up to conjugacy by $\partial\bar{N}$, are represented under $\bar{\rho}_0$ by parabolic elements of $PSL_2(\mathbf{C})$, i.e. images of elements of $SL_2(\mathbf{C})$ having trace ± 2. Let $PR_{\text{par}}(\Pi)$ denote the subset of $PR(\Pi)$ consisting of all representations which represent the peripheral elements of Π by parabolic elements of $PSL_2(\mathbf{C})$. Then we have the following refinement of Weil's theorem.

Proposition 1. *There is a neighborhood of $\bar{\rho}_0$ in $PR_{\text{par}}(\Pi)$ consisting of faithful representations with discrete images. These are all equivalent to $\bar{\rho}_0$ in $PSL_2(\mathbf{C})$.*

This is similar to a result stated by Thurston as [14, Prop. 5.1], with an indication of a proof. A proof of Proposition 1 along the lines suggested by Thurston has been written up by W.L. Lok and will appear in his thesis [7]. It seems likely that the argument given by Marden in [8, proof of Theorem 9.1] would also provide a proof of Proposition 1.

We shall show how to combine this result with the techniques of [1]. It is shown in [1, § 1] how to identify the representations (resp. characters of representations) of any finitely generated group Π with the points of a closed algebraic subset $R(\Pi)$ (resp. $X(\Pi)$) of a complex affine space. (See [10] for basic definitions in algebraic geometry.) It is also shown that the functions I_g: $X(\Pi) \to \mathbf{C}$ defined for $g \in \Pi$ by $I_g(\chi) = \chi(g)$, and the map t: $R(\Pi) \to X(\Pi)$ that assigns to each representation ρ its character χ_ρ, are given by (complex) polynomials in the ambient coordinates. If we take $\Pi = \pi_1(N)$ as above, then according to [1, Prop. 3.1.1], a result due to Thurston, the representation $\bar{\rho}_0$: $\Pi \to PSL_2(\mathbf{C})$ admits a lifting ρ_0: $\Pi \to SL_2(\mathbf{C})$. It is pointed out in § 3 of [1] that ρ_0 is irreducible (i.e. has no 1-dimensional invariant subspace). If R_0 denotes an irreducible component of $R(\Pi)$ containing ρ_0, it follows from [1, Prop. 1.4.4] that $X_0 = t(R_0) \subset X(\Pi)$ is a (closed) affine algebraic set.

Proposition 2. *Let $B_1, \ldots, B_n$ denote the components of $\partial\bar{N}$, and let γ_i be a non-contractible simple closed curve in B_i for $1 \leqq i \leqq n$. Let $g_i \in \Pi$ be an element whose conjugacy class corresponds to the free homotopy class of γ_i. Let k be an integer* *with* $0 \leqq k \leqq n$, *and let* V *be the algebraic subset of* X_0 *defined by the equations* $I_{g_i}(\chi)^2 = 4$, $k < i \leqq n$. *Let* V_0 *denote an irreducible component of* V *containing* χ_{ρ_0}. *Then if* χ *is a point of* V_0, i *is an integer with* $k < i \leqq n$, *and* g *is an element of the subgroup (defined up to conjugacy)* $\operatorname{im}(\pi_1(B_i) \to \pi_1(\bar{N}))$ *of* Π, *we have* $I_g(\chi) = \chi(g) = \pm 2$. *Furthermore, if* $r = 0$ *then* $V_0 = \{\chi_{\rho_0}\}$.

Proof. First note that $\pi_1(B_i) \to \pi_1(\bar{N})$ is injective for each i. For if not, then by the loop theorem [3, Theorem 4.2] and the irreducibility of the hyperbolic 3-manifold N, $\bar{N}$ would be a solid torus; and then N would have infinite volume. In particular, each γ_i is non-contractible in N, so that $g_i \neq 1$. Since ρ_0 is a lifting of the faithful representation $\bar{\rho}_0$, it follows that $\rho_0(g_i) \neq \pm 1$. Now by [1, Prop. 1.5.4] there is a Zariski neighborhood U of χ_{ρ_0} in X_0 such that $\rho(g_i) \neq \pm 1$ $(1 \leqq i \leqq n)$ for any $\rho \in t^{-1}(U)$. But if in addition $\rho \in t^{-1}(V)$ and $i > k$, then $\rho(g_i)$ has trace ± 2. It follows that for $i > k$ and $\rho \in t^{-1}(U \cap V_0)$, $\rho(g_i)$ is conjugate in $SL_2(\mathbf{C})$ to $\pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Hence any element of $SL_2(\mathbf{C})$ that commutes with $\rho(g_i)$ has trace ± 2. In particular, for $g \in \operatorname{im}(\pi_1(B_i) \to \pi_1(\bar{N}))$, we have $I_g(\chi) = \pm 2$ for $\chi \in U \cap V_0$ - and hence for all $\chi \in V_0$, since the Zariski-open subset $U \cap V_0$ of V_0 is automatically Zariski-dense. This proves the first assertion.

Now suppose that $k = 0$. Then every character in V_0 takes the value ± 2 on every peripheral element of Π. By Proposition 1 and the (obvious) continuity of the natural map $R(\Pi) \to PR(\Pi)$, there is a (classical) neighborhood W of ρ_0 in V_0 such that for every $\rho \in W$, the image $\bar{\rho}$ of ρ in $PSL_2(\mathbf{C})$ is equivalent to $\bar{\rho}_0$. Now a given representation of Π in $PSL_2(\mathbf{C})$ can admit only finitely many liftings to $SL_2(\mathbf{C})$ (in fact at most 2^m of them, if Π has m generators). Thus W is contained in the union of finitely many equivalence classes of representations. But the equivalence class of any representation $\rho \in R(\Pi)$ is contained in a subvariety of $R(\Pi)$ having (complex) dimension $\leqq 3$; this is because it is the image of the 3-dimensional variety $SL_2(\mathbf{C})$ under the (polynomial) map $A \to i_A \circ \rho$, where i_A denotes conjugation by the matrix A. Hence $\dim V_0 \leqq 3$.

On the other hand, it follows from [1, Prop. 1.2.1] that the points of $X(\Pi)$ which are characters of reducible representations form a Zariski-closed subset of $X(\Pi)$. Hence there is a Zariski-neighborhood U' of χ_{ρ_0} in V_0 consisting of characters of irreducible representations. The proof of [1, Corollary 1.5.3] shows that for each $\chi \in U'$, $t^{-1}(\chi)$ has dimension 3; hence $\dim t^{-1}(V_0) = 3 + \dim V_0$. This shows that $\dim V_0 = 0$, so that $V_0 = \{\chi_{\rho_0}\}$. The proof of Proposition 2 is now complete. □

The theory of graphs of groups is presented in [12] and [16], and reviewed in [1, §2]. Recall that a *graph of groups* is a pair $(G, \mathcal{G})$. Here G is a connected 1-dimensional CW complex $\mathcal{G}$ is a functor from the category whose objects are the edges and vertices of G, with a morphism from each edge to each of its endpoints, to the category of groups and monomorphisms. It is explained in each of the above references how to associate with $(G, \mathcal{G})$ a group $\pi_1(G, \mathcal{G})$, called the *fundamental group* of the graph of groups. For each vertex (resp. edge) c of G, $\pi_1(G, \mathcal{G})$ contains an isomorphic copy of $\mathcal{G}(c)$ which is canonical up to conjugacy; it is called a *vertex group* (resp. *edge group*). As in [1], we define a *splitting* of an abstract group Π to be an isomorphism of Π with the fundamental group of a graph of groups. A splitting of Π determines a set of vertex groups and edge groups in Π; by definition a conjugate of a vertex (resp. edge) group is a vertex (resp. edge) group. A splitting of Π is *non-trivial* if the vertex groups are all proper sungroups of Π.

Proof of Theorem 3. According to Thurston's uniformization theorem [15], if M is a simple, compact, orientable 3-manifold whose boundary is a non-empty union of tori, then with the notation used earlier in this section, M is homeomorphic to $\bar{N}$ for some hyperbolic 3-manifold N of finite volume. Set $\Pi = \pi_1(M) = \pi_1(N)$, and define ρ_0 and X_0 as above. Since $M = N$ has n boundary components, by [1, Prop. 3.2.1] (a result due to Thurston), we have $\dim X_0 \geqq n$.

The plan of the proof is to use [1, Theorem 2.2.1], which permits one to associate splittings of Π with affine curves in $X(\Pi)$, and [1, Prop. 2.3.1], which in turn permits one to associate systems of incompressible surfaces in M with splittings of $\Pi = \pi_1(M)$. The first step is therefore to define an appropriate curve $C \subset X_0 \subset X(\Pi)$.

For $k < i \leqq n$, choose an arbitrary non-contractible simple closed curve γ_i in B_i. For $1 \leqq i \leqq n$, let $g_i \in \pi_1(M)$ represent the conjugacy class defined by γ_i. Define a map $f\colon X_0 \to \mathbf{C}^{n-1}$ by $f(\chi) = (Z_i)_{1 \leqq i \leqq n-1}$, where $Z_i = I_{g_{i+1}}(\chi) - I_{g_1}(\chi)$ for $1 \leqq i < k$, and $Z_i = I_{g_{i+1}}(\chi)$ for $k \leqq i < n$. Since f is defined by polynomials in the ambient coordinates, it follows from [10, Theorem 3.13] that for each $p \in \mathbf{C}^{n-1}$, $f^{-1}(p)$ is a closed algebraic set whose irreducible components are all of dimension at least $(\dim X_0) - (\dim \mathbf{C}^{n-1}) > 0$. Thus through each point of X_0 there is an affine curve on which f is constant. Let C be such a curve through the point χ_{ρ_0}.

The function I_{g_1} is non-constant on C. Indeed, assume the contrary; since f is constant on C, it follows that I_{g_i} is constant on C for $1 \leqq i \leqq n$. This contradicts the last assertion of Proposition 2.

Similarly, since f is constant on C, I_{g_i} is indeed constant on C for $k < i \leqq n$. Hence, by the first assertion of Proposition 2, I_g is constant on C for every $g \in \operatorname{im}(\pi_1(B_i) \to \pi_1(M))$, $k < i \leqq n$.

We are now ready to apply [1, Theorem 2.2.1]. Let $\tilde{C}$ denote the smooth projective model of C. (It is obtained from the closure $\hat{C}$ of C in a projective space by resolving singularities; there is a natural map from $\tilde{C}$ to $\hat{C}$ which induces an isomorphism from the function field of $\tilde{C}$ to that of C.) As in [1], we let $\tilde{I}_g (g \in \Pi)$ denote the meromorphic function on $\tilde{C}$ corresponding to the function I_g on C. If $\tilde{x}$ is an ideal point of $\tilde{C}$, i.e. one that is mapped to a point of $\hat{C} - C$ under the natural map $\tilde{C} \to \hat{C}$, then [1, Theorem 2.2.1] asserts that there is a non-trivial splitting of Π such that for each element g of Π, $\tilde{I}_g$ is finite-valued at $\tilde{x}$ if and only if g belongs to some vertex group of the splitting.

Since I_{g_1} is non-constant on C, $\tilde{I}_{g_1}$ must have a pole at some ideal point $\tilde{x}$ of $\tilde{C}$. Since f is constant on $\tilde{C}$, it follows that $\tilde{I}_{g_i}$ has a pole at $\tilde{x}$ for $1 \leqq i \leqq k$. On the other hand, for any $g \in \operatorname{im}(\pi_1(B_i) \to \pi_1(M))$, $k < i \leqq n$, $\tilde{I}_g$ is finite-valued at $\tilde{x}$. Thus $\tilde{x}$ defines a splitting for which g_i does not lie in any vertex group when $1 \leqq i \leqq k$, but each element of $\operatorname{im}(\pi_1(B_i) \to \pi_1(M))\,(i > k)$ does lie in a vertex group. It is shown in [12, p. 90] that if every element of a finitely generated subgroup Γ of $\pi_1(G, \mathcal{G})$ is contained in a vertex group then Γ is itself contained in a vertex group. Thus $\operatorname{im}(\pi_1(B_i) \to \pi_1(M))$ is contained in a vertex group for $i > k$.

Now by [1, Proposition 2.3.1], one can associate with any non-trivial splitting of $\pi_1(M)$ a non-empty system of disjoint, orientable, incompressible surfaces $\Sigma_1, \ldots, \Sigma_r$ in M, none of them boundary-parallel, such that $\mathrm{im}(\pi_1(\Sigma_j) \to \pi_1(M))$ is contained in an edge group for $1 \leqq j \leqq r$ and $\mathrm{im}(\pi_1(P) \to \pi_1(M))$ is contained in a vertex group for each component P of $M - (\Sigma_1 \cup \ldots \cup \Sigma_r)$. Furthermore, if for a given component B of ∂M, $\mathrm{im}(\pi_1(B) \to \pi_1(M))$ is contained in a vertex group, we may take $\Sigma_1, \ldots, \Sigma_r$ to be disjoint from B. Applying this to the splitting we have obtained, we get a system of surfaces $\Sigma_1, \ldots, \Sigma_r$ that are disjoint from B_j for $k < i \leqq n$. On the other hand, some component of $\partial\Sigma_1 \cup \ldots \cup \partial\Sigma_r$ must lie in B_i for each $i \leqq k$, since otherwise γ_i would be disjoint from $\Sigma_1 \cup \ldots \cup \Sigma_r$, and g_i would belong to a vertex group. No component of $\partial\Sigma_1 \cup \ldots \cup \partial\Sigma_r$ can be a contractible curve in B_i, for then, since the simple manifold M is by definition irreducible, some Σ_i would be a boundary-parallel disc. Thus for $1 \leqq i \leqq k$, $(\partial\Sigma_1 \cup \ldots \cup \partial\Sigma_r) \cap B_i$ is a non-empty collection of non-contractible curves, which must all be homotopic since B_i is a torus. If they were homotopic to γ_i, g_i would lie in an edge group and hence in a vertex group, a contradiction. This proves Theorem 3.

We shall conclude this section by showing that Theorem 3 becomes false in an essential way if we drop the assumption that M is simple. In fact, $M = D^2 \times S^1$ and $M = S^1 \times S^1 \times I$, with $k=1$, are counterexamples, but we shall give more interesting ones. Let M be a Seifert fibered space with $n \geqq 2$ boundary components $B_1, \ldots, B_n$, and let k be chosen so that $1 \leqq k < n$. We may assume that $M \not\approx S^1 \times S^1 \times I$, and then, according to [5, Cor. II.2.11], up to isotopy there is a unique curve $\gamma_i \subset B_i$ which can appear as a fiber in a Seifert fibration of M. By [5, Lemma II.7.3], a non-boundary-parallel incompressible surface in M is either a union of fibers in some Seifert fibration (in which case any boundary curve lying in B_i, $1 \leqq i \leqq k$, must be homotopic to γ_i), or it meets every component of ∂M. Thus there is no surface satisfying the conditions stated in Theorem 3. Using this set of examples one can construct many others, for example by gluing a Seifert fibered space with three boundary tori to a simple manifold along a single boundary torus.

§ 2. Separating surfaces

We shall approach the proof of Theorem 2 via two special cases, of which the first is the crucial one.

Proof of Theorem 2 when M is simple and its boundary components are tori. It follows from Theorem 3 (with $k=1$ and $B_1 = B$) that there exist two non-homotopic, non-contractible simple closed curves γ_1, γ_2 in B such that for $i=1,2$, there is a non-boundary parallel, orientable, incompressible surface T_i whose boundary is non-empty, is contained in B, and consists of curves homotopic (in B) to γ_i. We shall show that one of the T_i must separate M.

Suppose that T_i does not separate M. Then an orientation of T_i determines a non-zero element $[T_i]$ of $H_2(M, \partial M; \mathbf{Q})$. The hypothesis of the Theorem implies, by Poincaré duality, that $H^1(\partial M; \mathbf{Q}) \to H^2(M, \partial M; \mathbf{Q})$ is surjective, whence (since $\mathbf{Q}$ is a field) $\partial: H_2(M, \partial M; \mathbf{Q}) \to H_1(\partial M; \mathbf{Q})$ is injective. Thus $\partial[T_i]$ is a non-zero multiple of $[\gamma_i] \in H_1(\partial M; \mathbf{Q})$, where $[\gamma_i]$ is defined by fixing an orientation of γ_i. From the exact homology sequence of $(M, \partial M)$ we know that $\partial[T_i] \in K = \ker(H_1(\partial M; \mathbf{Q}) \to H_1(M; \mathbf{Q}))$, and hence $[\gamma_i] \in K$. But K has rank $\leqq 1$ by a standard intersection-number argument; and since γ_1 and γ_2 are non-homotopic, $[\gamma_1]$ and $[\gamma_2]$ cannot both belong to K. This completes the proof. □

Proof of Theorem 2 when M is Seifert fibered. Fix a Seifert fibration of M with decomposition surface S and projection map $p: M \to S$. Since M is bounded and $H_1(\partial M; \mathbf{Q}) \to H_1(M; \mathbf{Q})$ is surjective, we have that S is bounded and $H_1(\partial S; \mathbf{Q}) \to H_1(S; \mathbf{Q})$ is surjective. Thus S can be obtained from a disc or Möbius band D by removing the interiors of disjoint discs $D_1, \ldots, D_l$, $l \geqq 0$. We may take $\partial D = p(B)$. Let $q_1, \ldots, q_m$ denote the images under p of the singular fibers of M. If α is a separating, properly embedded arc in D disjoint from the D_ι and $\{q_j\}$ then $A = p^{-1}(\alpha)$ is a separating, incompressible annulus in M; it is easy to see that A can be boundary-parallel only if α is the frontier of a disc containing no D_i or q_j. It follows at once that there is a separating, incompressible, non-boundary-parallel annulus with boundary contained in B unless either (i) D is a disc and $k + l \leqq 1$, or (ii) D is a Möbius band and $k = l = 0$. But (i) implies that M is homeomorphic to $D^2 \times S^1$ or $S^1 \times S^1 \times I$, and this is ruled out by the hypothesis. If (ii) holds then M is homeomorphic to a twisted I-bundle over a Klein bottle. But in this case M admits another Seifert fibration for which the decomposition surface is a disc and there are two singular fibers, and so neither (i) nor (ii) holds. This completes the proof. □

Proof of Theorem 2 in general. If M is reducible, then since $\partial M \neq \emptyset$, M contains a 2-sphere Σ not bounding a ball; and Σ cannot be boundary-parallel since the components of ∂M are tori. It follows at once that there is a non-boundary-parallel disc in M with boundary contained in B. Thus we may assume that M is irreducible. By [3, Lemma 13.2] (cf. [5], [6]), there is a finite collection $\{T_1, \ldots, T_l\}$ of disjoint incompressible tori in M such that each component of the manifold M' obtained by splitting M along $T_1 \cup \ldots \cup T_l$ is either simple or Seifert-fibered. Let us take this collection to be minimal among all such collections; then no T_i is boundary-parallel. Note that since $H_1(\partial M; \mathbf{Q}) \to H_1(M; \mathbf{Q})$ is surjective, each T_i must separate M. If $l = 0$ we are in one of the special cases treated above, so we may assume $l > 0$. Let M'_0 denote the component of M' containing B. Then since $l > 0$ and no T_i is boundary-parallel, M'_0 is not hemoemorphic to $D^2 \times S^1$ or $S^1 \times S^1 \times I$. Moreover, since $H_1(\partial M; \mathbf{Q}) \to H_1(M; \mathbf{Q})$ is surjective and M'_0 is homeomorphic to a submanifold of M, it is easy to see that $H_1(\partial M'_0; \mathbf{Q}) \to H_1(M'_0; \mathbf{Q})$ is surjective. By the above special cases of the Theorem, M'_0 contains a separating, non-boundary-parallel, incompressible surface Σ whose boundary is non-empty and is contained in B. Since each T_i separates M, the surface Σ is also separating, non-boundary-parallel, and incompressible when regarded as a surface in M, and this completes the proof. □

Proof of Theorem 1. The conclusion of the Theorem is trivial if some component of ∂M is a 2-sphere S; for then by hypothesis ∂M has a non-simply-connected component B; and if $\alpha \subset M$ is a properly embedded arc having one endpoint in S and one in B, then the frontier of a regular neighbourhood of $\alpha \cup S$ in M is a separating, incompressible disc which is not boundary-parallel. Thus to prove the Theorem in general it suffices to prove it under the assumption that no component of ∂M is a 2-sphere. Then the integer $l = -\chi(M) = -\frac{1}{2}\chi(\partial M)$ is non-negative; we shall prove the Theorem by induction on l. If $l=0$, each component of ∂M is a torus and in this case the conclusion follows from Theorem 2.

Now suppose that $l>0$, so that some component B of ∂M has genus >1. Let γ be any separating simple closed curve in B which is non-contractible in B. Let M' denote the 3-manifold obtained from M by attaching a 2-handle along γ. Then M' satisfies the hypotheses of the Theorem, and has at least two boundary components. Moreover, no component of M' is a 2-sphere, and $-\chi(M')<l$. Hence by the induction hypothesis, either M' contains a non-boundary-parallel, separating, incompressible surface with non-empty boundary, or M' is homeomorphic to $S^1 \times S^1 \times I$.

It is a special case of [13, Lemma 2.8] that if M is a compact, orientable 3-manifold, M' is obtained from adding a 2-handle to M, $H_1(\partial M; \mathbf{Q}) \to H_1(M; \mathbf{Q})$ is surjective, and no component of $\partial M'$ is a 2-sphere, then the existence of a non-boundary-parallel, separating, incompressible surface in M' implies the existence of one in M. Moreover, the proof of [13, Lemma 2.8] shows that if the given surface in M' has non-empty boundary, the surface in M may be taken to have non-empty boundary. Thus the induction step is complete unless M' is homeomorphic to $S^1 \times S^1 \times I$.

However, since γ separates T, M' cannot be homeomorphic to $S^1 \times S^1 \times I$ unless ∂M is connected and has genus 2. In this case, [13, Lemma 2.1] asserts that the separating, non-contractible curve $\gamma \subset \partial M$ may be chosen so that $H_1(A; \mathbf{Z}_2) \to H_1(M; \mathbf{Z}_2)$ has non-trivial kernel, where A is the closure of some component of $(\partial M) - \gamma$. This immediately implies that M' is not homeomorphic to $S^1 \times S^1 \times I$, and completes the proof of the Theorem. □

It is conceivable that a version of Theorem 1 is true without the hypothesis that the rational homology of M be carried by the boundary. However, without this hypothesis there is at least one more exceptional case: the manifold $S^1 \times (S^1 \times S^1 \# D^2)$ contains no separating incompressible surfaces.

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